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Supplementary materials for

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Proof S1 Proof of Lemma 4

Let the partial sum be $S_{N,M}=\sum_{n=-N}^{N}\sum_{m=-M}^{M}c_{n,m}\mathrm{e}^{\mathrm{j}my}\mathrm{e}^{\mathrm{i}nx}$. Then we have

$$\begin{aligned}
0 &\leq \int_{\mathbb{T}^2}|f(x,y)-S_{N,M}(x,y)|^2\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{T}^2}(f(x,y)-S_{N,M})(f^*(x,y)-S^*_{N,M})\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{T}^2}|f(x,y)|^2\mathrm{d}x\mathrm{d}y-\int_{\mathbb{T}^2}\sum_{n=-N}^{N}\sum_{m=-M}^{M}c_{n,m}\mathrm{e}^{\mathrm{j}my}\mathrm{e}^{\mathrm{i}nx}f^*(x,y)\mathrm{d}x\mathrm{d}y\\
&\quad-\int_{\mathbb{T}^2}f(x,y)\sum_{n=-N}^{N}\sum_{m=-M}^{M}\mathrm{e}^{-\mathrm{i}nx}\mathrm{e}^{-\mathrm{j}my}c^*_{n,m}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{T}^2}S_{N,M}S^*_{N,M}\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{T}^2}|f(x,y)|^2\mathrm{d}x\mathrm{d}y-\sum_{n=-N}^{N}\sum_{m=-M}^{M}c_{n,m}\int_{\mathbb{T}^2}(f(x,y)\mathrm{e}^{-\mathrm{i}nx}\mathrm{e}^{-\mathrm{j}my})^*\mathrm{d}x\mathrm{d}y\\
&\quad-\int_{\mathbb{T}^2}\sum_{n=-N}^{N}\sum_{m=-M}^{M}(f(x,y)\mathrm{e}^{-\mathrm{i}nx}\mathrm{e}^{-\mathrm{j}my})c^*_{n,m}\mathrm{d}x\mathrm{d}y+4\pi^2\sum_{n=-N}^{N}\sum_{m=-M}^{M}|c_{n,m}|^2\\
&=\int_{\mathbb{T}^2}|f(x,y)|^2\mathrm{d}x\mathrm{d}y-4\pi^2\sum_{n=-N}^{N}\sum_{m=-M}^{M}|c_{n,m}|^2.
\end{aligned}$$

Proof S2 Proof of Theorem 13

The error is given by

$$\begin{aligned}
&\mathrm{R1}_{N,M}(x,y)\\
&=\sum_{|n|>N}\sum_{m=-\infty}^{\infty}f(nh_1,mh_2)\frac{\sin(\sigma_1(x-nh_1))}{\sigma_1(x-nh_1)}\\
&\quad\cdot\frac{\sin(\sigma_2(y-mh_2))}{\sigma_2(y-mh_2)}+\sum_{|m|>M}\sum_{n=-\infty}^{\infty}f(nh_1,mh_2)\frac{\sin(\sigma_1(x-nh_1))}{\sigma_1(x-nh_1)}\frac{\sin(\sigma_2(y-mh_2))}{\sigma_2(y-mh_2)}\\
&\quad-\sum_{|m|>M}\sum_{|n|>N}f(nh_1,mh_2)\frac{\sin(\sigma_1(x-nh_1))}{\sigma_1(x-nh_1)}\cdot\frac{\sin(\sigma_2(y-mh_2))}{\sigma_2(y-mh_2)}.
\end{aligned}$$

Let E_n $(n=1,2,3)$ denote the three parts of the above equation. Applying the quaternion Cauchy inequality (3) to E_1 yields

$$|E_1| \le \left(\sum_{|n|>N} \sum_{m=-\infty}^{\infty} |f(nh_1, mh_2)|^2 \right)^{\frac{1}{2}} \left(\sum_{|n|>N} \sum_{m=-\infty}^{\infty} \left| \frac{\sin(\sigma_1(x-nh_1))}{\sigma_1(x-nh_1)} \frac{\sin(\sigma_2(y-mh_2))}{\sigma_2(y-mh_2)} \right|^2 \right)^{\frac{1}{2}}.$$

