Frontiers of Information Technology & Electronic Engineering
www.jzus.zju.edu.cn; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Cascading decomposition of Boolean control networks: a graph-theoretical method*

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Received Aug. 22, 2019; Revision accepted Oct. 29, 2019; Crosschecked Nov. 15, 2019; Published online Dec. 27, 2019

Abstract: Two types of cascading decomposition problems of Boolean control networks are investigated using a graph-theoretical method. A new graphic concept called nested perfect equal vertex partition (NPEVP) is proposed. Based on NPEVP, the necessary and sufficient graphic conditions for solvability of the cascading decomposition problems are obtained. Given the proposed graphic conditions, the logical coordinate transformations are constructively obtained to realize the corresponding cascading decomposition forms. Finally, two illustrative examples are provided to validate the results.

Key words: Boolean control networks; Semi-tensor product; Cascading decomposition; Graphic condition
https://doi.org/10.1631/FITEE.1900422 **CLC number:** O231

1 Introduction

Boolean network (BN), first proposed by Kauffman (1969), is a kind of dynamic system composed of logical variables and functions, which can be used to model and quantitatively describe cell regulation networks (Huang and Ingber, 2000; Huang, 2002; Farrow et al., 2004). BNs with external inputs are called Boolean control networks (BCNs) (Datta et al., 2004). In the past several decades, studies on BNs and BCNs have attracted great attention from biologists, physicists, and systems scientists. Consequently, a lot of excellent studies on the dynamic behaviors of BNs have been conducted (Albert and Othmer, 2003; Chaves et al., 2005; Klamt et al., 2006; Ching et al., 2007; Cheng and Qi, 2009). In the recent decade, a generalized matrix product called semi-tensor product (STP) was proposed by Cheng and Qi (2010a). This is a powerful tool to convert logical dynamic systems into algebraic systems. Based on STP, an algebraic state-space representation framework has been established for the analysis and control of BNs (Cheng et al., 2011). With the help of this novel theoretical framework, many fundamental results of BCNs have been obtained, including controllability (Cheng and Qi, 2009; Liu et al., 2015), optimal control (Laschov and Margaliot, 2011; Zhao et al., 2011; Li and Sun, 2012; Wu and Shen, 2015), observability (Fornasini and Valcher, 2013), stability or stabilization (Li et al., 2013; Li and Wang, 2017; Lu et al., 2018), disturbance decoupling (Cheng, 2011; Liu et al., 2017a,b; Li and Zhu, 2019; Yu et al., 2019), output regulation (Li et al., 2017), pinning control (Lu et al., 2016), l_1-gain problem (Meng et al., 2016), and so forth.

In traditional control theory, for either linear or nonlinear systems, system decomposition has been proved to be a powerful technique in system analysis and control design (Wonham, 1974). System decomposition is a natural way to deal with large-scale systems and one of the most efficient ways to

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economically realize BNs via circuits.

There have been many studies on system decomposition of BNs such as controllability decomposition (Cheng et al., 2010), observability decomposition (Cheng et al., 2010), decomposition with respect to (w.r.t.) inputs (Zou and Zhu, 2014), decomposition w.r.t. outputs (Zou and Zhu, 2017), and Kalman decomposition (Cheng et al., 2010; Zou and Zhu, 2015). Decomposition of multi-valued logical functions and an aggregation algorithm about large-scale BNs have been addressed in Cheng and Xu (2013) and Zhao et al. (2013), respectively.

