Frontiers of Information Technology & Electronic Engineering
www.jzus.zju.edu.cn; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Coherence analysis and Laplacian energy of recursive trees with controlled initial states*

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Received Mar. 8, 2019; Revision accepted June 23, 2019; Crosschecked Aug. 9, 2019; Published online Sept. 5, 2019

Abstract: We study the consensus of a family of recursive trees with novel features that include the initial states controlled by a parameter. The consensus problem in a linear system with additive noises is characterized as network coherence, which is defined by a Laplacian spectrum. Based on the structures of our recursive treelike model, we obtain the recursive relationships for Laplacian eigenvalues in two successive steps and further derive the exact solutions of first- and second-order coherences, which are calculated by the sum and square sum of the reciprocal of all nonzero Laplacian eigenvalues. For a large network size N, the scalings of the first- and second-order coherences are $\ln N$ and N, respectively. The smaller the number of initial nodes, the better the consensus bears. Finally, we numerically investigate the relationship between network coherence and Laplacian energy, showing that the first- and second-order coherences increase with the increase of Laplacian energy at approximately exponential and linear rates, respectively.

Key words: Consensus; Network coherence; Laplacian energy
https://doi.org/10.1631/FITEE.1900133 **CLC number:** O155; TP11

1 Introduction

Presently, complex networks (Strogatz, 2001; Dorogovtsev and Mendes, 2002; Newman, 2003) are considered as a powerful tool to understand the topology and dynamics of complex systems with a focus on some attributes, such as degree distribution (Albert and Barabási, 2002), random walks (Zhang et al., 2009), and synchronization (Russo and Shorten, 2018; Wang et al., 2018). Among network models, deterministic networks have attracted considerable attention because precise results can be well determined. This helps verify some random network models. An issue on deterministic fractal networks is how to obtain exact solutions of topology and dynamics, e.g., deriving analytical expressions of mean first-passage time measuring the efficiency of random walks (Zhang et al., 2009). On the other hand, the Laplacian spectrum of fractal networks has been widely studied, and is related to some structural properties and dynamical features (Farkas et al., 2001; Goh et al., 2001; Dorogovtsev et al., 2003).

The consensus problem in multi-agent systems with noise is to design a measurement-based distributed protocol such that the agents will reach consensus (Ma et al., 2010; Song et al., 2016). In coupled complex networks, noise also has much effect on the synchronization dynamics (Russo and Shorten, 2018). Recently, network coherence characterized by the Laplacian spectrum has been introduced to measure consensus errors (Patterson and Bamieh, 2014; Sun et al., 2014; Yi et al., 2015; Dai et al., 2018; Zong et al., 2018). Pioneering work studied

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* Project supported by the National Natural Science Foundation of China (No. 61673144) and the Program of China Scholarship Council (No. 201808330592)

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the network coherence of Vicsek fractals and showed that its fractal dimension dominates the scalings of coherence with network size (Patterson and Bamieh, 2014). However, the scalings of coherence in our treelike networks are not related to fractal dimension (Sun et al., 2014). Yi et al. (2015) investigated the consensus in a complex network with small-world topology and found that the small-world topological structure improves the performance of network coherence. Dai et al. (2018) and Zong et al. (2018) studied the effect of weights on the coherence in some families of weighted networks.

The Laplacian energy of a graph is defined by Laplacian eigenvalues (Gutman and Zhou, 2006), and is equal to the sum of singular values of a shift of the Laplacian matrix of the considered graph (Robbiano and Jimenez, 2009). This quantity has received wide attention in many mathematical and chemical fields such that it helps have a better understanding of π-electron energy of a molecule in theoretical chemistry (Chu et al., 2016). To the best of our knowledge, few results involve the relationship between network coherence and Laplacian energy.

Inspired by the above discussion, we will investigate this relationship and further study the effect of the initial states on network coherence. We first introduce a controlled network parameter into our recursive trees and propose a new method to calculate the network coherence for this family of recursive trees. Our results show that the consensus becomes worse with a larger initial number of nodes. Finally, we find that the first- and second-order coherences increase with the increase of Laplacian energy at approximately exponential and linear rates, respectively.

