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Supplementary materials for

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Proof S1 Proof of Theorem 1

The following proof is composed of two steps. First, we shall certify the boundedness of the estimate of the pseudo-partial-derivative (PPD) matrix $\hat{\mho}_i(h)$.

Step 1: boundedness of $\hat{\mho}_i(h)$

Define

$$\hat{\mho}_i(h) \triangleq \begin{bmatrix} \hat{\varpi}_{i,11}(h) & \hat{\varpi}_{i,12}(h) & \cdots & \hat{\varpi}_{i,1n}(h) \\ \hat{\varpi}_{i,21}(h) & \hat{\varpi}_{i,22}(h) & \cdots & \hat{\varpi}_{i,2n}(h) \\ \vdots & \vdots & & \vdots \\ \hat{\varpi}_{i,n1}(h) & \hat{\varpi}_{i,n2}(h) & \cdots & \hat{\varpi}_{i,nn}(h) \end{bmatrix}, \tag{S1}$$

which can also be rewritten as

$$\hat{\mho}_i(h) \triangleq \begin{bmatrix} \hat{\mho}_{i,1}(h) \\ \hat{\mho}_{i,2}(h) \\ \vdots \\ \hat{\mho}_{i,n}(h) \end{bmatrix}, \tag{S2}$$

where $\hat{\mho}_{i,j}(h) \triangleq \begin{bmatrix} \hat{\varpi}_{i,j1}(h) & \hat{\varpi}_{i,j2}(h) & \ldots & \hat{\varpi}_{i,jn}(h) \end{bmatrix}$. Then, at the triggering instant $h = h_s^i$, one has

$$\hat{\mho}_i(h_s^i - 1) = \hat{\mho}_i(h-1), u_i(h_s^i - 1) = u_i(h-1). \tag{S3}$$

The updating algorithm (16) is reformulated as

$$\hat{\mho}_{i,j}(h) = \hat{\mho}_{i,j}(h-1) + \frac{\gamma(\Delta y_{i,j}(h) - \hat{\mho}_{i,j}(h-1)\Delta u_i(h-1))\Delta u_i^{\mathrm{T}}(h-1)}{\nu + \|\Delta u_i(h-1)\|^2}, \tag{S4}$$

where $\Delta y_{i,j}(h) = \mho_{i,j}(h-1)\Delta u_i(h-1)$. Defining the estimation error $\tilde{\mho}_{i,j} = \mho_{i,j}(h) - \hat{\mho}_{i,j}(h)$ and combining with Lemma 1 and algorithm (16), we arrive at

$$\tilde{\mho}_{i,j}(h) = \tilde{\mho}_{i,j}(h-1) + \mho_{i,j}(h) - \mho_{i,j}(h-1) - \frac{\gamma\tilde{\mho}_{i,j}(h-1)\Delta u_i(h-1)\Delta u_i^{\mathrm{T}}(h-1)}{\nu + \|\Delta u_i(h-1)\|^2}. \tag{S5}$$

It is inferred from the fact that $\|\mho_i(h)\| \leq m$ in Lemma 1, and we can obtain $\|\mho_i(h) - \mho_i(h-1)\| \leq 2m$. Applying the basic inequality yields

$$\begin{aligned} \|\tilde{\mho}_{i,j}(h)\| &\leq \|\mho_{i,j}(h) - \mho_{i,j}(h-1)\| + \left\|\tilde{\mho}_{i,j}(h-1) - \frac{\gamma\tilde{\mho}_{i,j}(h-1)\Delta u_i(h-1)\Delta u_i^{\mathrm{T}}(h-1)}{\nu + \|\Delta u_i(h-1)\|^2}\right\| \\ &\leq \left\|\tilde{\mho}_{i,j}(h-1) - \frac{\gamma\tilde{\mho}_{i,j}(h-1)\Delta u_i(h-1)\Delta u_i^{\mathrm{T}}(h-1)}{\nu + \|\Delta u_i(h-1)\|^2}\right\| + 2m, \end{aligned} \tag{S6}$$

which is further organized as follows:

$$\begin{aligned}&\Big\|\tilde{\mho}_{i,j}(h-1)-\frac{\gamma\tilde{\mho}_{i,j}(h-1)\Delta u_i(h-1)\Delta u_i^{\mathrm{T}}(h-1)}{\nu+\|\Delta u_i(h-1)\|^2}\Big\|^2\\&=\Big(-2+\frac{\gamma\|\Delta u_i(h-1)\|}{\nu+\|\Delta u_i(h-1)\|^2}\Big)\frac{\gamma\|\tilde{\mho}_{i,j}(h-1)\Delta u_i(h-1)\|}{\nu+\|\Delta u_i(h-1)\|^2}+\|\tilde{\mho}_{i,j}(h-1)\|^2.\end{aligned}\tag{S7}$$

