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TODIM and TOPSIS with Z-numbers*

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Abstract: In this paper, we present an approach that can handle Z-numbers in the context of multi-criteria decision-making problems. The concept of Z-number as an ordered pair Z=(A, B) of fuzzy numbers A and B is used, where A is a linguistic value of a variable of interest and B is a linguistic value of the probability measure of A. As human beings, we communicate with each other by means of natural language using sentences like "the journey from home to university most likely takes about half an hour." The Z-numbers are converted to fuzzy numbers. Then the Z-TODIM and Z-TOPSIS are presented as a direct extension of the fuzzy TODIM and fuzzy TOPSIS, respectively. The proposed methods are applied to two case studies and compared with the standard approach using crisp values. The results obtained show the feasibility of the approach.

Key words: Multi-criteria decision-making; TODIM; TOPSIS; Fuzzy number; Z-number
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1 Introduction

Fuzzy set theory (Zadeh, 1965) is an effective approach for handling vagueness, lack of knowledge, and ambiguity inherent in the human decision-making process, and has been successfully applied to solve multi-criteria decision-making (MCDM) problems. For real-world problems, the decision matrix is affected by uncertainties, which may be modeled using fuzzy numbers. A fuzzy number (Dubois and Prade, 1980; Zimmermann, 1991) can be seen as an extension of an interval with a varied grade of membership; this means that each value in the interval has been associated with a real number that indicates its compatibility with the vague statement associated with a fuzzy number. In the last decades, MCDM methods, such as standard TOPSIS (Hwang and Yoon, 1981) and TODIM (Gomes and Lima, 1992), have been largely applied to a variety of MCDM problems.

TOPSIS has been generalized to deal with a variety of information types, such as interval numbers (Jahanshahloo et al., 2009; Dymova et al., 2013), probability distributions (Xiong and Qi, 2010), fuzzy information (Chen, 2000; Wang et al., 2005; Chen and Tsao, 2008; Mahdavi et al., 2008; Wang and Lee, 2009; Krohling and Campanharo, 2011), intuitionistic fuzzy information (Yue, 2014), and interval-valued intuitionistic fuzzy information (Ye, 2010; Park et al., 2011). For readers interested in TOPSIS with hybrid data types, he/she may refer to Lourenzutti and Krohling (2016).

The TODIM method has been extended to different types of information. Krohling and de Souza (2012) first proposed the fuzzy TODIM to treat fuzzy numbers. Another important aspect in the treatment of uncertainty is the reliability of information; thus, the fuzzy TODIM was extended to the intuitionistic fuzzy TODIM (Krohling et al., 2013; Lourenzutti and Krohling, 2013; Yu et al., 2018). The intuitionistic fuzzy TODIM can deal with uncertainty described by

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intuitionistic fuzzy numbers (Atanassov, 1986). Zhang and Xu (2014) extended the TODIM for hesitant fuzzy information. For information described by probability distributions, the Hellinger TODIM and Hellinger TOPSIS (Lourenzutti and Krohling, 2014) were developed.

In many decision-making problems, data come from different sources; data may be presented in a numerical format, interval, fuzzy, fuzzy intuitionist, or even described by probability distributions. Therefore, a method for decision-making should be able to simultaneously process hybrid information, i.e., information in different formats (Wang and Wang, 2008; Xiong and Qi, 2010; Fan et al., 2013; Lourenzutti and Krohling, 2016; Lourenzutti et al., 2017). In the last few years, other related approaches to MCDM have been presented. Zhou et al. (2017) approached the MCDM problem based on SMAA-ELECTRE with extended grey numbers. Peng et al. (2018) proposed probability multi-valued neutrosophic sets with applications to group decision-making. In addition, alternative methods have appeared to handle linguistic intuitionistic fuzzy numbers (Zhang et al., 2017), with multi-hesitant fuzzy linguistic information (Wang et al., 2017) or hesitant probabilistic fuzzy sets (Li and Wang, 2017).