2 Recursive trees with controlled initial states

In this study, we introduce a positive number r to control the different initial states of our considered recursive trees and denote T_g^r the recursive trees after g steps, where $r \geq 2$ and $g \geq 1$. These are built as follows:

At the initial state $g = 1$, T_1^r includes the number of r nodes connected by $(r-1)$ edges. For the subsequent steps $g \geq 2$, T_g^r is constructed by T_{g-1}^r by attaching one new node to each existing node in T_{g-1}^r (Fig. 1). It is clear that our model has different

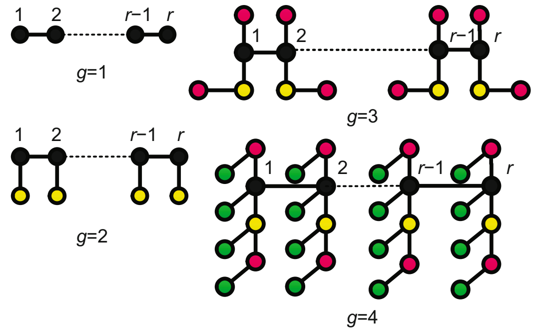

Fig. 1 Illustration of recursive trees T_g^r ($g = 1, 2, 3, 4$)

initial states by adjusting the parameter r. Based on the structures of T_g^r, we obtain the total number of vertices N_g^r and edges E_g^r as $N_g^r = r \cdot 2^{g-1}$ and $E_g^r = r \cdot 2^{g-1} - 1$, respectively.

3 Network coherence

In this study, we propose a method to calculate the first- and second-order coherences for this family of recursive trees, to obtain the scalings of network coherence with regard to the network size, and finally to investigate the relationship between network coherence and Laplacian energy.

3.1 First-order coherence

The first-order consensus dynamics in the presence of noise can be described by

$$\dot{x}_i(t) = -\sum_{j \in \Omega_i} L_{ij} x_j(t) + \omega_i(t), \tag{1}$$

where $x_i(t)$ denotes the state of vertex i at time t, Ω_i is the set of neighboring vertices of vertex i, L_{ij} is the element of Laplacian matrix $\boldsymbol{L}$, and $\omega_i(t)$ is a delta-correlated Gaussian noise imposed on vertex i. Let $\boldsymbol{x}(t) = \left(x_1(t),\ x_2(t),\ \ldots,\ x_N(t)\right)^{\mathrm{T}} \in \mathbb{R}^N$ denote the system state and $\boldsymbol{\omega}(t) = \left(\omega_1(t),\ \omega_2(t),\ \ldots,\ \omega_N(t)\right)^{\mathrm{T}} \in \mathbb{R}^N$ be the vector of uncorrelated variables of the noise. Then the consensus system (1) reads as

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}\boldsymbol{x}(t) + \boldsymbol{\omega}(t). \tag{2}$$

Definition 1 The first-order network coherence is given by the mean steady-state variance of the deviation from the average of all node values, which is expressed as

$$H^{(1)} := \lim_{t \to \infty} \frac{1}{N} \sum_{i=1}^{N} \operatorname{var}\left\{x_i(t) - \frac{1}{N} \sum_{j=1}^{N} x_j(t)\right\}.$$

Then we define the output of system (2), which is expressed as

$$\boldsymbol{y}(t) = \boldsymbol{P}\boldsymbol{x}(t), \tag{3}$$

where $\boldsymbol{P} = \boldsymbol{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is the projection operator with $\boldsymbol{I}$ an identity matrix and $\mathbf{1}$ an N-vector of all ones.

The first-order coherence $H^{(1)}$ is related to the H_2 norm of the above systems (2) and (3), that is,

$$H^{(1)} = \frac{1}{N}\mathrm{tr}\bigg(\int_0^{\infty} \mathrm{e}^{-\boldsymbol{L}^{\mathrm{T}}t}\boldsymbol{P}\mathrm{e}^{-\boldsymbol{L}t}\mathrm{d}t\bigg).$$

It has been shown that $H^{(1)}$ is completely determined by the $(N-1)$ nonzero eigenvalues of the Laplacian matrix (Xiao et al., 2007; Bamieh et al., 2012) in Eq. (2), which is given by

$$H^{(1)} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_N$ $(0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N)$ are the Laplacian eigenvalues.