In addition, it is not tricky to confirm that there exist $\gamma\in(0,1)$ and $\nu>0$ such that

$$-2+\frac{\gamma\|\Delta u_i(h-1)\|}{\nu+\|\Delta u_i(h-1)\|^2}<0,\tag{S8}$$

which is further concluded that there exists a scalar $\rho\in(0,1)$ satisfying the following condition:

$$\Big\|\tilde{\mho}_{i,j}(h-1)-\frac{\gamma\tilde{\mho}_{i,j}(h-1)\Delta u_i(h-1)\Delta u_i^{\mathrm{T}}(h-1)}{\nu+\|\Delta u_i(h-1)\|^2}\Big\|\leq\ \rho\|\tilde{\mho}_{i,j}(h-1)\|.\tag{S9}$$

Substituting inequality (S9) into inequality (S6) yields

$$\|\tilde{\mho}_{i,j}(h)\|\leq\rho\|\tilde{\mho}_{i,j}(h-1)\|+2m\leq\cdots\leq\rho^{h-1}\|\tilde{\mho}_{i,j}(1)\|+\frac{2m(1-\rho^{h-1})}{1-\rho},\tag{S10}$$

which implies that $\tilde{\mho}_{i,j}(h)$ is bounded. Since $\|\mho_i(h)\|\leq m$, it is readily seen from inequality (S10) that both $\tilde{\mho}_i(h)$ and $\hat{\mho}_i(h)$ are bounded. In addition, it is obvious that $\hat{\mho}_i(h)$ remains unchanged over the interval $h\in(h_s^i,h_{s+1}^i)$. Thus, it can be calculated that $\hat{\mho}_i(h)$ is bounded at all instants. Because both $\mho_i(h)$ and $\hat{\mho}_i(h)$ are bounded, one has that $Q(h)$, $A(h)$, and $M(h)$ are bounded matrices. Thus, there exist Nn-dimensional matrices $\bar{Q}$, $\bar{A}$, and $\bar{M}$ satisfying $Q^{\mathrm{T}}(h)Q(h)\leq\bar{Q}$, $A^{\mathrm{T}}(h)A(h)\leq\bar{A}$, and $M^{\mathrm{T}}(h)M(h)\leq\bar{M}$.

Step 2: consensus analysis

Construct the following Lyapunov function:

$$V_1(h)=\tilde{y}^{\mathrm{T}}(h)\tilde{y}(h).\tag{S11}$$

Along the trajectory of system (25), the difference of $V_1(h)$ can be evaluated as follows:

$$\begin{aligned}&\Delta V_1(h+1)\\=&V_1(h+1)-V_1(h)\\=&[M(h)\tilde{y}(h)+Q(h)\tilde{d}(h-1)+\eta(h+1)+A(h)\beta(h)e(h)]^{\mathrm{T}}\\&\cdot[M(h)\tilde{y}(h)+Q(h)\tilde{d}(h-1)+\eta(h+1)+A(h)\beta(h)e(h)]-\tilde{y}^{\mathrm{T}}(h)\tilde{y}(h)\\=&\tilde{y}^{\mathrm{T}}(h)(M^{\mathrm{T}}(h)M(h)-I+\tau)\tilde{y}(h)+\tilde{d}^{\mathrm{T}}(h-1)Q^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)+\eta^{\mathrm{T}}(h+1)\eta(h+1)\\&+\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)e(h)+2\tilde{d}^{\mathrm{T}}(h-1)Q^{\mathrm{T}}(h)M(h)\tilde{y}(h)+2\eta^{\mathrm{T}}(h+1)M(h)\tilde{y}(h)\\&+2\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)M(h)\tilde{y}(h)+2\eta^{\mathrm{T}}(h+1)Q(h)\tilde{d}(h-1)+2\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)\eta(h+1)\\&+2\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)-\tau\tilde{y}^{\mathrm{T}}(h)\tilde{y}(h).\end{aligned}\tag{S12}$$

