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Construction of a new Lyapunov function for a dissipative gyroscopic system using the residual energy function

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Abstract: In a dissipative gyroscopic system with four degrees of freedom and tensorial variables in contravariant (right upper index) and covariant (right lower index) forms, a Lagrangian-dissipative model, i.e., $\{L, D\}$-model, is obtained using second-order linear differential equations. The generalized elements are determined using the $\{L, D\}$-model of the system. When the prerequisite of a Legendre transform is fulfilled, the Hamiltonian is found. The Lyapunov function is obtained as a residual energy function (REF). The REF consists of the sum of Hamiltonian and losses or dissipative energies (which are negative), and can be used for stability by Lyapunov's second method. Stability conditions are mathematically proven.

Key words: Lyapunov function; Residual energy function; Stability of dissipative gyroscopic system
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1 Introduction

Stability of physical systems is important when considering the continuity of physical systems. There are many methods related to stability. One of them is Lyapunov's second (or direct) method, which was investigated by Barbashin and Krasovsky (1952), Krasovskii (1959), Lasalle (1960), Yoshizawa (1966), Hahn (1967), Rouche et al. (1977), Marino and Nicosia (1983), Lyapunov (1992), McLachlan et al. (1998), and Maschke et al. (2000). Mathematical and physical foundations of Lagrangian and Hamiltonian were studied by Heil and Kitzka (1984) and Arnold (1989). In contrast, a physical or an engineering system can be modeled by a Lagrangian L and a generalized velocity proportional Rayleigh dissipation function D, i.e., $\{L, D\}$-model, and a Hamiltonian H depending on the tensorial variables in covariant and contravariant forms (Susse and Civelek, 2003). Extended Hamiltonians in different tensorial forms to directly obtain equations of generalized motion in dissipative systems proposed by Süsse and Civelek (2013) include the higher-order Lagrangian and nonconservative Hamiltonians.

The Lyapunov exponent and the almost sure asymptotic stability of quasi-linear gyroscopic systems were studied by Huang and Zhu (2000). Exact stationary solutions of the stochastically excited and dissipated gyroscopic systems were studied by Ying and Zhu (2000). Chen LQ et al. (2004) focused on a continuous gyroscopic system with certain small nonlinear terms and parameter excitation terms. Other related works include Ao (2004), Kwon et al. (2005), and Yin and Ao (2006). The limitations of traditional approaches for constructing Lyapunov functions and the necessity of a new approach are well-known. New approaches are required to construct Lyapunov functions for traditional systems within this context. Construction of a Lyapunov function in general cases remains an unsolved problem in physical/engineering science. Dissipative gyroscopic systems are

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considered to be interesting examples that can help solve this problem. A Lyapunov function for a dissipative dynamic system has not been constructed, and the systematic approach would be a great contribution to the field. Xu et al. (2011) obtained a Lyapunov function for a dissipative gyroscopic system, where some engineering scientists and applied mathematicians could obtain such a system through the Lyapunov function. Dynamic equivalence among Hamiltonian and Lyapunov functions was studied by Yuan et al. (2014). The stability of equilibrium for non-complex and non-holonomic mechanical systems (gradient systems) was investigated in Chen J et al. (2018), where a generalized skew symmetric matrix is a special form of a Lyapunov function.

A linear dynamic approach is studied in this work, and intrinsic nonlinear dynamic approaches can be found in Yuan et al. (2013), where a potential function (defined as an energy function, a generalized Hamiltonian, or a Lyapunov function) describes a nonlinear dynamic system for the deterministic dynamics. Another work related to the nonlinear case is Ma et al. (2014), where for the first time potential functions were constructed in a continuous dissipative chaotic system and used to reveal its dynamic properties. Ma et al. (2014) proposed that a potential function is not unique for a deterministic system.

The use of the sum of kinetic and potential energies together with the generalized velocity proportional (Rayleigh) dissipation function (in tensorial forms) was described for the stability analysis by Civelek and Diemar (2003). Civelek (2018) obtained Lyapunov functions as residual energy functions (REFs) in a systematic way different from other approaches.