Methods such as TOPSIS and TODIM and their extensions have been successfully applied to tackle different kinds of information to solve MCDM problems. However, a very important aspect in decision-making is the reliability of the information involved in the process. Z-numbers were proposed by Zadeh (2011) as a new way to treat uncertainty and reliability of information. Z-numbers are composed of two parts: one part is a restriction on values that can be assumed, and the other part is the reliability of the information. Z-numbers may be used to model sentences like "the temperature in summer will surely be very high," or "the journey from home to university most likely takes about half an hour." As human beings, we communicate with each other by means of natural language using sentences like the ones mentioned above. An approach for working with Z-numbers was presented by Kang et al. (2012b), where a procedure to convert Z-numbers into fuzzy numbers was proposed. Then Kang et al. (2012a) applied the procedure to solve MCDM problems after converting Z-numbers to crisp values.

Azadeh et al. (2013) presented a new analytic hierarchy process (AHP) method based on Z-numbers to deal with linguistic decision-making problems. Xiao (2014) proposed a method for fuzzy multi-criteria decision-making with Z-numbers, where the evaluation of each alternative with respect to each criterion was described by Z-numbers. Patel et al. (2015) provided a model of Z-numbers based on certain realistic assumptions regarding probability distributions. Aliev et al. (2015) proposed an arithmetic method to discrete Z-numbers as addition, subtraction, multiplication, division, square root, and other operations. Then Aliev et al. (2016) proposed an arithmetic method for continuous Z-numbers. Aliev et al. (2016) stated that problems involving computation with *Z*-numbers were far from easy to solve. Peng and Wang (2017) presented an approach for handling hesitant and uncertain linguistic Z-numbers with applications to group decision-making problems.

2 Fuzzy multi-criteria decision-making

In this section, we provide some basic knowledge about fuzzy sets and fuzzy numbers.

2.1 Preliminaries on fuzzy numbers

Definition 1 (Fuzzy sets) A fuzzy set $\tilde{A}$ in a universe of discourse X is characterized by a membership function $\mu_{\tilde{A}}(x)$ that assigns each element x in X to a real number in the interval [0, 1]. Numeric value $\mu_{\tilde{A}}(x)$ stands for the grade of membership of x in $\tilde{A}$ (Zadeh, 1965).

Definition 2 (Fuzzy numbers) A trapezoidal fuzzy number $\tilde{a}$ is defined by a quadruplet $\tilde{a}=(a_1, a_2, a_3, a_4)$ with membership function given as (Mahdavi et al., 2008)

$$\mu_{\tilde{a}}(x)=\begin{cases}0, & x<a_1,\\ \dfrac{x-a_1}{a_2-a_1}, & a_1\le x\le a_2,\\ 1, & a_2\le x\le a_3,\\ \dfrac{x-a_3}{a_3-a_4}, & a_3\le x\le a_4,\\ 0, & x>a_4.\end{cases} \tag{1}$$

Remark 1 A triangular fuzzy number is a special case of a trapezoidal fuzzy number when $a_2=a_3$.

Definition 3 (Defuzzified value) Let a trapezoidal fuzzy number be $\tilde{a}=(a_1, a_2, a_3, a_4)$. Then the defuzzified value $m(\tilde{a})$ is calculated as (Mahdavi et al., 2008)

$$m(\tilde{a})=\frac{1}{4}(a_1+a_2+a_3+a_4). \quad (2)$$

Definition 4 (Distance between trapezoidal fuzzy numbers) Let two trapezoidal fuzzy numbers be $\tilde{a}=(a_1, a_2, a_3, a_4)$ and $\tilde{b}=(b_1, b_2, b_3, b_4)$. Then the distance between them is calculated as (Mahdavi et al., 2008)

$$d(\tilde{a},\tilde{b})=\sqrt{\frac{1}{6}\left[\sum_{i=1}^{4}(b_i-a_i)^2+\sum_{i=1}^{3}(b_i-a_i)(b_{i+1}-a_{i+1})\right]}. \quad (3)$$

