Electronic Supplementary Materials

For https://doi.org/10.1631/jzus.A2100560

Flow-induced vibration characteristics of the U-type Coriolis mass flowmeter with liquid hydrogen

Xiang-xiang PEI, Xiang LI, Hao-hao XU, Xuan-hong YE, Xiao-bin ZHANG

Data S1

Analysis of curved beams with rectangular cross-sections can be used to establish the vibration equation of the curved tube. A micro segment separated from the curved beam was analyzed for its general characteristics. As shown in Fig. S1, establish a coordinate system with the tangent direction of the beam axis as the z-axis, the radial direction of the beam as the x-axis, and the vertical downward as the y-axis direction.

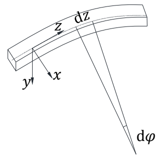

Fig. S1 The three-dimensional coordinate system of curved beam

The internal forces of the micro-segment include axial force N, torque T, shear forces Q_x and Q_y, and force couple M_x and M_y. Corresponding the differential length dz will produce corresponding the differential increment, as shown in Fig. S2.

The dynamic equations of force and force couple can be obtained through the force analysis of the microsegment, assuming that the mass of the analyzed micro-segment is dm=ρA·dz. Along the x-axis, the force and force couple equations can be established as Eq. (S1).

$$\sum F_x = \mathrm{d}m \cdot a_x:$$
$$-Q_x\cos\frac{\mathrm{d}\varphi}{2} + N sin\frac{\mathrm{d}\varphi}{2} + \left(Q_x + \frac{\partial Q_x}{\partial z}\mathrm{d}z\right) cos\frac{\mathrm{d}\varphi}{2} + \left(N + \frac{\partial N}{\partial z}\mathrm{d}z\right) sin\frac{\mathrm{d}\varphi}{2} = \rho A \cdot \mathrm{d}z \cdot a_x \tag{S1}$$

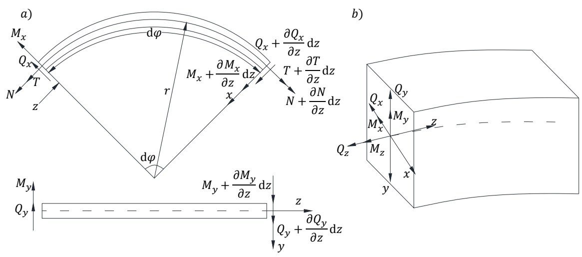

Fig. S2 The internal force of curved beam micro-segment

In the formula, cos(dφ/2)≈1, sin(dφ/2)≈dφ/2, dz=r·dφ, and the high-order trace is omitted, the following form can be obtained.

$$\frac{\partial Q_x}{\partial z} + \frac{N}{r} = \rho A a_x \tag{S2}$$

$$\sum M_x = J_x \cdot \alpha_x:$$
$$-M_x\cos\frac{\mathrm{d}\varphi}{2} + T\sin\frac{\mathrm{d}\varphi}{2} + \left(M_x + \frac{\partial M_x}{\partial z}\mathrm{d}z\right)\cos\frac{\mathrm{d}\varphi}{2} + \left(T + \frac{\partial T}{\partial z}\mathrm{d}z\right)\sin\frac{\mathrm{d}\varphi}{2} - Q_y\mathrm{d}z = J_x \cdot \alpha_x \tag{S3}$$

Where, the moment of inertia J_x can be obtained by Eq. (S4).

$$J_x = \int_m r^2 \mathrm{d}m \tag{S4}$$

For homogeneous objects, Eq. (S4) has the following form.

$$J_x = \int_m y^2\mathrm{d}m = \frac{m}{V}\int_V y^2\mathrm{d}V = \mathrm{d}z \cdot \frac{m}{V}\int_A y^2\mathrm{d}A = \mathrm{d}z \cdot \rho I_x \alpha_x \tag{S5}$$