By Lemmas 10 and 11, we have

$$|E_1| \le \frac{2K_N|\sin(\sigma_1 x)|}{\pi} \left(\frac{2N}{(Nh_1)^2 - x^2} \right)^{\frac{1}{2}}.$$

Similarly,

$$|E_2| \le \frac{2L_M|\sin(\sigma_2 y)|}{\pi} \left(\frac{2M}{(Mh_2)^2 - y^2} \right)^{\frac{1}{2}},$$

$$|E_3| \le \frac{2\sqrt{MN}}{\pi^2} \frac{J_{N,M}}{\sqrt{((Nh_1)^2 - x^2)((Mh_2)^2 - y^2)}}.$$

Proof S3 Proof of Theorem 14

Because the BL quaternionic function $f(z_1, z_2)$ is holomorphic in two variables (z_1, z_2), then the residue theorem can be applied to the evaluation of the integrals in Eq. (25), and we have

$$\begin{aligned}
&\oint_{C_{\rm i}} \frac{f(x,z_2)}{(z_2-y)\sin^2(\frac{z_2\pi}{2h_2})} \frac{\sin^2(\frac{y\pi}{2h_2})}{2\pi\mathrm{j}} \mathrm{d}z_2 \\
&= \sin^2\left(\frac{y_1\pi}{2h_2}\right) \left(\frac{f(x,z_2)}{\sin^2(\frac{y\pi}{2h_2})}\bigg|_{z_2=y} \sum_{m=K_2(y)-M}^{m=K_2(y)+M} \frac{f(x,z_2)(z_2-m_2h_2)^2}{\sin^2(\frac{y\pi}{2h_2})} \frac{\partial}{\partial z_2}\bigg|_{z_2=m_2h_2} \right) \\
&= f(x,y) - \sum_{m=K_2(y)-M}^{K_2(y)+M} \left(\mathrm{sinc}\left(\frac{y}{2h_2} - m \right) \right)^2 \left(f(x, m_2h_2) + (y - m_2h_2) \frac{\partial f}{\partial y}(x, m_2h_2) \right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\sin^2(\frac{x\pi}{2h_1})}{2\pi\mathrm{i}} \oint_{C_{\rm j}} \frac{f(z_1,y)}{(z_1-x)\sin^2\left(\frac{z_1\pi}{2h_1}\right)} \mathrm{d}z_1 \\
&= \sin^2\left(\frac{x\pi}{2h_1}\right) \left(\frac{f(z_1,y)}{\sin^2(\frac{z_1\pi}{2h_1})}\bigg|_{z_1=x} + \sum_{m=K_1(x)-N}^{n=K_1(x)+N} \frac{\partial}{\partial z_1} \frac{(z_1-n_2h_1)^2 f(z_1,y)}{\sin^2(\frac{z_1\pi}{2h_1})}\bigg|_{z_1=m_2h_1} \right) \\
&= f(x,y) - \sum_{m=K_1(x)-N}^{n=K_1(x)+N} \left(\mathrm{sinc}\left(\frac{x}{2h_1} - n \right) \right)^2 \left(f(n_2h_1, y) + (x - n_2h_1) \frac{\partial f}{\partial x}(n_2h_1, y) \right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\sin^2(\frac{x\pi}{2h_1})}{2\pi\mathrm{i}} \oint_{C_{\rm j}} \oint_{C_{\rm i}} \frac{f(z_1,z_2)}{\sin^2(\frac{z_1\pi}{2h_1})(z_1-x)(z_2-y)\sin^2(\frac{z_2\pi}{2h_2})} \mathrm{d}z_1 \mathrm{d}z_2 \frac{\sin^2(\frac{y\pi}{2h_2})}{2\pi\mathrm{j}} \\
&= f(x,y) - \sum_{m=K_2(y)-M}^{K_2(y)+M} \left(\mathrm{sinc}\left(\frac{y}{2h_2} - m \right) \right)^2 \left(f(x, m_2h_2) + (y - m_2h_2) \frac{\partial f}{\partial y}(x, m_2h_2) \right) \\
&\quad - \sum_{m=K_1(x)-N}^{n=K_1(x)+N} \left(\mathrm{sinc}\left(\frac{x}{2h_1} - n \right) \right)^2 \left(f(n_2h_1, y) + (x - n_2h_1) \frac{\partial f}{\partial x}(n_2h_1, y) \right)
\end{aligned}$$