2.2 Decision matrix described by fuzzy numbers

Let us consider the fuzzy decision matrix $\boldsymbol{A}$, which consists of alternatives and criteria, and it is described as

$$\boldsymbol{A}=\begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & \cdots & C_n \end{array} \\ \begin{pmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \vdots & & \vdots \\ \tilde{x}_{m1} & \cdots & \tilde{x}_{mn} \end{pmatrix} \end{array}, \quad (4)$$

where A_i (i=1, 2, …, m) are alternatives, C_j (j=1, 2, …, n) are criteria, and $\tilde{x}_{ij}$ are fuzzy numbers that indicate the rating of alternative A_i with respect to criterion C_j. The weight vector $\boldsymbol{W}=(w_1, w_2, \dots, w_n)$ composed of the individual weights w_j (j=1, 2, …, n) for each criterion C_j satisfies $\sum_{j=1}^{n} w_j=1$.

2.3 Fuzzy TODIM

The fuzzy TODIM method (Krohling and de Souza, 2012), which is an extension of TODIM (Gomes and Lima, 1992; Gomes and Rangel, 2009), is described in the following steps:

Step 1: The criteria are normally classified into two types: benefit and cost. The fuzzy decision matrix $\tilde{\boldsymbol{A}}=[\tilde{x}_{ij}]_{m\times n}$ (i=1, 2, …, m, j=1, 2, …, n) is normalized resulting in the corresponding fuzzy decision matrix $\tilde{\boldsymbol{R}}=[\tilde{r}_{ij}]_{m\times n}$. The fuzzy normalized value $\tilde{r}_{ij}$ is calculated as

$$\begin{cases} r_{ij}{}^{k}=\dfrac{\max(a_{ij}{}^{4})-a_{ij}{}^{k}}{\max_i(a_{ij}{}^{4})-\min_i a_{ij}}, & \text{for cost criteria,} \\ r_{ij}{}^{k}=\dfrac{a_{ij}{}^{k}-\min(a_{ij})}{\max_i(a_{ij}{}^{4})-\min_i a_{ij}}, & \text{for benefit criteria,} \end{cases} \quad (5)$$

where k=1, 2, 3, 4.

Step 2: Calculate the dominance of each alternative $\tilde{A}_i$ over each alternative $\tilde{A}_j$ by (Lourenzutti and Krohling, 2013)

$$\delta(\tilde{A}_i,\tilde{A}_j)=\sum_{c=1}^{m}\phi_c(\tilde{A}_i,\tilde{A}_j) \quad \forall(i,j), \quad (6)$$

where

$$\phi_c(\tilde{A}_i,\tilde{A}_j)=\begin{cases} \sqrt{w_c\cdot d(\tilde{x}_{ic},\tilde{x}_{jc})}, & m(\tilde{x}_{ic})-m(\tilde{x}_{jc})>0, \\ 0, & m(\tilde{x}_{ic})-m(\tilde{x}_{jc})=0, \\ -\dfrac{1}{\theta}\sqrt{w_c\cdot d(\tilde{x}_{ic},\tilde{x}_{jc})}, & m(\tilde{x}_{ic})-m(\tilde{x}_{jc})<0. \end{cases} \quad (7)$$

Note that $\phi_c(\tilde{A}_i, \tilde{A}_j)$ represents the contribution of criterion c to function $\delta(\tilde{A}_i, \tilde{A}_j)$ when comparing alternative i with alternative j. Parameter θ represents the attenuation factor of the losses, which can be tuned according to the problem at hand. In Eq. (7), $m(\tilde{x}_{ic})$ and $m(\tilde{x}_{jc})$ stand for the defuzzified values of fuzzy numbers $\tilde{x}_{ic}$ and $\tilde{x}_{jc}$, respectively. $d(\tilde{x}_{ic},\tilde{x}_{jc})$ designates the distance between two fuzzy numbers $\tilde{x}_{ic}$ and $\tilde{x}_{jc}$, as defined in Eq. (3). There are three cases that can occur in Eq. (7): (1) If $m(\tilde{x}_{ic})-m(\tilde{x}_{jc})>0$, it represents a gain; (2) If $m(\tilde{x}_{ic})-m(\tilde{x}_{jc})=0$, there is no gain or loss; (3) If $m(\tilde{x}_{ic})-m(\tilde{x}_{jc})<0$, it represents a loss. The final matrix of the dominance is obtained by summing up the partial matrices of dominance for each criterion.