Substituting the moment of inertia J_x into the Eq. (S3), and the formula can be simplified as

$$\frac{\partial M_x}{\partial z} + \frac{T}{r} - Q_y = \rho I_x \alpha_x \tag{S6}$$

where, I_x is the moment of inertia around the x-axis. Along the y-axis, the force and force couple equations can be established as:

$$\sum F_y = \mathrm{d}m \cdot a_y:$$
$$-Q_y + \left(Q_y + \frac{\partial Q_y}{\partial z}\mathrm{d}z\right) = \rho A \cdot \mathrm{d}z \cdot a_y \Rightarrow \frac{\partial Q_y}{\partial z} = \rho A a_y \tag{S7}$$

$$\sum M_y = J_y \cdot \alpha_y:$$
$$-M_y + \left(M_y + \frac{\partial M_y}{\partial z}\mathrm{d}z\right) + Q_x\cos\frac{\mathrm{d}\varphi}{2} \cdot r\sin\frac{\mathrm{d}\varphi}{2} + Q_x\sin\frac{\mathrm{d}\varphi}{2} \cdot \left(r - r\cos\frac{\mathrm{d}\varphi}{2}\right) + \left(Q_x + \frac{\partial Q_x}{\partial z}\mathrm{d}z\right)\cos\frac{\mathrm{d}\varphi}{2} \cdot r\sin\frac{\mathrm{d}\varphi}{2} + \left(Q_x + \frac{\partial Q_x}{\partial z}\mathrm{d}z\right)\sin\frac{\mathrm{d}\varphi}{2} \cdot \left(r - r\cos\frac{\mathrm{d}\varphi}{2}\right) = \mathrm{d}z \cdot \rho I_y \alpha_y \Rightarrow$$
$$\frac{\partial M_y}{\partial z} + Q_x = \rho I_y \alpha_y \tag{S8}$$

Along the z-axis, the force and force couple equations can be established as:

$$\sum F_z = \mathrm{d}m \cdot a_z:$$
$$-Q_x\sin\frac{\mathrm{d}\varphi}{2} - N\cos\frac{\mathrm{d}\varphi}{2} - \left(Q_x + \frac{\partial Q_x}{\partial z}\mathrm{d}z\right)\sin\frac{\mathrm{d}\varphi}{2} + \left(N + \frac{\partial N}{\partial z}\mathrm{d}z\right)\cos\frac{\mathrm{d}\varphi}{2} = \rho A \cdot \mathrm{d}z \cdot a_z \Rightarrow \frac{\partial N}{\partial z} - \frac{Q_x}{r} = \rho A a_z \tag{S9}$$

$$\sum M_z = J_z \cdot \alpha_z:$$
$$-M_x\sin\frac{\mathrm{d}\varphi}{2} - T\cos\frac{\mathrm{d}\varphi}{2} - \left(M_x + \frac{\partial M_x}{\partial z}\mathrm{d}z\right)\sin\frac{\mathrm{d}\varphi}{2} + \left(T + \frac{\partial T}{\partial z}\mathrm{d}z\right)\cos\frac{\mathrm{d}\varphi}{2} - Q_y\left(r - r\cos\frac{\mathrm{d}\varphi}{2}\right) + \left(Q_y + \frac{\partial Q_y}{\partial z}\mathrm{d}z\right)\left(r - r\cos\frac{\mathrm{d}\varphi}{2}\right) = \mathrm{d}z \cdot \rho I_z \alpha_z \Rightarrow$$
$$\frac{\partial T}{\partial z} - \frac{M_x}{r} = \rho I_z \alpha_z \tag{S10}$$

In summary, the dynamic equation of the curved beam is shown in Eq. (S11).

$$\frac{\partial Q_x}{\partial z} + \frac{N}{r} = \rho A a_x;\ \frac{\partial M_x}{\partial z} + \frac{T}{r} - Q_y = \rho I_x \alpha_x \tag{S11}$$
$$\frac{\partial Q_y}{\partial z} = \rho A a_y;\ \frac{\partial M_y}{\partial z} + Q_x = \rho I_y \alpha_y \tag{S12}$$
$$\frac{\partial N}{\partial z} - \frac{Q_x}{r} = \rho A a_z;\ \frac{\partial T}{\partial z} - \frac{M_x}{r} = \rho I_z \alpha_z \tag{S13}$$

In order to establish the relationship between internal force and deformation, it is necessary to construct the geometric relationship of micro-segment deformation. The deformation of the curved beam is shown in Fig. S3. The axial displacement is $u(z)$, the radial displacement is $v(z)$, the out-of-plane displacement is $w(z)$, and the torsion angle of the cross-section is $\Phi(z)$.