3.2 Second-order coherence

In the second-order consensus problem, every vertex has two state variables, $x_{1,i}(t)$ and $x_{2,i}(t)$. Then the whole network can be denoted by two state vectors, $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$, where $\boldsymbol{x}_1(t) \in \mathbb{R}^N$ is the position vector and $\boldsymbol{x}_2(t) \in \mathbb{R}^N$ is the velocity vector. The second-order consensus dynamics subject to noises are given by

$$\begin{bmatrix}\dot{\boldsymbol{x}}_1(t)\\ \dot{\boldsymbol{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \boldsymbol{I}\\ -\boldsymbol{L} & -\boldsymbol{L}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1(t)\\ \boldsymbol{x}_2(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \boldsymbol{I}\end{bmatrix}\boldsymbol{\omega}(t),$$

where $\boldsymbol{\omega}(t)$ is a $2N$-vector of zero-mean noise processes.

Definition 2 The second-order network coherence is defined as the mean steady-state variance of the deviation from the average of position vector $\boldsymbol{x}_1(t)$, which is expressed as

$$H^{(2)} := \lim_{t\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mathrm{var}\bigg\{x_{1,i}(t) - \frac{1}{N}\sum_{j=1}^{N}x_{1,j}(t)\bigg\}.$$

In the same way, the second-order network coherence is determined by the nonzero Laplacian eigenvalues, as

$$H^{(2)} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i^2}.$$

3.3 Laplacian eigenvalues

Let $\boldsymbol{A}_g^r = [A_{ij}]_{N_g^r\times N_g^r}$ be the adjacency matrix of T_g^r, where $A_{ij} = A_{ji} = 1$ if nodes i and j are connected; otherwise, $A_{ij} = A_{ji} = 0$. $\boldsymbol{H}_g^r = \mathrm{diag}(h_1, h_2, \ldots, h_{N_g^r})$ is the diagonal degree matrix of T_g^r, where h_i represents the degree of node i. Then the Laplacian matrix $\boldsymbol{L}_g^r$ of T_g^r is defined by $\boldsymbol{L}_g^r = \boldsymbol{H}_g^r - \boldsymbol{A}_g^r$.

Based on the structures of our recursive trees, the adjacency matrix $\boldsymbol{A}_g^r$ and diagonal degree matrix $\boldsymbol{H}_g^r$ read as

$$\boldsymbol{A}_g^r = \begin{pmatrix}\boldsymbol{A}_{g-1}^r & \boldsymbol{I}_{g-1}^r\\ \boldsymbol{I}_{g-1}^r & \mathbf{0}\end{pmatrix}$$

and

$$\boldsymbol{H}_g^r = \begin{pmatrix}\boldsymbol{H}_{g-1}^r + \boldsymbol{I}_{g-1}^r & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_{g-1}^r\end{pmatrix},$$

where each block is an $(r\cdot 2^{g-2})\times(r\cdot 2^{g-2})$ matrix and $\boldsymbol{I}_{g-1}^r$ is an identity matrix. Then the Laplacian matrix $\boldsymbol{L}_g^r$ is

$$\boldsymbol{L}_g^r = \boldsymbol{H}_g^r - \boldsymbol{A}_g^r = \begin{pmatrix}\boldsymbol{L}_{g-1}^r + I_{g-1}^r & -\boldsymbol{I}_{g-1}^r\\ -\boldsymbol{I}_{g-1}^r & \boldsymbol{I}_{g-1}^r\end{pmatrix}.$$

To find the spectrum of $\boldsymbol{L}_g^r$, we need to solve the roots of $\lambda_g^r(x)$, that is

$$\begin{aligned}\lambda_g^r(x) &= \det(x\boldsymbol{I}_g^r - \boldsymbol{L}_g^r)\\ &= \det\begin{pmatrix}(x-1)\boldsymbol{I}_{g-1}^r - \boldsymbol{L}_{g-1}^r & \boldsymbol{I}_{g-1}^r\\ \boldsymbol{I}_{g-1}^r & (x-1)\boldsymbol{I}_{g-1}^r\end{pmatrix}.\end{aligned}$$