$$+\sum_{n=K_1(x)-N}^{K_1(x)+N}\sum_{m=K_2(y)-M}^{K_2(y)+M}\Bigg[f(n_2h_1,m_2h_2)(x-n_2h_1)\frac{\partial f}{\partial x}(n_2h_1,m_2h_2)$$
$$+(y-m_2h_2)\frac{\partial f}{\partial y}(n_2h_1,m_2h_2)+(x-n_2h_1)(y-m_2h_2)\frac{\partial^2 f}{\partial x\partial y}(n_2h_1,m_2h_2)\Bigg]$$
$$\cdot\left(\operatorname{sinc}\left(\frac{x}{2h_1}-n\right)\right)^2\left(\operatorname{sinc}\left(\frac{y}{2h_2}-m\right)\right)^2.$$

Proof S4 Proof of Lemma 12

First, we show that $\mathcal{F}_{\mathrm{R}}$ belongs to $L^1(\mathbb{R}^2,\mathbb{H})$ and $\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|\mathrm{d}v\mathrm{d}u<\infty$. Indeed, the quaternion Cauchy-Schwarz inequality (3) and Parseval Eq. (4) give

$$\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|\mathrm{d}v\mathrm{d}u<4\sigma_1\sigma_2\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|^2\mathrm{d}v\mathrm{d}u<\infty.$$

Second, if $|v|\le\sigma_1$ and $|u|\le\sigma_2$, then $|\mathrm{e}^{\mathrm{i}vz_1}|\le\mathrm{e}^{\sigma_1|z_1|}$ and $|\mathrm{e}^{\mathrm{j}uz_2}|\le\mathrm{e}^{\sigma_2|z_2|}$. Therefore, we have

$$\begin{aligned}&|f(z_1,z_2)|\\=&\frac{1}{4\pi^2}\left|\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathcal{F}_{\mathrm{R}}[f](v,u)\mathrm{e}^{\mathrm{j}uz_2}\mathrm{e}^{\mathrm{i}vz_1}\mathrm{d}u\mathrm{d}v\right|\\\le&\frac{1}{4\pi^2}\mathrm{e}^{\sigma_1|z_1|}\mathrm{e}^{\sigma_2|z_2|}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|\mathrm{d}v\mathrm{d}u,\end{aligned}$$

$$\begin{aligned}&|f(z_1,y)|\\=&\frac{1}{4\pi^2}\left|\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathcal{F}_{\mathrm{R}}[f](v,u)\mathrm{e}^{\mathrm{j}uy}\mathrm{e}^{\mathrm{i}vz_1}\mathrm{d}u\mathrm{d}v\right|\\\le&\frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}[f](v,u)||\mathrm{e}^{\mathrm{i}vz_1}|\mathrm{d}u\mathrm{d}v\\\le&\frac{1}{4\pi^2}\mathrm{e}^{\sigma_1|z_1|}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|\mathrm{d}v\mathrm{d}u,\\&|f(x,z_2)|\\=&\frac{1}{4\pi^2}\left|\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\mathcal{F}_{\mathrm{R}}[f](v,u)\mathrm{e}^{\mathrm{j}uz_2}\mathrm{e}^{\mathrm{i}vx}\mathrm{d}u\mathrm{d}v\right|\\\le&\frac{1}{4\pi^2}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}[f](v,u)||\mathrm{e}^{\mathrm{j}uz_2}|\mathrm{d}u\mathrm{d}v\\\le&\frac{1}{4\pi^2}\mathrm{e}^{\sigma_2|z_2|}\int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}|\mathcal{F}_{\mathrm{R}}(v,u)|\mathrm{d}v\mathrm{d}u.\end{aligned}$$