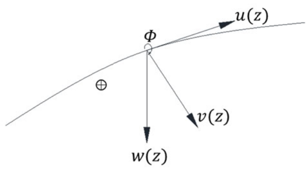

Fig. S3 Displacement and rotation angle of the micro-segment

The deformation of the micro-segment in the xOz plane is shown in Fig. S4.

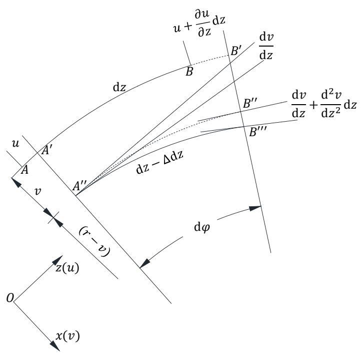

Fig. S4 Deformation of the micro-segment in the *xOz* plane

The micro-segment AB is deformed to maintain the equilibrium position of $A''B'''$. The total deformation can be decomposed into axial deformation ($AB \rightarrow A'B'$), radial deformation ($A'B' \rightarrow A''B''$), and flexural rotation around the y-axis ($A''B'' \rightarrow A''B'''$). If the axial deformation is neglected to be $A'B' \approx \mathrm{d}z$, when the initial radius of curvature is r and only the radial displacement is considered, the micro-segment will be shortened to $\mathrm{d}z$-$\Delta\mathrm{d}z$, where $\Delta\mathrm{d}z$ is shown in Eq. (S14).

$$\Delta\mathrm{d}z = r\mathrm{d}\varphi - (r - v)\mathrm{d}\varphi = v\mathrm{d}\varphi = \frac{v}{r}\mathrm{d}z, (r\mathrm{d}\varphi = \mathrm{d}z) \quad \text{(S14)}$$

Therefore, the axial strain ε_z of the micro-segment is

$$\varepsilon_z = \frac{\left(u+\frac{\mathrm{d}u}{\mathrm{d}z}\mathrm{d}z\right) - u - \frac{v}{r}\mathrm{d}z}{\mathrm{d}z} = \frac{\mathrm{d}u}{\mathrm{d}z} - \frac{v}{r} \quad \text{(S15)}$$

The curvature of the micro-segment around the y-axis after deformation can be calculated by the angular increment per unit arc length. The total curvature of the beam segment when it is deformed to the position $A''B'''$ can be deduced from Fig. S4.

$$\frac{1}{r'} = \frac{\mathrm{d}\varphi + \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}^2v}{\mathrm{d}z^2}\mathrm{d}z\right) - \frac{\mathrm{d}v}{\mathrm{d}z}}{\mathrm{d}z - \Delta\mathrm{d}z} = \frac{\frac{1}{r}\mathrm{d}z + \frac{\mathrm{d}^2v}{\mathrm{d}z^2}\mathrm{d}z}{\mathrm{d}z - \frac{v}{r}\mathrm{d}z} = \frac{\frac{1}{r} + \frac{\mathrm{d}^2v}{\mathrm{d}z^2}}{1 - \frac{v}{r}} = \frac{\left(\frac{1}{r} + \frac{\mathrm{d}^2v}{\mathrm{d}z^2}\right)\left(1 + \frac{v}{r}\right)}{1 - \left(\frac{v}{r}\right)^2} \approx \frac{1}{r} + \frac{\mathrm{d}^2v}{\mathrm{d}z^2} + \frac{v}{r^2} \quad \text{(S16)}$$

The change in curvature of the micro-segment along the y-axis due to deformation is k_y.

$$k_y = \frac{1}{r'} - \frac{1}{r} = \frac{\mathrm{d}^2v}{\mathrm{d}z^2} + \frac{v}{r^2} \quad \text{(S17)}$$

In order to establish the torsion equations around the x-axis and z-axis, it is necessary to investigate the coupling relationship between the bending deformation and the torsion deformation of the beam section shown in Fig. S5.