Step 3: Calculate the global value of alternative i by normalizing the final matrix of dominance as

$$\xi_i=\frac{\sum_j\delta(i,j)-\min\sum_j\delta(i,j)}{\max\sum_j\delta(i,j)-\min\sum_j\delta(i,j)}. \quad (8)$$

Ranking the values ξ_i provides the rank of each alternative, and the best alternatives are those that have higher values of ξ_i.

2.4 Fuzzy TOPSIS

In this subsection, we describe the fuzzy TOPSIS (Krohling and Campanharo, 2011), since we will apply it. In this case, the weighted normalized fuzzy decision matrix $\tilde{\boldsymbol{P}}=\left[\tilde{p}_{ij}\right]_{m\times n}$ (i=1, 2, …, m, j=1, 2, …, n) is constructed by multiplying the normalized fuzzy decision matrix by its associated weights. The weighted fuzzy normalized value $\tilde{p}_{ij}$ is calculated as $\tilde{p}_{ij}=w_i\cdot\tilde{r}_{ij}$ (i=1, 2, …, m, j=1, 2, …, n).

The fuzzy TOPSIS is described as the following steps:

Step 1: Identify the positive ideal solution A^+ (benefits) and negative ideal solution A^- (costs) as

$$A^+=(\tilde{p}_1^+,\tilde{p}_2^+,...,\tilde{p}_m^+), \tag{9}$$

$$A^-=(\tilde{p}_1^-,\tilde{p}_2^-,...,\tilde{p}_m^-), \tag{10}$$

where

$$\tilde{p}_j^+=(\max_i \tilde{p}_{ij}, j\in J_1; \min_i \tilde{p}_{ij}, j\in J_2), \tag{11}$$

$$\tilde{p}_j^-=(\min_i \tilde{p}_{ij}, j\in J_1; \max_i \tilde{p}_{ij}, j\in J_2), \tag{12}$$

where J_1 and J_2 represent the criteria benefits and costs, respectively.

Step 2: Calculate the Euclidean distances from the positive ideal solution A^+ and the negative ideal solution A^- of each alternative A_i as

$$d_i^+=\sum\nolimits_{j=1}^n d(\tilde{p}_{ij},\tilde{p}_j^+), \tag{13}$$

$$d_i^-=\sum\nolimits_{j=1}^n d(\tilde{p}_{ij},\tilde{p}_j^-), \tag{14}$$

where i=1, 2, …, m, and the distance $d(\tilde{p}_{ij},\tilde{p}_j^+)$ between two fuzzy numbers is calculated according to Eq. (3).

Step 3: Calculate the relative closeness ξ_i for each alternative A_i with respect to a positive ideal solution as

$$\xi_i=d_i^-\Big/(d_i^++d_i^-). \tag{15}$$

Step 4: Rank the alternatives according to the relative closeness. The best alternatives are those that have higher values of ξ_i, and therefore should be chosen because they are closer to the positive ideal solution.

3 Decision-making with Z-numbers

In this section, we provide some basic knowledge of Z-numbers (Zadeh, 2011; Kang et al., 2012a, 2012b).