According to the knowledge of the matrix, the characteristic polynomial $\lambda_g^r(x)$ of $\boldsymbol{L}_g^r$ is expressed as

$$\begin{aligned}\lambda_g^r(x) =& \det\bigg((x-1-\frac{1}{x-1})\boldsymbol{I}_{g-1}^r - \boldsymbol{L}_{g-1}^r\bigg)\\ &\cdot\det\bigg((x-1)\boldsymbol{I}_{g-1}^r\bigg)\\ =&(x-1)^{r\cdot 2^{g-2}}\\ &\cdot\det\bigg((x-1-\frac{1}{x-1})\boldsymbol{I}_{g-1}^r - \boldsymbol{L}_{g-1}^r\bigg).\end{aligned}\tag{4}$$

From Eq. (4), $\lambda_g^r(x)$ can be recursively recast as

$$\lambda_g^r(x) = (x-1)^{r\cdot 2^{g-2}}\lambda_{g-1}\Big(f(x)\Big), \tag{5}$$

where

$$f(x) = x - 1 - \frac{1}{x-1}.$$

The recurrent relationship between $\lambda_g^r(x)$ and $\lambda_{g-1}^r(x)$ in Eq. (5) is important for obtaining the first- and second-order coherences. Note that the matrix $\boldsymbol{L}_g^r$ has $r\cdot 2^{g-1}$ Laplacian eigenvalues. Denote $\Lambda_g^r = \{\lambda_1^g, \lambda_2^g, \ldots, \lambda_{r\cdot 2^{g-1}}^g\}$ as the set of Laplacian eigenvalues. To facilitate the following calculations, the quantity $x-1-\dfrac{1}{x-1}=\lambda_i^{g-1}$ is equivalent to

$$x^2-(\lambda_i^{g-1}+2)x+\lambda_i^{g-1}=0. \tag{6}$$

Thus, Eq. (6) shows that it will give birth to two eigenvalues in Λ_g^r when an eigenvalue is taken from Λ_{g-1}^r. For convenience, let the first eigenvalue be $\lambda_1^g=0$.

3.4 Detailed calculations of $\boldsymbol{H}^{(1)}$ and $\boldsymbol{H}^{(2)}$

In what follows, we introduce two quantities τ_g^r and ϕ_g^r to obtain the values of $H^{(1)}$ and $H^{(2)}$, i.e.,

$$\tau_g^r=\sum_{i=2}^{N_g^r}\frac{1}{\lambda_i^g}$$

and

$$\phi_g^r=\sum_{i=2}^{N_g^r}\frac{1}{(\lambda_i^g)^2}.$$

Using Eq. (6) and Vieta's formulae, we obtain

$$\lambda_{2i-1}^g+\lambda_{2i}^g=\lambda_i^{g-1}+2, \tag{7}$$
$$\lambda_{2i-1}^g\cdot\lambda_{2i}^g=\lambda_i^{g-1}. \tag{8}$$