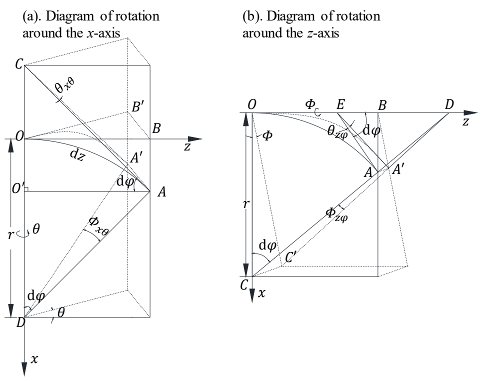

Fig. S5 The coupling relationship between bending and torsion deformation

When the micro-segment OA causes a slight angle θ around the x-axis, the section A will produce a rotation angle $\theta_{x\theta}$ and a torsion angle $\Phi_{x\theta}$. As shown in Fig. S5(a), OB, OB' and CA, CA' are the corresponding tangents when points O and A are not rotated and when rotated by θ.

$$\frac{\theta_{x\theta}}{\theta} = \frac{AA'/CA}{BB'/OB} \quad \text{(S18)}$$

Since $AA'=BB'$, the following formula can be obtained.

$$\theta_{x\theta} = \theta \cdot \frac{OB}{CA} = \theta \cdot \frac{O'A}{CA} = \theta\cos d\varphi \; (\theta_{x\theta} \approx \theta) \quad \text{(S19)}$$

The corresponding radius of point A deformed from DA to DA', then, $\Phi_{x\theta}$ can be obtained.

$$\Phi_{x\theta} = \frac{AA'}{r} = \frac{1}{r}CA \cdot \theta_{x\theta} = \frac{1}{r} \cdot r \cdot \mathrm{tg}d\varphi \cdot \theta\cos d\varphi = \theta\sin d\varphi \; (\Phi_{x\theta} \approx \theta d\varphi) \quad \text{(S20)}$$

When the torsion angle of the micro-segment OA around the z-axis is Φ, the torsion angle $\Phi_{z\varphi}$ and the rotation angle $\theta_{z\varphi}$ will be generated in the section A. As shown in Fig. S5(b), EA, EA' and DC, DC' are the tangent and diameter of the A section before and after deformation, respectively.

$$\frac{\Phi_{z\varphi}}{\Phi}=\frac{\frac{CC'}{CD}}{\frac{CC'}{r}}=\frac{r}{CD}=\frac{r}{\frac{r}{\mathrm{cosd}\varphi}}=\mathrm{cosd}\varphi;$$

$$\Phi_{z\varphi}=\Phi\mathrm{cosd}\varphi\approx\Phi \tag{S21}$$

$$\theta_{z\varphi}=-\frac{AA'}{AE}=-\frac{AD\Phi_{z\varphi}}{\frac{AD}{\mathrm{tgd}\varphi}}=-\Phi\mathrm{cosd}\varphi\cdot\mathrm{tgd}\varphi$$

$$=-\Phi\mathrm{sind}\varphi\approx-\Phi\mathrm{d}\varphi \tag{S22}$$

Then, the actual curvature k_x of the micro-segment around the x-axis can be obtained by Eq. (S23).

$$k_x=\frac{\left[\frac{\mathrm{d}w}{\mathrm{d}z}+\frac{\mathrm{d}^2w}{\mathrm{d}z^2}\mathrm{d}z-\Phi\mathrm{d}\varphi\right]-\frac{\mathrm{d}w}{\mathrm{d}z}}{\mathrm{d}z}=\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\frac{\Phi}{r} \tag{S23}$$

The twist rate k_z of the micro-segment around the z-axis is shown in Eq. (S24).

$$k_z=\frac{\left[\Phi+\mathrm{d}\Phi+\frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{d}\varphi\right]-\Phi}{\mathrm{d}z}=\frac{\mathrm{d}\Phi}{\mathrm{d}z}+\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}z} \tag{S24}$$