3.1 Preliminaries on Z-number

Zadeh (2011) introduced the concept of Z-number as an ordered pair Z=(A, B) of fuzzy numbers A and B, where A is a linguistic value of a variable of interest and B is a linguistic value of the probability measure of A. The ordered triple (X, A, B) is referred to as a Z-valuation (Aliev et al., 2015). A Z-valuation defined by a tuple of the form (X, A, B) can be understood that X is (A, B), where X is a variable, A is a fuzzy set used to describe the constraint, and B is a fuzzy number to describe a constraint on a partial reliability of A. Z-valuations can be used to model sentences like "the temperature in summer will surely be very high," where X is the variable "the temperature," A is a fuzzy set "very high," and B is a fuzzy constriction "surely". In this example, you can obtain a better idea of the meaning of A and B by interpreting A as a restriction of the values that temperature can assume, i.e., "very high," and B as a fuzzy restriction on the probability of X being A, i.e., Prob (X is A) is B. Since Z-numbers represent a relatively new concept, some operations, such as the distance between two Z-numbers, are not defined yet. Then we will present the procedure to convert Z-numbers to fuzzy numbers proposed by Kang et al. (2012b).

3.2 Conversion of a Z-number into a fuzzy number

Kang et al. (2012b) defined an operation to convert Z-numbers to fuzzy numbers by means of the calculation of fuzzy expectation. This method can be summarized in the following two steps:

Step 1: Given a Z-number X(A, B), the reliability B is transformed into a crisp number using the centroid method as

$$\alpha=\frac{\int x\mu_B(x)\mathrm{d}x}{\int \mu_B(x)\mathrm{d}x}. \tag{16}$$

For trapezoidal fuzzy numbers, the centroid is calculated as

$$\alpha = (b_1 + b_2 + b_3 + b_4)/4. \quad (17)$$

Step 2: Calculating the fuzzy number Z' from the Z-number (A, B) as

$$Z' = \langle x, \mu_{Z'}(x)\rangle \big| \mu_{Z'}(x) = \mu_A(x/\sqrt{\alpha}). \quad (18)$$

If A is a trapezoidal fuzzy number, then Z' is calculated as

$$Z' = \left(\sqrt{\alpha}a_1, \sqrt{\alpha}a_2, \sqrt{\alpha}a_3, \sqrt{\alpha}a_4\right). \quad (19)$$

The advantage of the suggested approach is its low analytical and computational complexity, allowing a large spectrum of applications (Aliev et al., 2015). However, it is worth mentioning that converting Z-numbers to classical fuzzy numbers leads to a loss of the original information.

3.3 Z-TODIM

The method can be summarized in the following steps:

Step 1: Define the decision matrix in terms of Z-numbers $X(A, B)$.

Step 2: Convert the decision matrix with Z-numbers $X(A, B)$ into the corresponding fuzzy number Z'.

Step 3: Apply the fuzzy TODIM.

In the case of Z-TOPSIS, the only necessary change is to replace the application of fuzzy TODIM by fuzzy TOPSIS in step 3. In Section 4, we will illustrate the approach by means of two case studies.

4 Experimental results

4.1 Case study 1—vehicle choice

This case study proposed by Kang et al. (2012a) consists of the selections of a vehicle for a journey among three possible choices: A_1 (car), A_2 (taxi), and A_3 (train). The criteria are: C_1 (price), C_2 (journey time), and C_3 (comfort). Price and journey time are cost criteria, and comfort is a benefit criterion. In this study, the ratings and criteria weights are described by Z-numbers (Table 1). The linguistic values VH, H, and M stand for very high, high, and medium, respectively. The description of Z-numbers expressed as numeric values is shown in Table 2.

The Z-TODIM and Z-TOPSIS were applied to this problem after converting the Z-numbers to trapezoidal fuzzy numbers (Table 3). The fuzzy weights for the three criteria were converted to crisp values (Kang et al., 2012a). In Table 4, we present the results obtained by Z-TODIM, Z-TOPSIS, and TOPSIS with crisp values provided by Kang et al. (2012a). The final ranking is shown in Fig. 1. We can note that the alternative A_1 (car) is the best alternative, followed by A_3 (train), and the worst is A_2 (taxi) obtained by Z-TODIM and Z-TOPSIS. We can observe that in the approach proposed by Kang et al. (2012a) using TOPSIS with crisp values (simplified version), a change occurs in the order of the alternative A_2 (taxi) with alternative A_3 (train).