From Eqs. (7) and (8), we rewrite τ_g^r and ϕ_g^r as

$$\begin{aligned}\tau_g^r=&\sum_{i=2}^{N_g^r}\frac{1}{\lambda_i^g}\\ =&\frac{1}{\lambda_2^g}+\left[\frac{1}{\lambda_3^g}+\frac{1}{\lambda_4^g}\right]\\ &+\cdots+\left[\frac{1}{\lambda_{r\cdot 2^{g-1}-1}^g}+\frac{1}{\lambda_{r\cdot 2^{g-1}}^g}\right]\\ =&\frac{1}{2}+\frac{\lambda_3^g+\lambda_4^g}{\lambda_3^g\cdot\lambda_4^g}\\ &+\cdots+\frac{\lambda_{r\cdot 2^{g-1}-1}^g+\lambda_{r\cdot 2^{g-1}}^g}{\lambda_{r\cdot 2^{g-1}-1}^g\cdot\lambda_{r\cdot 2^{g-1}}^g}\\ =&\frac{1}{2}+\frac{\lambda_2^{g-1}+2}{\lambda_2^{g-1}}+\cdots+\frac{\lambda_{r\cdot 2^{g-2}}^{g-1}+2}{\lambda_{r\cdot 2^{g-2}}^{g-1}}\\ =&r\cdot 2^{g-2}+2\tau_{g-1}^r-\frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\phi_g^r=&\sum_{i=2}^{N_g^r}\frac{1}{(\lambda_i^g)^2}\\ =&\frac{1}{(\lambda_2^g)^2}+\left[\frac{1}{(\lambda_3^g)^2}+\frac{1}{(\lambda_4^g)^2}\right]\\ &+\cdots+\left[\frac{1}{(\lambda_{r\cdot 2^{g-1}-1}^g)^2}+\frac{1}{(\lambda_{r\cdot 2^{g-1}}^g)^2}\right]\\ =&\frac{1}{4}+\frac{(\lambda_3^g+\lambda_4^g)^2-2\lambda_3^g\lambda_4^g}{(\lambda_3^g\lambda_4^g)^2}\\ &+\cdots+\frac{(\lambda_{r\cdot 2^{g-1}-1}^g+\lambda_{r\cdot 2^{g-1}}^g)^2-2\lambda_{r\cdot 2^{g-1}-1}^g\lambda_{r\cdot 2^{g-1}}^g}{(\lambda_{r\cdot 2^{g-1}-1}^g\lambda_{r\cdot 2^{g-1}}^g)^2}\\ =&\frac{1}{4}+\frac{(\lambda_2^{g-1}+2)^2-2\lambda_2^{g-1}}{(\lambda_2^{g-1})^2}\\ &+\cdots+\frac{(\lambda_{r\cdot 2^{g-2}}^{g-1}+2)^2-2\lambda_{r\cdot 2^{g-2}}^{g-1}}{(\lambda_{r\cdot 2^{g-2}}^{g-1})^2}\\ =&2\tau_{g-1}^r+4\phi_{g-1}^r+r\cdot 2^{g-2}-\frac{3}{4}.\end{aligned}$$

Furthermore, we obtain the recursive relationships of τ_g^r and ϕ_g^r, that is,

$$\begin{cases}\tau_g^r=r\cdot 2^{g-2}-\dfrac{1}{2}+2\tau_{g-1}^r,\\ 2\tau_{g-1}^r=2\cdot(r\cdot 2^{g-3})-2\times\dfrac{1}{2}+2^2\tau_{g-2}^r,\\ 2^2\tau_{g-2}^r=2^2\cdot(r\cdot 2^{g-4})-2^2\times\dfrac{1}{2}+2^3\tau_{g-3}^r,\\ \cdots\\ 2^{g-2}\tau_2^r=2^{g-2}\cdot(r\cdot 2^0)-2^{g-2}\times\dfrac{1}{2}+2^{g-1}\tau_1^r.\end{cases}$$

Solving the above equations gives

$$\tau_g^r=(rg-r-1)\cdot 2^{g-2}+2^{g-1}\tau_1^r+\frac{1}{2}. \tag{9}$$

Hence, we need to calculate this quantity τ_1^r, and it becomes

$$\tau_1^r=\sum_{i=2}^{N_1^r}\frac{1}{\lambda_i^1}.$$

According to the initial states of recursive trees, we obtain its Laplacian matrix, i.e.,

$$\boldsymbol{L}_1^r=\boldsymbol{H}_1^r-\boldsymbol{A}_1^r$$
$$=\begin{pmatrix}1&-1&0&\cdots&0&0\\-1&2&-1&\cdots&0&0\\0&-1&2&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&2&-1\\0&0&0&\cdots&-1&1\end{pmatrix},$$

where $\boldsymbol{H}_1^r$ and $\boldsymbol{A}_1^r$ are

$$\boldsymbol{H}_1^r = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{A}_1^r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Solving $\boldsymbol{L}_1^r x = \lambda x$ gives