In summary, the geometric equation of the curved beam can be obtained.

$$\varepsilon_z=\frac{\mathrm{d}u}{\mathrm{d}z}-\frac{v}{r} \tag{S25}$$

$$k_x=\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\frac{\Phi}{r} \tag{S26}$$

$$k_y=\frac{\mathrm{d}^2v}{\mathrm{d}z^2}+\frac{v}{r^2} \tag{S27}$$

$$k_z=\frac{\mathrm{d}\Phi}{\mathrm{d}z}+\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}z} \tag{S28}$$

According to the principle of material mechanics of general elastomers, the force of the cross section can be related to the deformation of the material.

$$N_z=EA\varepsilon_z \tag{S29}$$

$$M_x=-EI_xk_x \tag{S30}$$

$$M_y=EI_yk_y \tag{S31}$$

$$T=-EI_\omega\frac{\mathrm{d}^2k_z}{\mathrm{d}z^2}+GI_dk_z \tag{S32}$$

Where, A is the area of the cross section, E is the Young's modulus, G is the shear modulus, I_x and I_y are the moment of inertia around the x-axis and y-axis, I_d is the moment of inertia around the z-axis, and the warping constant of the beam is I_ω, which warping will occur when the non-circular section is twisted. For a circular cross-section, no warpage occurs during torsion, and the relationships between material deformation and force are shown in Eqs. (S33~S36).

$$N_z=\mathrm{EA}\left(\frac{\mathrm{d}u}{\mathrm{d}z}-\frac{v}{r}\right) \tag{S33}$$

$$M_x=-EI_x\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\frac{\Phi}{r}\right) \tag{S34}$$

$$M_y=EI_y\left(\frac{\mathrm{d}^2v}{\mathrm{d}z^2}+\frac{v}{r^2}\right) \tag{S35}$$

$$T=GI_d\left(\frac{\mathrm{d}\Phi}{\mathrm{d}z}+\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}z}\right) \tag{S36}$$

The analyzed curved tube and its cross section are shown in Fig. S2 (in manuscript), the average radius of the tube is r_t, the wall thickness of the tube is t_t, where $t_t << r_t$. Denote the cross-sectional area of the pipe section as A, the second moment of area as I, and the moment of inertia as J. In the process of modeling the measuring tube, assuming that the influence of axial stress is not considered, the axial strain is zero, that is

$$\varepsilon_z=0\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}z}-\frac{v}{r}=0\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}z}=\frac{v}{r} \tag{S37}$$

Considering that the curved tube only vibrates in the low frequency range, and $r_t<<r$, the moment of inertia of the micro-segment can be ignored. The dynamic control equation derived above and the relationship between deformation and force can be abbreviated as Eqs. (S38~S43).

$$\frac{\partial Q_x}{\partial z}+\frac{N}{r}=\rho Aa_x=\rho A\frac{\partial^2v}{\partial t^2};\ \frac{\partial M_x}{\partial z}+\frac{T}{r}-Q_y=0 \tag{S38}$$

$$\frac{\partial Q_y}{\partial z}=\rho Aa_y=\rho A\frac{\partial^2w}{\partial t^2};\ \frac{\partial M_y}{\partial z}+Q_x=0 \tag{S39}$$

$$\frac{\partial N}{\partial z}-\frac{Q_x}{r}=\rho Aa_z=\rho A\frac{\partial^2u}{\partial t^2};\ \frac{\partial T}{\partial z}-\frac{M_x}{r}=0 \tag{S40}$$

$$M_x=-EI_x\left(\frac{\partial^2w}{\partial z^2}-\frac{\Phi}{r}\right) \tag{S41}$$

$$M_y=EI_y\left(\frac{\partial^2v}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial z}\right) \tag{S42}$$

$$T=GI_d\left(\frac{\partial\Phi}{\partial z}+\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}z}\right) \tag{S43}$$

Combining the above equations and eliminating the force and force couple, the out-of-plane vibration equation of curved tube can be obtained as shown in Eq. (S44) and Eq. (S45).