Cascading decomposition, as an essential and special decomposition form of BCNs, is a theoretically interesting and practically useful issue. If a large-scale BCN can be decomposed into cascading form, then the problems of topological structure, controllability, and stabilization can probably be solved by analyzing each subsystem separately. Cheng et al. (2011) first investigated the cascading state-space decomposition problem (SSDP) and gave a necessary and sufficient algebraic condition. However, it is still difficult to constructively compute the logical coordinate transformation to realize the cascading state-space decomposition. Zou et al. (2018, 2019) studied the cascading SSDP of BCNs using the nested method and designed controllers realizing the cascading state-space decomposition. However, how to construct the logical coordinate transformation to solve cascading SSDP is still an open problem. Moreover, as far as we know, the cascading SSDP of BCNs discussed in the literature does not involve the decomposition of inputs. The cascading SSDP with cascading inputs is also very interesting and important. Once a large-scale BCN can be decomposed into cascading state-space decomposition in a cascading input form, based on its decomposition form, it not only helps solve the problems of topological structure, controllability, and stabilization by analyzing each subsystem separately but also helps analyze the control capability of BCNs. Actually, the controllability decomposition in Cheng et al. (2010) or decomposition w.r.t. inputs in Zou and Zhu (2014) can be regarded as a special case of the cascading SSDP with cascading inputs. In addition, if we replace inputs by disturbances in cascading state-space decomposition with cascading inputs, some related disturbance decoupling problems, such as triangular decoupling, can be analyzed. Based on the above motivations, we propose a new cascading decomposition form of BCNs. To distinguish the two cascading decomposition problems, we call the original and new cascading SSDP with cascading inputs Type-I and Type-II cascading decomposition problems, respectively.

In this study, we investigate two types of cascading decomposition problems of BCNs in a different way from the existing methods. Using vertex partition theory and analyzing the state transition diagram of BCNs, some interesting results are obtained. The main contributions of this study are as follows:

1. This study provides a graph perspective to study cascading SSDP of BCNs for the first time, and gives a simple and clear graphic description for the solvability of cascading SSDP.

2. To realize cascading state-space decomposition, we design an algorithm to construct a logical coordinate transformation $\boldsymbol{z}=\boldsymbol{Tx}$ based on the graphic condition.

3. We propose a new cascading SSDP with cascading inputs called the Type-II cascading decomposition problem, and derive the similar results.

Throughout the paper, we use the notations as shown in Table 1.

2 Preliminaries

First, we give some necessary preliminaries on the STP of matrices, the theory of vertex partition, and the graphic structure of BNs.

Definition 1 Set $\boldsymbol{A}=(a_{ij})\in\mathcal{R}_{m\times n}$, $\boldsymbol{B}=(b_{ij})\in\mathcal{R}_{p\times q}$. Let $\alpha=\mathrm{lcm}(n,p)$ be the least common

Table 1 Notations used in this paper

Notation	Description
$\|S\|$	Number of elements in set S
$\mathcal{D}=\{1,\ 0\}$	Set of values of logical variables
$\mathbf{Col}_i(\boldsymbol{A})$	i^{th} column of matrix $\boldsymbol{A}$
$\mathbf{Col}(\boldsymbol{A})$	Set of columns of matrix $\boldsymbol{A}$
$\boldsymbol{A}^{\mathrm{T}}$	Transpose of matrix $\boldsymbol{A}$
$\mathbf{1}_n$	$[\underbrace{1,1,\ldots,1}_{n}]^{\mathrm{T}}$
$\boldsymbol{\delta}_k^i$	i^{th} column of $\boldsymbol{I}_k$
Δ_k	$\Delta_k:=\{\boldsymbol{\delta}_k^i \| i=1,2,\ldots,k\}$
$\mathcal{L}_{m\times n}$	$\mathcal{L}_{m\times n}:=\{\boldsymbol{L}\in\mathcal{R}_{m\times n}\ \big\|\ \mathbf{Col}(\boldsymbol{L})\subset\Delta_m\}$, $\boldsymbol{L}\in\mathcal{L}_{m\times n}$ is called a logical matrix
$\boldsymbol{\delta}_m$	$\boldsymbol{\delta}_m[i_1,\ i_2,\ldots,i_r]:=\ [\boldsymbol{\delta}_m^{i_1},\ \boldsymbol{\delta}_m^{i_2},\ldots,\boldsymbol{\delta}_m^{i_r}]$
$\ltimes$	STP of matrices
N_i	$N_i:=n_1+n_2+\ldots+n_i$
$[a,b]$	$[a,b]:=\{a,a+1,\ldots,b\}$, where a and b are two integers

multiple of n and p. Then the STP of $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as (Cheng and Qi, 2010a)

$$\boldsymbol{A} \ltimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{I}_{\frac{\alpha}{n}})(\boldsymbol{B} \otimes \boldsymbol{I}_{\frac{\alpha}{p}}). \tag{1}$$

In Definition 1, when $n = p$, the STP becomes the conventional matrix product. Hence, the STP is a generalization of the traditional matrix product. In the following discussion, $\boldsymbol{A} \ltimes \boldsymbol{B}$ is denoted by $\boldsymbol{AB}$.