$$\begin{cases} x_1 \quad -x_2 & = \lambda x_1, \\ \quad -x_1 \quad +2x_2 \quad -x_3 & = \lambda x_2, \\ \quad\quad -x_2 \quad +2x_3 \quad -x_4 & = \lambda x_3, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \\ \quad\quad -x_{r-2} \quad +2x_{r-1} \quad -x_r & = \lambda x_{r-1}, \\ \quad\quad\quad -x_{r-1} \quad +x_r & = \lambda x_r, \end{cases}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)^{\mathrm{T}}$. Furthermore,

$$Q_1^r(\lambda) x_1 = 0, \tag{10}$$

where

$$Q_1^r(\lambda) = \lambda - 1 - \frac{1}{(\lambda-2) - \frac{1}{(\lambda-2) - \frac{1}{\cdots(\lambda-2) - \frac{1}{\lambda-1}}}}.$$

Since $x_1 \neq 0$, Eq. (10) is

$$Q_1^r(\lambda) = 0. \tag{11}$$

From Eq. (11), we know that $Q_1^r(\lambda) = 0$ has a unique root 0. In the following, we introduce a new quantity $U_1^r(\lambda)$ to find the other Laplacian eigenvalues, i.e.,

$$U_1^r(\lambda) = \frac{1}{\lambda} Q_1^r(\lambda).$$

We denote two polynomials $M_1^r(\lambda)$ and $D_1^r(\lambda)$ satisfying $U_1^r(\lambda) = \dfrac{M_1^r(\lambda)}{D_1^r(\lambda)}$ and $\gcd[M_1^r(\lambda), D_1^r(\lambda)] = 1$, where gcd is the greatest common divisor of two polynomials, and $m_1^r(0)$ and $d_1^r(0)$ are the constant terms of $M_1^r(\lambda)$ and $D_1^r(\lambda)$, respectively. Then $M_1^r(\lambda)$ and $D_1^r(\lambda)$ are

$$M_1^r(\lambda) = (\lambda - 1) M_1^{r-1}(\lambda) - D_1^{r-1}(\lambda)$$

and

$$D_1^r(\lambda) = \lambda M_1^{r-1}(\lambda) - D_1^{r-1}(\lambda).$$

By the initial conditions of $m_1^2(0) = -2$ and $d_1^2(0) = -1$, we obtain

$$m_1^r(0) = (-1)^{r-1} \cdot r$$

and

$$d_1^r(0) = (-1)^{r-1}.$$

We continue to calculate the coefficients of the first- and second-order terms of λ in $M_1^r(\lambda)$ and $D_1^r(\lambda)$, denoted by $m_1^r(1)$, $m_1^r(2)$, $d_1^r(1)$, and $d_1^r(2)$, respectively. Using the initial conditions of $m_1^2(1) = 1$, $d_1^2(1) = 1$, $m_1^2(2) = 0$, and $d_1^2(2) = 0$, we obtain

$$\begin{aligned} m_1^r(1) =& (-1)^r \frac{r(r-1)(r+1)}{6}, \\ m_1^r(2) =& (-1)^{r-1} \frac{(r-2)(r-1)r(r+1)(r+2)}{120}, \\ d_1^r(1) =& (-1)^r \frac{(r-1)r}{2}, \end{aligned}$$

and

$$d_1^r(2) = (-1)^{r-1} \frac{(r-2)(r-1)r(r+1)}{24}.$$

Since $U_1^r(\lambda)$ has $(r-1)$ roots, we introduce a new polynomial $W_1^r(\lambda)$ as

$$\begin{aligned} W_1^r(\lambda) =& M_1^r(\lambda) - 0 \cdot (D_1^r(\lambda)) \\ =& \left(\lambda - \lambda_2^1\right)\left(\lambda - \lambda_3^1\right) \ldots \left(\lambda - \lambda_r^1\right). \end{aligned} \tag{12}$$

Based on Eq. (12), the coefficients $w_1^r(0)$, $w_1^r(1)$, and $w_1^r(2)$ of $W_1^r(\lambda)$ are

$$\begin{cases} w_1^r(0) = (-1)^{r-1} \cdot r, \\ w_1^r(1) = (-1)^r \dfrac{r(r-1)(r+1)}{6}, \\ w_1^r(2) = (-1)^{r-1} \\ \qquad \cdot \dfrac{(r-2)(r-1)r(r+1)(r+2)}{120}. \end{cases}$$