By means of Assumption 3, one has $\|\tilde{d}_i(h)\|\leq\alpha(h)d$. With Lemma 4, Eq. (S12) is further manipulated as follows:

$$
\begin{aligned}
&\Delta V_1(h+1)\\
&\le \tilde{y}^{\mathrm{T}}(h)\big(\breve{\ell}_1 M^{\mathrm{T}}(h)M(h)-(1-\tau)I\big)\tilde{y}(h)+\breve{\ell}_2\tilde{d}^{\mathrm{T}}(h-1)Q^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)\\
&\quad+\breve{\ell}_3\eta^{\mathrm{T}}(h+1)\eta(h+1)+\breve{\ell}_4\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)e(h)\\
&\le \tilde{y}^{\mathrm{T}}(h)\big(\breve{\ell}_1\bar{M}-(1-\tau)I\big)\tilde{y}(h)+\breve{\ell}_2\tilde{d}^{\mathrm{T}}(h-1)\bar{Q}\tilde{d}(h-1)\\
&\quad+\breve{\ell}_3\eta^{\mathrm{T}}(h+1)\eta(h+1)+\breve{\ell}_4\beta(h)e^{\mathrm{T}}(h)\bar{A}\beta(h)e(h)\\
&\quad-\varepsilon_1\tilde{d}^{\mathrm{T}}(h-1)\tilde{d}(h-1)+\varepsilon_1\alpha^2(h-1)d^2\\
&\quad-\varepsilon_2\eta^{\mathrm{T}}(h+1)\eta(h+1)+\varepsilon_2\eta^{\mathrm{T}}(h+1)\eta(h+1)\\
&\quad+\varepsilon_3\beta^2(h)\left(\sum_{i=1}^{N}\theta_i-e^{\mathrm{T}}(h)e(h)\right)-\tau\tilde{y}^{\mathrm{T}}(h)\tilde{y}(h)\\
&=\varOmega_1^{\mathrm{T}}(h)\varPi_1\varOmega_1(h)-\tau\tilde{y}^{\mathrm{T}}(h)\tilde{y}(h)+\varUpsilon_1(h),
\end{aligned}
\tag{S13}
$$

where $\varOmega_1(h)\triangleq[\tilde{y}^{\mathrm{T}}(h)\ \ \tilde{d}^{\mathrm{T}}(h-1)\ \ \eta^{\mathrm{T}}(h+1)\ \ \beta(h)e^{\mathrm{T}}(h)]^{\mathrm{T}}$ and $\varUpsilon_1(h)\triangleq\varepsilon_1\alpha^2(h-1)d^2+\varepsilon_2\eta^{\mathrm{T}}(h+1)\eta(h+1)+\varepsilon_3\beta^2(h)\sum_{i=1}^{N}\theta_i$ with ε_1–ε_3 and ℓ_1–ℓ_6 being positive constants.

It follows from inequality (S13) that

$$
V_1(h+1)\le(1-\tau)V_1(h)+\varUpsilon_1(h). \tag{S14}
$$

Noting that $0<\tau<1$, $\sum_{h=0}^{\infty}\tau=\infty$, and $\lim_{h\to\infty}\frac{\varUpsilon_1(h)}{\tau}=0$, it is simple to deduce from Lemma 2 that $\lim_{h\to\infty}V_1(h)=0$. Thus, we can draw the conclusion that $\lim_{h\to\infty}\|\bar{y}(h)-y_i(h)\|=0$. The proof is complete.

Proof S2 Proof of Theorem 2

Construct a Lyapunov function as follows:

$$
V_2(h)=\hat{y}^{\mathrm{T}}(h)\hat{y}(h). \tag{S15}
$$

Then, calculating the difference of $V_2(h)$ results in

$$
\begin{aligned}
&\Delta V_2(h+1)\\
&=V_2(h+1)-V_2(h)\\
&=[M(h)\hat{y}(h)+Q(h)\tilde{d}(h-1)+A(h)\beta(h)\delta(h)+A(h)\beta(h)e(h)]^{\mathrm{T}}\\
&\quad\cdot[M(h)\hat{y}(h)+Q(h)\tilde{d}(h-1)+A(h)\beta(h)\delta(h)+A(h)\beta(h)e(h)]-\hat{y}^{\mathrm{T}}(h)\hat{y}(h)\\
&=\hat{y}^{\mathrm{T}}(h)(M^{\mathrm{T}}(h)M(h)-I)\hat{y}(h)+\tilde{d}^{\mathrm{T}}(h-1)Q^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)\\
&\quad+\beta(h)\delta^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)\delta(h)+\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)e(h)\\
&\quad+2\tilde{d}^{\mathrm{T}}(h-1)Q^{\mathrm{T}}(h)M(h)\hat{y}(h)+2\beta(h)\delta^{\mathrm{T}}(h)A^{\mathrm{T}}(h)M(h)\hat{y}(h)\\
&\quad+2\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)M(h)\hat{y}(h)+2\beta(h)\delta^{\mathrm{T}}(h)A^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)\\
&\quad+2\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)\delta(h)+2\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)-\hat{y}^{\mathrm{T}}(h)\hat{y}(h),
\end{aligned}
\tag{S16}
$$