Definition 2 The k-dimensional power-reducing matrix is defined as (Cheng and Qi, 2010a)

$$\boldsymbol{M}_{r,k} = [\boldsymbol{\delta}_k^1 \otimes \boldsymbol{\delta}_k^1, \boldsymbol{\delta}_k^2 \otimes \boldsymbol{\delta}_k^2, \ldots, \boldsymbol{\delta}_k^k \otimes \boldsymbol{\delta}_k^k]. \tag{2}$$

Set $\boldsymbol{X} \in \Delta_k$. Then $\boldsymbol{XX} = \boldsymbol{M}_{r,k}\boldsymbol{X}$. Let $\boldsymbol{X} \in \Delta_m, \boldsymbol{Y} \in \Delta_n$. Then $\mathbf{1}_m^{\mathrm{T}}\boldsymbol{X} = 1$, $\mathbf{1}_n^{\mathrm{T}}\boldsymbol{Y} = 1$, and

$$\boldsymbol{XY} = (\boldsymbol{X} \otimes \boldsymbol{I}_n)(\boldsymbol{I}_1 \otimes \boldsymbol{Y}) = \boldsymbol{X} \otimes \boldsymbol{Y}, \tag{3}$$

$$\boldsymbol{X} = (\boldsymbol{I}_m\boldsymbol{X}) \otimes (\mathbf{1}_n^{\mathrm{T}}\boldsymbol{Y}) = (\boldsymbol{I}_m \otimes \mathbf{1}_n^{\mathrm{T}})\boldsymbol{XY}, \tag{4}$$

$$\boldsymbol{Y} = (\mathbf{1}_m^{\mathrm{T}}\boldsymbol{X}) \otimes (\boldsymbol{I}_n\boldsymbol{Y}) = (\mathbf{1}_m^{\mathrm{T}} \otimes \boldsymbol{I}_n)\boldsymbol{XY}. \tag{5}$$

Consider the logical mapping $g : \mathcal{D}^n \to \mathcal{D}^n$ defined by

$$\boldsymbol{z}_i = g_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n), \quad i = 1, 2, \ldots, n. \tag{6}$$

If $g : \mathcal{D}^n \to \mathcal{D}^n$ is a bijection, it is called a logical coordinate transformation (Cheng and Qi, 2010b). Let $\boldsymbol{z} = \boldsymbol{Tx}$ be the algebraic form of the logical coordinate transformation (6), where $\boldsymbol{T} \in \mathcal{L}_{2^n \times 2^n}$ is the structure matrix of g. It is clear that g is a logical coordinate transformation if and only if $\boldsymbol{T}$ is a nonsingular logical matrix, e.g., a permutation matrix (Cheng and Qi, 2010b).

A logical variable takes value from $\mathcal{D} = \{1, 0\}$, where 1 and 0 represent true and false, respectively. In Cheng and Qi (2010a), each element of $\mathcal{D}$ is identified with the corresponding logical vector in Δ_2 as $1 \sim \boldsymbol{\delta}_2^1$ and $0 \sim \boldsymbol{\delta}_2^2$.

Consider a BCN described by the logical form

$$\begin{cases} \boldsymbol{x}_1(t+1) = f_1\Big(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t), \\ \qquad \boldsymbol{u}_1(t), \boldsymbol{u}_2(t), \ldots, \boldsymbol{u}_m(t)\Big), \\ \boldsymbol{x}_2(t+1) = f_2\Big(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t), \\ \qquad \boldsymbol{u}_1(t), \boldsymbol{u}_2(t), \ldots, \boldsymbol{u}_m(t)\Big), \\ \quad \vdots \\ \boldsymbol{x}_n(t+1) = f_n\Big(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t), \\ \qquad \boldsymbol{u}_1(t), \boldsymbol{u}_2(t), \ldots, \boldsymbol{u}_m(t)\Big), \end{cases} \tag{7}$$