Then we obtain the solutions of τ_1^r and ϕ_1^r as

$$\tau_1^r = \sum_{i=2}^{N_1^r} \frac{1}{\lambda_i^1} = -\frac{w_1^r(1)}{w_1^r(0)} = \frac{(r-1)(r+1)}{6}$$

and

$$\phi_1^r = \sum_{i=2}^{N_1^r} \frac{1}{(\lambda_i^1)^2} = \left(\frac{w_1^r(1)}{w_1^r(0)}\right)^2 - 2\frac{w_1^r(2)}{w_1^r(0)} = \frac{(r-1)(r+1)(2r^2+7)}{180}.$$

Substituting τ_1^r into Eq. (9) yields

$$\tau_g^r = (rg-r-1)\cdot 2^{g-2} + 2^{g-1}\tau_1^r + \frac{1}{2} = (rg-r-1)\cdot 2^{g-2} + \frac{2^{g-2}(r^2-1)}{3} + \frac{1}{2}.$$

In the same way, we obtain ϕ_g^r, i.e.,

$$\phi_g^r = 4^{g-1}\left[\frac{(r-1)(r+1)(2r^2+7)}{180}\right] + (2^g-g-3)2^{g-2}r + 2^{g-2}(2^{g-1}-1)\frac{r^2-4}{3} + \frac{4^{g-1}-1}{12}.$$

Finally, we obtain the first- and second-order network coherences as

$$H^{(1)} = \frac{\tau_g^r}{2N_g^r} = \frac{(rg-r-1)\cdot 2^{g-2} + \frac{1}{2} + \frac{2^{g-2}(r^2-1)}{3}}{r\cdot 2^g} = \frac{rg-r-1}{4r} + \frac{r^2-1}{12r} + \frac{1}{r\cdot 2^{g+1}} \tag{13}$$

and

$$H^{(2)} = \frac{\phi_g^r}{2N_g^r} = 2^{g-2}\left[\frac{(r-1)(r+1)(2r^2+7)}{180r}\right] + \frac{2^g-g-3}{4} + \frac{(2^{g-1}-1)(r^2-4)}{12r} + \frac{4^{g-1}-1}{3r\cdot 2^{g+2}}. \tag{14}$$

Via $N_g^r = r\cdot 2^{g-1}$, we have $g = (\ln N_g^r - \ln r)/\ln 2$. Therefore, for a large network (i.e., $N_g^r \to \infty$), we obtain the first- and second-order network coherences with regard to network order N_g^r as $H^{(1)} - \ln N_g^r$ and $H^{(2)} - N_g^r$, respectively. Fig. 2 plots the network coherence versus network size N_g^r with different initial states ($r = 5, 10, 15$), implying that the consensus becomes better with the smaller number of nodes in the initial states.

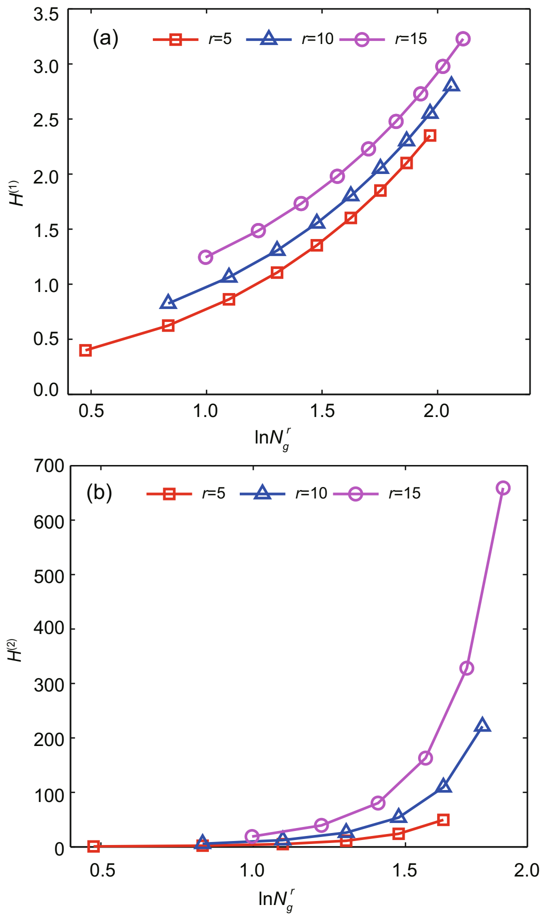

Fig. 2 Semilogarithmic graph of network coherence with regard to network size N_g^r at different initial states ($r = 5,\ 10,\ 15$): (a) $H^{(1)}$; (b) $H^{(2)}$