which further implies that

$$
\begin{aligned}
&\Delta V_2(h+1)\\
&\leq \hat{y}^{\mathrm{T}}(h)\big(\breve{\ell}_5 M^{\mathrm{T}}(h)M(h)-I\big)\hat{y}(h)+\breve{\ell}_6\tilde{d}^{\mathrm{T}}(h-1)Q^{\mathrm{T}}(h)Q(h)\tilde{d}(h-1)\\
&\quad+\breve{\ell}_7\beta(h)\delta^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)\delta(h)+\breve{\ell}_8\beta(h)e^{\mathrm{T}}(h)A^{\mathrm{T}}(h)A(h)\beta(h)e(h)\\
&\leq \hat{y}^{\mathrm{T}}(h)\big(\breve{\ell}_5\bar{M}-I\big)\hat{y}(h)+\breve{\ell}_6\tilde{d}^{\mathrm{T}}(h-1)\bar{Q}\tilde{d}(h-1)+\breve{\ell}_7\beta(h)\delta^{\mathrm{T}}(h)\bar{A}\beta(h)\delta(h)\\
&\quad+\breve{\ell}_8\beta(h)e^{\mathrm{T}}(h)\bar{A}\beta(h)e(h)-\omega_1\tilde{d}^{\mathrm{T}}(h-1)\tilde{d}(h-1)+\omega_1\alpha^2(h-1)d^2\\
&\quad-\omega_2\beta(h)\delta^{\mathrm{T}}(h)\beta(h)\delta(h)+\omega_2\beta^2(h)l^2+\omega_3\beta^2(h)\left(\sum_{i=1}^{N}\theta_i-e^{\mathrm{T}}(h)e(h)\right)\\
&=\Omega_2^{\mathrm{T}}(h)\Pi_2\Omega_2(h)+\Upsilon_2(h),
\end{aligned}
\tag{S17}
$$

where $\Omega_2(h)\triangleq[\hat{y}^{\mathrm{T}}(h)\ \ \tilde{d}^{\mathrm{T}}(h-1)\ \ \beta(h)\delta^{\mathrm{T}}(h)\ \ \beta(h)e^{\mathrm{T}}(h)]^{\mathrm{T}}$ and $\Upsilon_2(h)\triangleq\omega_1\alpha^2(h-1)d^2+\omega_2\beta^2(h)l^2+\omega_3\beta^2(h)\sum_{i=1}^{N}\theta_i$ with ω_1–ω_3 and ℓ_7–ℓ_{12} being positive constants. Then, we can reasonably calculate that

$$
\Omega_2^{\mathrm{T}}(h)\Pi_2\Omega_2(h)\leq-\hbar\Omega_2^{\mathrm{T}}(h)\Omega_2(h),
\tag{S18}
$$

where $\hbar\triangleq\rho_{\min}\{-\Pi_2\}>0$. Taking inequality (S17) into account, we can obtain

$$
V_2(h+1)\leq\ V_2(h)-\hbar\Omega_2^{\mathrm{T}}(h)\Omega_2(h)+\Upsilon_2(h).
\tag{S19}
$$

Note that $\lim_{h\to\infty}\Upsilon_2(h)=0$ and that $\Upsilon_2(h)$ is bounded, which indicates that $\sum_{h=0}^{\infty}\Upsilon_2(h)<\infty$. It can be lightly derived from Lemma 3 that $V(h)$ converges to 0. Hence, we arrive at $\lim_{h\to\infty}V_2(h)=0$ and $\lim_{h\to\infty}\|y^*-y_i(h)\|=0$. According to the definition of limit, it can be shown that the limits of $y_i(h)$ and $\bar{y}(h)$ exist. Since the limit point of the sequence $\bar{y}(h)$ is unique, based on Theorem 1, one can deduce that $\lim_{h\to\infty}y_i(h)=\bar{y}(h)$. Thus, we have $\lim_{h\to\infty}\bar{y}(h)=y^*$. The proof of Theorem 2 is complete.