where $\boldsymbol{x}_i$ $(i = 1, 2, \ldots, n)$ are state variables and $\boldsymbol{u}_j$ $(j = 1, 2, \ldots, m)$ are inputs. Let $\boldsymbol{x}(t) = \ltimes_{i=1}^n \boldsymbol{x}_i$, $\boldsymbol{u}(t) = \ltimes_{j=1}^q \boldsymbol{u}_j$, where all the logical variables take values in Δ_2. Then Eq. (7) is converted into the following algebraic form:

$$\boldsymbol{x}(t+1) = \boldsymbol{Lu}(t)\boldsymbol{x}(t), \tag{8}$$

where $\boldsymbol{L} \in \mathcal{L}_{2^n \times 2^{n+m}}$.

Remark 1 The conversion process between Eqs. (7) and (8) can be found in Cheng and Qi (2010a).

Let $\boldsymbol{L} = [\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_i, \ldots, \boldsymbol{L}_{2^m}]$ with $\boldsymbol{L}_i \in \mathcal{L}_{2^n \times 2^n}$ for $i = 1, 2, \ldots, q$. Set $\boldsymbol{B} = \sum_{i=1}^{2^m} \boldsymbol{L}_i$. Then $\boldsymbol{B} = (b_{pq}) \in \mathcal{N}_{2^n \times 2^n}$ is a non-negative matrix, which can be regarded as an adjacency matrix of a weighted directed graph $\mathcal{G}$ with vertex set $V = \{1, 2, \ldots, 2^n\}$, where $\mathcal{G}$ has a directed edge $(q,\ p)$ if and only if $b_{pq} \neq 0$. Here, we call $\mathcal{G}$ the state transition diagram of Eq. (8). We say that p is an out-neighbor of q if $b_{pq} \neq 0$. Let $C_q = \{\ p \mid b_{pq} \neq 0\ \}$. Then C_q is the set of all the out-neighbors of q. Let S be a subset of V and set $\mathcal{N}(S) = \{\ p \mid \exists\ q \in S,\ b_{pq} \neq 0\ \} = \bigcup_{q \in S} C_q$, where $\mathcal{N}(S)$ is the out-neighborhood of S. Clearly, $C_q = \mathcal{N}(\{q\})$.

Definition 3 Let S_l $(l = 1, 2, \ldots, \mu)$ be some subsets of V. $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$ is called a vertex partition of $\mathcal{G}$ if $\bigcup\limits_{l=1}^{\mu} S_l = V$ and $S_i \cap S_j = \varnothing$ for any $i \neq j$. A vertex partition $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$ of $\mathcal{G}$ is called an equal vertex partition if $|S_l| = |V|/\mu$, where $l = 1, 2, \ldots, \mu$ and $|\cdot|$ denotes the number of elements in a set (Borůvka, 1974).

We give a simple example to illustrate the above definitions.

Example 1 For a nonnegative matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 2 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix},$$

the state transition diagram of $\boldsymbol{A}$ is shown in Fig. 1. Let $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$. Then $\mathcal{S} = \{S_1, S_2\}$ is an equal vertex partition.

Definition 4 Consider a state transition diagram with vertex set V. An equal vertex partition $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$ of V is called a perfect equal vertex partition (PEVP) if for any $l \in \{1, 2, \ldots, \mu\}$, there exists an α_l such that $\mathcal{N}(S_l) \subset S_{\alpha_l}$ (Zou and Zhu, 2014).

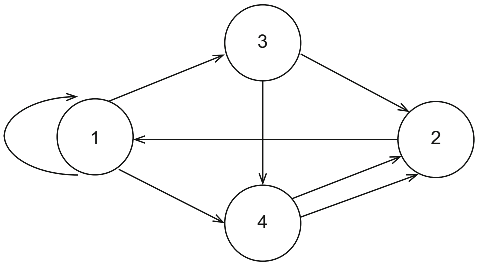

Fig. 1 State transition diagram of matrix A

Definition 5 Let $\mathcal{R}$ and $\mathcal{S}$ be two partitions of a set V. Assume that for every $R \in \mathcal{R}$, there exists an $S \in \mathcal{S}$ such that $R \subset S$. Then the partition $\mathcal{R}$ is said to be a finer partition than $\mathcal{S}$, and the partition $\mathcal{S}$ is said to be a coarser partition than $\mathcal{R}$, denoted by $\mathcal{R} \sqsubset \mathcal{S}$ (Potůček, 2014).