A four-dimensional linearized dissipative gyroscopic system with a perturbed differential equation in Xu et al. (2011) has the following matrix:

$$\ddot{\boldsymbol{q}}+(\boldsymbol{G}+\boldsymbol{B})\dot{\boldsymbol{q}}+(\boldsymbol{C}+\boldsymbol{R})\boldsymbol{q}=\boldsymbol{0}. \tag{1}$$

Considering vectors and matrixes expressed below:

$$\begin{aligned}&\boldsymbol{q}=\begin{bmatrix}q^1\\q^2\end{bmatrix},\quad \boldsymbol{G}=\begin{bmatrix}0&g\\-g&0\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}b&0\\0&b\end{bmatrix},\\&\boldsymbol{R}=\begin{bmatrix}0&r\\-r&0\end{bmatrix},\quad \boldsymbol{C}=\begin{bmatrix}c&0\\0&c\end{bmatrix},\end{aligned} \tag{2}$$

Eq. (1) can be rewritten as the following scalar differential equations:

$$\begin{cases}\ddot{q}^1+b\dot{q}^1+cq^1+g\dot{q}^2+rq^2=0\\ \quad\Rightarrow g\dot{q}^2+rq^2=-\left(\ddot{q}^1+b\dot{q}^1+cq^1\right),\\ \ddot{q}^2+b\dot{q}^2+cq^2-g\dot{q}^1-rq^1=0\\ \quad\Rightarrow g\dot{q}^1+rq^1=\ddot{q}^2+b\dot{q}^2+cq^2.\end{cases} \tag{3}$$

2 {*L*, *D*}-model

Using the $\{L, D\}$-model of the system, these linear differential equations can be obtained through the extended Euler-Lagrangian differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^k}-\frac{\partial L}{\partial q^k}+\frac{\partial D}{\partial\dot{q}^k}=F^k,\quad k=1,\,2,\ldots,f, \tag{4}$$

where F^k is the generalized external force and k the number of degrees of freedom. The linear differential equation of a dissipative generalized motion in its most general form is expressed as

$${}_kM\,\ddot{q}^k+{}_kR\dot{q}^k+{}_kKq^k+{}_kC=F^k, \tag{5}$$

$$\ddot{q}^k+\frac{{}_kR}{{}_kM}\dot{q}^k+\frac{{}_kK}{{}_kM}q^k+\frac{{}_kC}{{}_kM}=\frac{F^k}{{}_kM}, \tag{6}$$

where the left lower index is the usual index, ${}_kM\neq0$ the generalized mass, ${}_kR$ the resistive loss, ${}_kK$ the potential elements, ${}_kC$ a constant, and k=1, 2. By comparing Eqs. (3) and (6), we can determine the generalized elements, constants, and external forces as follows:

$$\begin{cases}{}_1R={}_1Mb,\ {}_1K={}_1Mc,\ F^1=0,\\ {}_1C={}_1M\left(g\dot{q}^2+rq^2\right)=-{}_1M\left(\ddot{q}^1+b\dot{q}^1+cq^1\right),\\ {}_2R={}_2Mb,\ {}_2K={}_2Mc,\ F^2=0,\\ {}_2C=-{}_2M\left(g\dot{q}^1+rq^1\right)=-{}_2M\left(\ddot{q}^2+b\dot{q}^2+cq^2\right).\end{cases} \tag{7}$$

Using Eq. (3), the generalized velocity proportional Rayleigh dissipation functions including the constant term ${}_kC$ take the following forms:

$$\frac{\partial D}{\partial\dot{q}^1}=-{}_1M\left(\ddot{q}^1+cq^1\right),\ \frac{\partial D}{\partial\dot{q}^2}=-{}_2M\left(\ddot{q}^2+cq^2\right). \tag{8}$$

Therefore, the $\{L, D\}$-model with an autonomous Lagrangian L has the forms below:

$$\begin{cases} L=\sum_{k=1}^{2}\left[\underbrace{\frac{{}_kM}{2}\left(\dot{q}^k\right)^2}_{T\left(\dot{q}^k\right)}-\underbrace{\frac{{}_kMc}{2}\left(q^k\right)^2}_{U\left(q^k\right)}\right], \\ D\left(\dot{q}^1,\dot{q}^2\right)=-{}_1M\left(\ddot{q}^1+cq^1\right)\dot{q}^1-{}_2M\left(\ddot{q}^2+cq^2\right)\dot{q}^2, \end{cases} \tag{9}$$

where $T\left(\dot{q}^k\right)$ and $U(q^k)$ are kinetic and potential energy parts of the Lagrangian, respectively.