$$\frac{\partial^2}{\partial z^2}\left(\frac{GJ}{EI}\frac{w}{r^2}-\frac{\partial^2w}{\partial z^2}\right)+\frac{1}{r}\left(1+\frac{GJ}{EI}\right)\frac{\partial^2\Phi}{\partial z^2}=\frac{\rho A}{EI}\frac{\partial^2w}{\partial t^2} \tag{S44}$$

$$\frac{1}{r}\left(1+\frac{GJ}{EI}\right)\frac{\partial^2w}{\partial z^2}+\left(\frac{GJ}{EI}\frac{\partial^2}{\partial z^2}-\frac{1}{r^2}\right)\Phi=0 \tag{S45}$$

When there is fluid flow inside the pipe, not only additional inertia terms will be introduced, but also the pressure of fluid. To simplify the analysis, the influence of fluid pressure is ignored and the fluid is considered to flow along the pipe at the same average velocity. The total mass per unit length is the sum of the tube mass M_t and the fluid additional mass M_f. In the process of establishing the control equation, only the movement of the micro-element segment along the z-direction is analyzed. The local acceleration of the fluid in this direction is equal to the local acceleration of the tube. Therefore, the mass in the equation of motion should be M_f+M_t instead of M_t. As indicated in Fig. S6, assuming that the fluid velocity along the tube is U and the velocity of the fluid in the second normal direction is dw/dt.

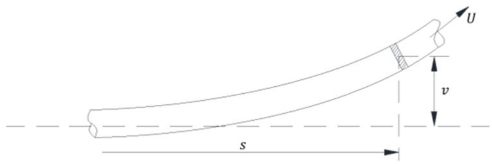

Fig. S6 Schematic diagram of fluid micro-element movement

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial z}U + \frac{\partial w}{\partial t} \tag{S46}$$

The acceleration of fluid in the second normal direction can be written as:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \frac{\partial^2 w}{\partial z^2}U^2 + 2\frac{\partial^2 w}{\partial z\,\partial t}U + \frac{\partial^2 w}{\partial t^2} + \frac{\partial U}{\partial t}\frac{\partial w}{\partial z} \tag{S47}$$

Assuming that the fluid is stably, the $\partial U/\partial t$ term can be ignored and the inertia term in the equation of motion is shown in Eq. (S48).

$$M_{\mathrm{t}}\frac{\partial^2 w}{\partial t^2} + M_{\mathrm{f}}(\frac{\partial^2 w}{\partial z^2}U^2 + 2\frac{\partial^2 w}{\partial z\,\partial t}U + \frac{\partial^2 w}{\partial t^2}) \tag{S48}$$

Substituting the total inertia term for the original inertia term in the control equation, the control equations of straight tube and curved tube with fluid flowing inside can be obtained as Eqs. (S49~S52), respectively.

$$\frac{\partial^4 w}{\partial z^4} + \frac{2M_{\mathrm{f}}}{EI}U\frac{\partial^2 w}{\partial z\,\partial t} + \frac{M_f}{EI}U^2\frac{\partial^2 w}{\partial z^2} + \frac{M_{\mathrm{t}}+M_{\mathrm{f}}}{EI}\frac{\partial^2 w}{\partial t^2} = 0 \tag{S49}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{S50}$$

$$\frac{\partial^4 w}{\partial z^4} - \frac{GJ}{EIr^2}\frac{\partial^2 w}{\partial z^2} - \frac{1+GJ/EI}{r}\frac{\partial^2 \Phi}{\partial z^2} + \frac{2M_{\mathrm{f}}}{EI}U\frac{\partial^2 w}{\partial z\,\partial t} + \frac{M_{\mathrm{f}}}{EI}U^2\frac{\partial^2 w}{\partial z^2} + \frac{M_{\mathrm{t}}+M_{\mathrm{f}}}{EI}\frac{\partial^2 w}{\partial t^2} = 0 \tag{S51}$$

$$\frac{\partial^2 \Phi}{\partial z^2} - \frac{EI}{r^2 GJ}\Phi + \frac{1}{r}\left(1 + \frac{EI}{GJ}\right)\frac{\partial^2 w}{\partial z^2} = 0 \tag{S52}$$