4 Laplacian energy

Gutman and Zhou (2006) proposed the concept of Laplacian energy, defined as its Laplacian eigenvalues, i.e.,

$$\mathrm{LE}(T_g^r) = \sum_{i=1}^{N_g^r}\left|\lambda_i - \frac{2E_g^r}{N_g^r}\right| = \sum_{i=1}^{N_g^r}\left|\lambda_i + \frac{1}{r\cdot 2^{g-2}} - 2\right|. \tag{15}$$

As an example, we numerically calculate the Laplacian energy in Eq. (15) and the network coherences $H^{(1)}$ and $H^{(2)}$ in Eqs. (13) and (14). The relationship between Laplacian energy and network coherence at different initial states ($r = 5,\ 10,\ 15$) is given in Fig. 3. We can observe that the first-order coherence increases with the increase of Laplacian energy at an exponential rate, while the second-order

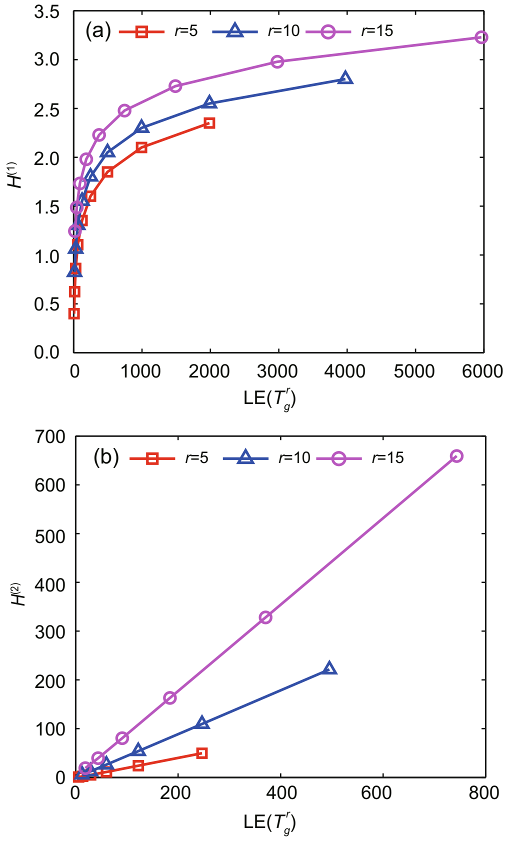

Fig. 3 Network coherence versus Laplacian energy with different initial states ($r = 5, 10, 15$): (a) $H^{(1)}$; (b) $H^{(2)}$

coherence increases at a linear rate. Furthermore, the second-order consensus is worse than the first-order one. However, they are both better with a smaller number of nodes at the initial states.

5 Conclusions

We have investigated the consensus dynamics with additive noise in a family of deterministic recursive trees and have obtained the scalings of network coherence with regard to the network size, showing that the first- and second-order coherences increase with the increase of the network size at the logarithmic and linear rates for a large network size, respectively. Based on the structures of our considered recursive trees, we have proposed a method to derive the exact solutions of network coherence. The results showed that the consensus is better with a smaller number of nodes at the initial states. Finally, we have studied the relationship between network coherence and Laplacian energy. The first- and second-order coherences showed exponential and linear relationship with the Laplacian energy, respectively. It is noted that we only numerically studied this relationship in this type of recursive tree, so whether the conclusions hold for other recursive trees, even some complex networks, needs further investigation.

Compliance with ethics guidelines

Mei-du HONG, Wei-gang SUN, Su-yu LIU, and Teng-fei XUAN declare that they have no conflict of interest.

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