3 Generalized momenta, Hamiltonian, and configuration space

The generalized momenta and their first-time derivative are expressed as

$$\begin{cases} p_k=\dfrac{\partial L}{\partial \dot{q}^k}=\dfrac{\partial T\left[\dot{q}^k(t)\right]}{\partial \dot{q}^k}={}_kM\dot{q}^k\neq 0,\ \forall\, {}_kM\neq 0, \\ \dot{q}^k\neq 0\Rightarrow q^k\neq \text{constant}\big|_t . \end{cases} \tag{10}$$

The prerequisite of a Legendre transform is expressed as

$$\det\left(\frac{\partial L}{\partial \dot{q}^j\partial \dot{q}^k}\right)=\det\left(\frac{\partial p_j}{\partial \dot{q}^k}\right)\neq 0,\ j, k=1, 2, \ldots, f. \tag{11}$$

If Eq. (11) is fulfilled, the Hamiltonian H^+ related to Lagrangian L is expressed as

$$H^+\left(p_k, \dot{q}^k\right)=\sum_{k=1}^{f}p_k\dot{q}^k-L. \tag{12}$$

For this case, the condition is expressed as

$$\det\begin{pmatrix} {}_1M & 0 \\ 0 & {}_2M \end{pmatrix}={}_1M\,{}_2M\neq 0,\ \forall\, {}_kM\neq 0,\ k=1, 2, \tag{13}$$

where the metric tensor and its inverse are

$$\begin{cases} [g_{ij}]=\dfrac{1}{K}\begin{pmatrix} {}_1M & 0 \\ 0 & {}_2M \end{pmatrix}, \\ [g^{ij}]=[g_{ij}]^{-1}=K\begin{pmatrix} 1/{}_1M & 0 \\ 0 & 1/{}_2M \end{pmatrix}, \end{cases} \tag{14}$$

where K is the normalization constant, the right lower index is in covariant form, and the right upper index is in contravariant tensorial form.

The metric tensor has a nonzero determinant, and its inverse consists of symmetric constant elements. Consequently, the generalized motion takes place in the Euclidean space. However, this does not make any difference in our considerations for this case. The generalized coordinates and momenta can be transformed between the covariant and contravariant forms as follows:

$$\begin{cases} q_1=\dfrac{{}_1M}{K}q^1,\ q^1=\dfrac{K}{{}_1M}q_1,\ q_2=\dfrac{{}_2M}{K}q^2,\ q^2=\dfrac{K}{{}_2M}q_2, \\ p_1=\dfrac{{}_1M}{K}p^1,\ p^1=\dfrac{K}{{}_1M}p_1, p_2=\dfrac{{}_2M}{K}p^2,\ p^2=\dfrac{K}{{}_2M}p_2. \end{cases} \tag{15}$$

For our case, the Hamiltonian is expressed as

$$\begin{aligned} H^+&=\sum_{k=1}^{2}\left[\underbrace{\frac{{}_kM}{2}\left(\dot{q}^k\right)^2}_{T\left(\dot{q}^k\right)}+\underbrace{\frac{{}_kMc}{2}\left(q^k\right)^2}_{U\left(q^k\right)}\right] \\ &=\sum_{k=1}^{2}\left[\underbrace{\frac{(p_k)^2}{2\,{}_kM}}_{T(p_k)}+\underbrace{\frac{{}_kMc}{2}(q^k)^2}_{U(q^k)}\right]. \end{aligned} \tag{16}$$

4 Residual energy function and stability

In this section, we will prove that the REF appears as an autonomous Lyapunov function.