Here, we propose a concept which plays an important role in the rest of this study.

Definition 6 Let $\mathcal{S}^i$ be a vertex partition, $i = 1, 2, \ldots, p$. $\{\mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^p\}$ is said to be a set of nested vertex partitions if

$$\mathcal{S}^p \sqsubset \mathcal{S}^{p-1} \sqsubset \ldots \sqsubset \mathcal{S}^2 \sqsubset \mathcal{S}^1. \tag{9}$$

Furthermore, if $\mathcal{S}^i$ in expression (9) is a PEVP for each $i = 1, 2, \ldots, p$, then $\{\mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^p\}$ is said to be a set of nested PEVPs (NPEVPs). An example is given to illustrate Definition 6.

Example 2 Consider a state transition diagram with vertex set $V = \{1, 2, \ldots, 16\}$. Construct three vertex partitions. $\mathcal{S}^1 = \{S_i^1\}_{i=1}^4$ with $S_1^1 = \{9, 11, 13, 15\}$, $S_2^1 = \{1, 3, 5, 7\}$, $S_3^1 = \{2, 4, 6, 8\}$, and $S_4^1 = \{10, 12, 14, 16\}$. $\mathcal{S}^2 = \{S_i^2\}_{i=1}^8$ with $S_1^2 = \{9, 11\}$, $S_2^2 = \{13, 15\}$, $S_3^2 = \{1, 3\}$, $S_4^2 = \{5, 7\}$, $S_5^2 = \{2, 4\}$, $S_6^2 = \{6, 8\}$, $S_7^2 = \{10, 12\}$, and $S_8^2 = \{14, 16\}$. $\mathcal{S}^3 = \{S_i\}_{i=1}^{16}$ with $S_i^3 = \{i\}$. Then we have $\mathcal{S}^3 \sqsubset \mathcal{S}^2 \sqsubset \mathcal{S}^1$. So, $\{\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^3\}$ is a set of nested vertex partitions.

3 Type-I cascading decomposition

Type-I cascading decomposition of BCNs was first proposed by Cheng et al. (2011). This is helpful in analyzing the topological structure, controllability, and stabilization problems of the original system. The definition is described as follows:

Definition 7 Consider Eq. (7) with Eq. (8) (Cheng et al., 2011). The Type-I cascading decomposition problem is solvable if there exists a logical coordinate transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ such that under the $\boldsymbol{z}$ coordinate frame, the system can be converted into

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1) = \boldsymbol{G}_1 \boldsymbol{u}(t) \boldsymbol{z}^{[1]}(t), \\ \boldsymbol{z}^{[2]}(t+1) = \boldsymbol{G}_2 \boldsymbol{u}(t) \boldsymbol{z}^{[1]}(t) \boldsymbol{z}^{[2]}(t), \\ \quad \vdots \\ \boldsymbol{z}^{[p]}(t+1) = \boldsymbol{G}_p \boldsymbol{u}(t) \boldsymbol{z}^{[1]}(t) \boldsymbol{z}^{[2]}(t) \ldots \boldsymbol{z}^{[p]}(t), \\ \boldsymbol{z}^{[p+1]}(t+1) = \boldsymbol{G}_{p+1} \boldsymbol{u}(t) \boldsymbol{z}(t), \end{cases} \tag{10}$$

where $\boldsymbol{z} = \ltimes_{i=1}^{p+1} \boldsymbol{z}^{[i]}$, $\boldsymbol{z}^{[i]} = \ltimes_{l=N_{i-1}+1}^{N_i} \boldsymbol{z}_l \in \Delta_{2^{n_i}}$, $\boldsymbol{u} \in \Delta_{2^m}$, $\boldsymbol{G}_i \in \mathcal{L}_{2^{n_i} \times 2^{N_i+m}}$, $N_0 = 0$, $N_i := n_1 + n_2 + \ldots + n_i$, $n_1 + n_2 + \ldots + n_{p+1} = n$, and $l = 1, 2, \ldots, p+1$.