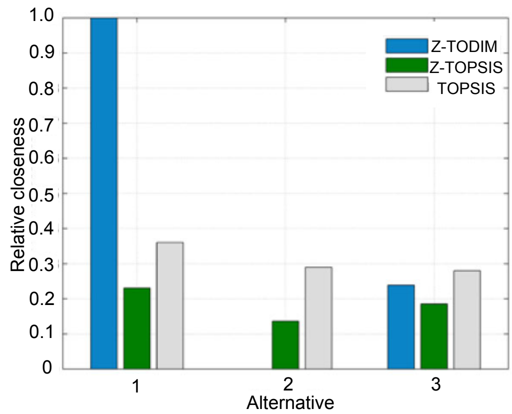

Fig. 1 Final ranking of the alternatives for case study 1

To investigate the influence of θ, we carried out a sensitivity study simulating the Z-TODIM for several values of θ (0.5, 0.8, 1.0, 1.2, 1.5, 1.8, 2.0, 2.5, and 5.0), and the ranking is shown in Table 5. We can notice that the final ranking does not change with θ.

4.2 Case study 2—clothing evaluation

This case study proposed by Xiao (2014) consists of evaluations of a certain type of clothing by male customers. The alternatives are: A_1 (like very much), A_2 (like), A_3 (ordinary), and A_4 (dislike). The criteria are: C_1 (color and style), C_2 (durability), and C_3 (price). All criteria are benefits. The rating of the alternatives is described by Z-numbers (Table 6). The linguistic values VS, S, and NVS stand for very

Table 1 Decision matrix described by Z-numbers expressed as linguistic values (Kang et al., 2012a)

Alternative	C_1 (pound), (VH, VH)	C_2 (min), (H, VH)	C_3, (M, VH)
A_1 (car)	(9, 10, 12), VH	(70, 100, 120), M	(4, 5, 6), H
A_2 (taxi)	(20, 24, 25), H	(60, 70, 100), VH	(7, 8, 10), H
A_3 (train)	(15, 15, 15), H	(70, 80, 90), H	(1, 4, 7), H

C_1: price; C_2: journey time; C_3: comfort. VH: very high; H: high; M: medium

Table 2 Decision matrix described by Z-numbers expressed as fuzzy numbers (Kang et al., 2012a)

Alternative	C_1 (pound), ((0.75, 1.00, 1.00), (0.75, 1.00, 1.00))	C_2 (min), ((0.50, 0.75, 1.00), (0.75, 1.00, 1.00))	C_3, ((0.25, 0.50, 0.75), (0.75, 1.00, 1.00))
A_1 (car)	((9, 10, 12), (0.75, 1.00, 1.00))	((70, 100, 120), (0.75, 1.00, 1.00))	((4, 5, 6), (0.50, 0.75, 1.00))
A_2 (taxi)	((20, 24, 25), (0.50, 0.75, 1.00))	((60, 70, 100), (0.75, 1.00, 1.00))	((7, 8, 10), (0.50, 0.75, 1.00))
A_3 (train)	((15, 15, 15), (0.50, 0.75, 1.00))	((70, 80, 90), (0.50, 0.75, 1.00))	((1, 4, 7), (0.50, 0.75, 1.00))

C_1: price; C_2: journey time; C_3: comfort

Table 3 Decision matrix after converting Z-numbers into fuzzy numbers (Kang et al., 2012a)

Alternative	C_1 (pound), (0.72, 0.96, 0.96)	C_2 (min), ((0.50, 0.75, 1.00), (0.75, 1.00, 1.00))	C_3, ((0.25, 0.50, 0.75), (0.75, 1.00, 1.00))
A_1 (car)	(8.62, 9.57, 11.49)	(49.50, 70.71, 84.85)	(3.46, 4.33, 5.20)
A_2 (taxi)	(17.32, 20.79, 21.65)	(57.45, 67.02, 95.75)	(6.06, 6.93, 8.660)
A_3 (train)	(12.99, 12.99, 12.99)	(60.62, 69.28, 77.94)	(0.87, 3.46, 6.06)