The REF is defined as

$$\begin{gathered} \mathscr{H}^+=H^+-\int\sum_{k=1}^{f}\frac{\partial D\left(\dot{q}^1,\dot{q}^1,\ldots,\dot{q}^f\right)}{\partial \dot{q}^k}\dot{q}^k\mathrm{d}t, \\ \mathscr{H}^+>0,\ \forall\, t\in\mathbb{R}_0^+. \end{gathered} \tag{17}$$

Thus, the REF here is expressed as

$$\begin{aligned} \mathscr{H}^+=&\sum_{k=1}^{2}\left[\underbrace{\frac{(p_k)^2}{2\,{}_kM}}_{T(p_k)}+\underbrace{\frac{{}_kMc}{2}(q^k)^2}_{U(q^k)}\right] \\ &+\int\left[{}_1M\left(\ddot{q}^1+cq^1\right)\dot{q}^1+{}_2M\left(\ddot{q}^2+cq^2\right)\dot{q}^2\right]\mathrm{d}t. \end{aligned} \tag{18}$$

The first term of the right-hand side of Eq. (18) is an autonomous Hamiltonian, and therefore its first-time derivative is zero. Hence, considering the first-time derivative of the REF, we can obtain

$$\dot{\mathscr{K}}^{+} = {}_{1}M\left(\ddot{q}^{1} + cq^{1}\right)\dot{q}^{1} + {}_{2}M\left(\ddot{q}^{2} + cq^{2}\right)\dot{q}^{2}. \quad (19)$$

For marginal or asymptotic stability, the necessary condition is expressed as

$${}_{1}M\left(\ddot{q}^{1} + cq^{1}\right)\dot{q}^{1} + {}_{2}M\left(\ddot{q}^{2} + cq^{2}\right)\dot{q}^{2} \le 0. \quad (20)$$

For condition (20) to be negative semidefinite, it must satisfy

$${}_{r}M\left(\ddot{q}^{r} + cq^{r}\right)\dot{q}^{r} \le 0 \Rightarrow \left(\ddot{q}^{r} + cq^{r}\right)\dot{q}^{r} \le 0. \quad (21)$$

4.1 Marginal stability

Because $\dot{q}^{r} \neq 0 \Rightarrow q^{r}(t) \neq \text{constant}$, we can obtain

$$\ddot{q}^{r} + cq^{r} = 0. \quad (22)$$

Eq. (22) is an undamped oscillator, and the only fix point can be found through the equivalent first-order system, which is expressed as

$$\begin{cases} \dot{q}^{r} = v^{r}, \\ \dot{v}^{r} = -cq^{r}. \end{cases} \quad (23)$$

The fix point is given as (0, 0), which is a center type. The solution of the differential equation for the equality case, i.e., Eq. (22), is expressed as

$$\mathrm{e}^{\lambda t}(\lambda^{2} + c) = 0 \Rightarrow {}_{1,2}\lambda = \pm \mathrm{i}\sqrt{c}. \quad (24)$$

Thus, the related variable is expressed as

$$q^{r}(t) = \mathrm{e}^{\pm \mathrm{i}\sqrt{c}t} = \cos\left(\sqrt{c}t\right) \pm \mathrm{i}\sin\left(\sqrt{c}t\right), \quad (25)$$

representing a complex oscillation. This means that the solution must be in the form of sinusoidal oscillation with an angular velocity of $\omega_{c} = \sqrt{c}$:

$$\begin{cases} \mathrm{Re}\{\mathrm{e}^{\pm \mathrm{i}\omega_{c}t}\} = \cos(\omega_{c}t), \\ \mathrm{Im}\{\mathrm{e}^{\pm \mathrm{i}\omega_{c}t}\} = \pm\sin(\omega_{c}t). \end{cases} \quad (26)$$

The related trajectories in the state space are closed curves in the form of ellipse, and the motion is clockwise as $\omega_{c} = \sqrt{c} > 0$ and is periodic, expressed as

$$[q^{r}(t)]^{2} + [v^{r}(t)]^{2}/c = 1. \quad (27)$$

Thus, the marginal stability condition is proven.