First, according to the theory of STP of matrices, we have the following proposition:

Proposition 1 Considering Eq. (8), the Type-I cascading decomposition problem is solvable if and only if there exists a logical coordinate transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ such that under the $\boldsymbol{z}$ coordinate frame, for any $i \in [1, p]$, the system can be converted into

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1) \boldsymbol{z}^{[2]}(t+1) \ldots \boldsymbol{z}^{[i]}(t+1) \\ \quad = \boldsymbol{G}_{1,2,\ldots,i} \boldsymbol{u}(t) \boldsymbol{z}^{[1]}(t) \boldsymbol{z}^{[2]}(t) \ldots \boldsymbol{z}^{[i]}(t), \\ \boldsymbol{z}^{[i+1]}(t+1) \boldsymbol{z}^{[i+2]}(t+1) \ldots \boldsymbol{z}^{[p+1]}(t+1) \\ \quad = \boldsymbol{G}_{i+1,i+2,\ldots,p+1} \boldsymbol{u}(t) \boldsymbol{z}(t), \end{cases} \tag{11}$$

where $\boldsymbol{z}^{[1]}(t), \boldsymbol{z}^{[2]}(t), \ldots, \boldsymbol{z}^{[i]}(t) \in \Delta_{2^{N_i}}$, $\boldsymbol{G}_{1,2,\ldots,i} \in \mathcal{L}_{2^{N_i} \times 2^{N_i+m}}$, and $\boldsymbol{G}_{i+1,i+2,\ldots,p+1} \in \mathcal{L}_{2^{n-N_i} \times 2^{n+m}}$.

Proof Necessity: By multiplying the first i equations and the last $(p+1-i)$ equations of Eq. (10), we obtain Eq. (11) with a straight computation.

Sufficiency: Comparing the first equation of Eq. (11) with the first equation of

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1) \boldsymbol{z}^{[2]}(t+1) \ldots \boldsymbol{z}^{[i+1]}(t+1) \\ \quad = \boldsymbol{G}_{1,2,\ldots,i+1} \boldsymbol{u}(t) \boldsymbol{z}^{[1]}(t) \boldsymbol{z}^{[2]}(t) \ldots \boldsymbol{z}^{[i+1]}(t), \\ \boldsymbol{z}^{[i+2]}(t+1) \boldsymbol{z}^{[i+3]}(t+1) \ldots \boldsymbol{z}^{[p+1]}(t+1) \\ \quad = \boldsymbol{G}_{i+2,i+3,\ldots,p+1} \boldsymbol{u}(t) \boldsymbol{z}(t), \end{cases} \tag{12}$$

we have that $\boldsymbol{z}^{[i+1]}(t+1)$ is the logical function of $\boldsymbol{z}^{[i+1]}(t)$. Then, $\boldsymbol{z}^{[i+1]}(t+1)$ can be written as

$$\boldsymbol{z}^{[i+1]}(t+1) = f\left(\boldsymbol{u}(t), \boldsymbol{z}^{[1]}(t), \boldsymbol{z}^{[2]}(t), \ldots, \boldsymbol{z}^{[i+1]}(t)\right). \tag{13}$$

From Eqs. (11) and (12), there exists a logical matrix $\boldsymbol{G}_{i+1}$ such that

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1)\boldsymbol{z}^{[2]}(t+1)\ldots\boldsymbol{z}^{[i]}(t+1) \\ \quad =\boldsymbol{G}_{1,2,\ldots,i}\boldsymbol{u}(t)\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[i]}(t), \\ \boldsymbol{z}^{[i+1]}(t+1)=\boldsymbol{G}_{i+1}\boldsymbol{u}(t)\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[i+1]}(t), \\ \boldsymbol{z}^{[i+2]}(t+1)\boldsymbol{z}^{[i+3]}(t+1)\ldots\boldsymbol{z}^{[p+1]}(t+1) \\ \quad =\boldsymbol{G}_{i+2,i+3,\ldots,p+1}\boldsymbol{u}(t)\boldsymbol{z}(t). \end{cases} \tag{14}$$

Eq. (10) can be derived by taking $i=1,2,\ldots,p-1$ in Eq. (14) separately. Considering Eq. (8), let $\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2,\ldots,\boldsymbol{L}_{2^m}]$ and $\mathcal{G}_j$ be the state transition diagram of $\boldsymbol{L}_j$, $j=1,2,\ldots,2^m$. The following lemma are contained in Theorem 4.2 in Zou and Zhu (2017):