C_1: price; C_2: journey time; C_3: comfort

Table 4 Ranking of the alternatives for case study 1

Alternative	*Z*-TODIM	*Z*-TOPSIS	TOPSIS*
A_1 (car)	1.000	0.2305	0.3600
A_2 (taxi)	0	0.1363	0.2900
A_3 (train)	0.2388	0.1856	0.2800

* Proposed by Kang et al. (2012a)

Table 5 Sensitivity study for case study 1 regarding θ

Alternative	Rank	θ=0.5	θ=0.8	θ=1.0	θ=1.2
A_1 (car)	1	1.0	1.0	1.0	1.0
A_2 (taxi)	3	0	0	0	0
A_3 (train)	2	0.3012	0.2722	0.2570	0.2442

Alternative	Rank	θ=1.5	θ=1.8	θ=2.5	θ=5.0
A_1 (car)	1	1.0	1.0	1.0	1.0
A_2 (taxi)	3	0	0	0	0
A_3 (train)	2	0.2283	0.2155	0.1933	0.1540

sure, sure, and not very sure, respectively. The Z-numbers expressed as numeric values are shown in Table 7. The criteria weights are crisp values.

The Z-TODIM and Z-TOPSIS were applied to this problem after converting the Z-numbers to trapezoidal fuzzy numbers (Kang et al., 2012a) (Table 8). In Table 9, we present the results obtained by Z-TODIM, Z-TOPSIS, and those provided by Xiao (2014). The final ranking is illustrated in Fig. 2. We can note that the alternative A_2 (like) is the best alternative followed by A_3 (ordinary), and the worst are A_4 (dislike) and A_1 (like very much) obtained by Z-TODIM. The results are very similar compared with those obtained by Z-TOPSIS and TOPSIS (Xiao, 2014). The small differences rest in the worst alternatives A_1 and A_4. To investigate the influence of θ, we carried out a sensitivity study simulating the Z-TODIM for several values of θ (0.5, 0.8, 1.0, 1.2, 1.5, 1.8, 2.0, 2.5, and 5), and the ranking is shown in Table 10. We can notice that the final ranking does not change with θ.

Table 6 Decision matrix described by Z-numbers expressed as linguistic values (Xiao, 2014)

Alternative	C_1 (0.35)	C_2 (0.50)	C_3 (0.15)
A_1 (like very much)	about 20%, VS	nearly 10%, S	about 10%, VS
A_2 (like)	nearly 40%, S	nearly 60%, S	over 40%, NVS
A_3 (ordinary)	about 30%, NVS	about 20%, S	over 30%, VS
A_4 (dislike)	about 1%, VS	about 10%, VS	about 20%, S

C_1: color and style; C_2: durability; C_3: price. VS: very sure; S: sure; NVS: not very sure

Table 7 Decision matrix described by Z-numbers expressed as fuzzy numbers (Xiao, 2014)

Alternative	C_1 (0.35)	C_2 (0.5)	C_3 (0.15)
A_1 (like very much)	(0, 0.15, 0.25, 0.35), (0, 0.20, 0.35; 0.80)	(0, 0.03, 0.12, 0.20), (0, 0.10, 0.20; 0.80)	(0, 0.08, 0.16, 0.20), (0, 0.10, 0.20; 0.90)
A_2 (like)	(0.25, 0.35, 0.42, 0.50), (0.30, 0.40, 0.50; 0.80)	(0.40, 0.50, 0.65, 0.75), (0.40, 0.60, 0.75; 0.80)	(0.30, 0.35, 0.45, 0.55), (0.30, 0.40, 0.55; 0.70)
A_3 (ordinary)	(0.20, 0.25, 0.35, 0.45), (0.20, 0.30, 0.45; 0.70)	(0.10, 0.15, 0.25, 0.35), (0.10, 0.20, 0.35; 0.90)	(0.25, 0.30, 0.38, 0.45), (0.25, 0.3, 0.45; 0.90)
A_4 (dislike)	(0, 0.08, 0.10, 0.20), (0, 0.10, 0.20; 0.80)	(0, 0.07, 0.16, 0.20), (0, 0.10, 0.20; 0.80)	(0.10, 0.15, 0.25, 0.35), (0.10, 0.20 ,0.35; 0.90)