4.2 Asymptotic stability

In contrast to marginal stability, for asymptotic stability, the differential inequality case of inequality (21) is valid. The condition of asymptotic stability is given when the related solutions

$$(\ddot{q}^{r} + cq^{r})\dot{q}^{r} < 0 \quad (28)$$

of inequality (21) are fulfilled. Instead of attempting to solve this nonlinear differential inequality, the following approach is preferred:

For inequality (21), two possibilities exist:

1. $\ddot{q}^{r} + cq^{r} > 0,\ \dot{q}^{r} < 0.$

In this case, we have

$$\begin{aligned} &\ddot{q}^{r} + cq^{r} > 0 \Rightarrow \mathrm{e}^{\lambda t}(\lambda^{2} + c) > 0 \\ &\Rightarrow \lambda > \mathrm{i}\sqrt{c} \text{ and } \lambda < -\mathrm{i}\sqrt{c} \Rightarrow \text{no solution.} \end{aligned} \quad (29)$$

2. $\ddot{q}^{r} + cq^{r} < 0,\ \dot{q}^{r} > 0.$

In this case, we can obtain

$$\mathrm{e}^{\lambda t}(\lambda^{2} + c) < 0 \Rightarrow -\mathrm{i}\sqrt{c} < \lambda < \mathrm{i}\sqrt{c}. \quad (30)$$

In this interval, the term is strictly negative. Hence, the results are expressed as follows:

$$\begin{aligned} \mathrm{e}^{\pm \mathrm{i}\sqrt{c}t - \alpha t} &= \mathrm{e}^{-\alpha t}\left[\cos\left(t\sqrt{c}\right) \pm \mathrm{i}\sin\left(t\sqrt{c}\right)\right] \\ &\le \mathrm{e}^{\pm \mathrm{i}\sqrt{c}t} = q^{k}(t),\ \ \alpha \ge 0, \end{aligned} \quad (31)$$

where the equality case to $q^{k}(t)$ is valid when $\alpha=0$. The term of $\mathrm{e}^{-\alpha t}\left[\cos\left(\sqrt{c}t\right) \pm \mathrm{i}\sin\left(\sqrt{c}t\right)\right]$ represents a spiral point sinking toward the origin, i.e., asymptotic stability.

The factor λ can be considered the angular velocity, i.e., $\omega=\lambda$. This means that for sinusoidal oscillations among the angular frequencies, i.e., $-\omega_c<-\omega<\omega_c$, the system is asymptotically stable. Therefore, the condition of asymptotic stability is proven.

Using the results above, conditions of marginal and asymptotic stability can be combined and rewritten as an angular frequency interval:

$$-\omega_c \le \omega \le \omega_c. \tag{32}$$

Thus, the conditions of negative definiteness of the REF, i.e., Eq. (18), are proven, and the REF fulfills all the properties of an autonomous Lyapunov function. As we can see, the prerequisite of the condition of negative semidefiniteness of the REF can be proven, and thus the REF can be considered an autonomous Lyapunov function.

5 Conclusions

In this study, a classical dissipative gyroscopic system has been investigated in a systematic manner within a Lagrangian-dissipative system. Using the Hamiltonian and dissipation functions of the system, a residual energy function (REF) related to the system has been developed. Through the first-time derivative of the REF, stability conditions have been proven mathematically for marginal and asymptotic cases of this function, i.e., positive semidefiniteness and angular frequency interval. An autonomous Lyapunov function could be constructed as an REF, and the stability of the concomitant system has been discussed.

The method proposed can be easily applied to different systems including coupled ones. Lagrangian and Hamiltonian can be applied to nonlinear systems, and the application of such an REF approach to nonlinear systems requires further research.

Contributors

Cem CİVELEK designed the research and drafted the manuscript. Özge CİHANBEĞENDİ helped organize the manuscript. Cem CİVELEK revised and finalized the paper.

Compliance with ethics guidelines

Cem CİVELEK and Özge CİHANBEĞENDİ declare that they have no conflict of interest.

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