C_1: color and style; C_2: durability; C_3: price. VS: very sure; S: sure; NVS: not very sure

Table 8 Decision matrix after converting Z-numbers into a fuzzy number

Alternative	C_1 (0.35)	C_2 (0.50)	C_3 (0.15)
A_1 (like very much)	(0, 0.15, 0.24, 0.34)	(0, 0.03, 0.10, 0.17)	(0, 0.08, 0.15, 0.19)
A_2 (like)	(0.22, 0.30, 0.36, 0.43)	(0.34, 0.43, 0.56, 0.65)	(0.21, 0.25, 0.32, 0.39)
A_3 (ordinary)	(0.14, 0.18, 0.25, 0.32)	(0.09, 0.13, 0.22, 0.30)	(0.24, 0.29, 0.37, 0.44)
A_4 (dislike)	(0, 0.08, 0.10, 0.19)	(0, 0.07, 0.15, 0.19)	(0.09, 0.13, 0.22, 0.30)

VS: very sure; S: sure; NVS: not very sure

Table 9 Ranking of the alternatives for case study 2

Alternative	Z-TODIM	Z-TOPSIS	TOPSIS*
A_1 (like very much)	0	0.0429	0.1362
A_2 (like)	1.0000	0.2539	0.4821
A_3 (ordinary)	0.4942	0.1207	0.2629
A_4 (dislike)	0.0451	0.0348	0.1242

* Proposed by Xiao (2014)

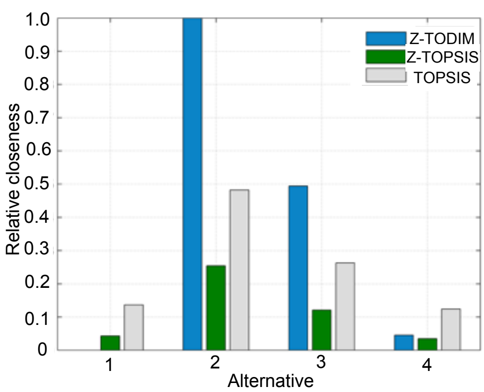

Fig. 2 Final ranking of the alternatives for case study 2

Table 10 Sensitivity study for case study 2 regarding θ

Alternative	Rank	θ=0.5	θ=0.8	θ=1.0	θ=1.2
A_1 (like very much)	4	0	0	0	0
A_2 (like)	1	1.0	1.0	1.0	1.0
A_3 (ordinary)	2	0.6095	0.5872	0.5766	0.5682
A_4 (dislike)	3	0.0058	0.0075	0.0084	0.0090
Alternative	Rank	θ=1.5	θ=1.8	θ=2.5	θ=5.0
A_1 (like very much)	4	0	0	0	0
A_2 (like)	1	1.0	1.0	1.0	1.0
A_3 (ordinary)	2	0.5584	0.5509	0.5388	0.5197
A_4 (dislike)	3	0.0098	0.0103	0.0113	0.0127

5 Conclusions

In this paper, we have presented the Z-TODIM and Z-TOPSIS, which can tackle MCDM problems modeled by Z-numbers. The method is based on the conversion of Z-numbers into trapezoidal/triangular fuzzy numbers and in turn, the application of the fuzzy TODIM or fuzzy TOPSIS. The methods have been applied to two case studies with promising results. The ranks of the alternatives obtained by Z-TODIM and Z-TOPSIS were almost the same. Since there is no closed expression to calculate the distance between Z-numbers, the approach presented turns out to be a promising way. We are currently expanding Z-TODIM and Z-TOPSIS for application to other challenging MCDM problems modeled by Z-numbers.

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