Lemma 1 Considering Eq. (8), for any $i\in[1,p]$, there exists a logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}_i\boldsymbol{x}$ such that under the $\boldsymbol{z}$ coordinate frame the system becomes Eq. (11) if and only if the diagrams $\mathcal{G}_j$ $(j=1,2\ldots,2^m)$ have a common PEVP $\mathcal{S}^i=\{S_l^i\}_{l=1}^{2^{N_i}}$ with $|S_l^i|=2^{n-N_i}$. Moreover, given a common PEVP $\mathcal{S}^i=\{S_l^i\}_{l=1}^{2^{N_i}}$, the permutation matrix $\boldsymbol{T}_i$ can be obtained from (Zou and Zhu, 2017)

$$(\boldsymbol{I}_{2^{N_i}}\otimes\boldsymbol{1}_{2^{n-N_i}}^{\mathrm{T}})\boldsymbol{T}_i=\boldsymbol{Q}_i, \tag{15}$$

where $\boldsymbol{Q}_i=\boldsymbol{\delta}_{2^{N_i}}[j_1^i,j_2^i,\ldots,j_q^i\ldots,j_{2^n}^i]$ with $j_q^i=l$ for any $q\in S_l^i$.

Remark 2 We give an explanation for Eq. (15). By the construction of $\boldsymbol{Q}_i$ and $|S_l^i|=2^{n-N_i}$, we see that all the columns of $\boldsymbol{Q}_i$ are just the columns of $\boldsymbol{I}_{2^{N_i}}\otimes\boldsymbol{1}_{2^{n-N_i}}^{\mathrm{T}}$. Reordering the columns of $\boldsymbol{I}_{2^{N_i}}\otimes\boldsymbol{1}_{2^{n-N_i}}^{T}$ yields $\boldsymbol{Q}_i$, which implies that there exists T_i satisfying Eq. (15).

Let $i\sim\boldsymbol{\delta}_{2^n}^i$ for any $i\in V$. Then $\{\boldsymbol{\delta}_{2^n}^1,\boldsymbol{\delta}_{2^n}^2,\ldots,\boldsymbol{\delta}_{2^n}^{2^n}\}\sim\{1,2,\ldots,2^n\}$. For simplicity, we call $\{\boldsymbol{\delta}_{2^n}^1,\boldsymbol{\delta}_{2^n}^2,\ldots,\boldsymbol{\delta}_{2^n}^{2^n}\}$ a vertex set in the following.

Next, we give a graphic condition related only to the structure matrix of BCNs for the solvability of the Type-I cascading decomposition problem.

Theorem 1 Considering Eq. (8), the Type-I cascading decomposition problem is solvable by a coordinate transformation $\boldsymbol{z}=\boldsymbol{Tx}$ if and only if the state transition diagram of Eq. (8) has a set of NPEVPs $\{\mathcal{S}^{1,2,\ldots,p},\mathcal{S}^{1,2,\ldots,p-1},\ldots,\mathcal{S}^{1,2},\mathcal{S}^1\}$ satisfying $\mathcal{S}^{1,2\ldots p}\sqsubset\mathcal{S}^{1,2,\ldots,p-1}\sqsubset\ldots\sqsubset\mathcal{S}^{1,2}\sqsubset\mathcal{S}^1$, where for any $i\in[1,p]$, $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$ with $|S_l^{1,2,\ldots,i}|=2^{n-N_i}$ is a common PEVP of $\mathcal{G}_j$, $j=1,2\ldots,2^m$.

Proof Necessity: Suppose that the Type-I cascading decomposition problem is solvable. From Proposition 1, for any $i\in[1,p]$, we have Eq. (11). Fix $\boldsymbol{z}^{[1]}\boldsymbol{z}^{[2]}\ldots\boldsymbol{z}^{[i]}=\boldsymbol{\delta}_{2^{N_i}}^l$ and define $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$ with

$$S_l^{1,2,\ldots,i}=\{\boldsymbol{T}^{\mathrm{T}}\boldsymbol{z}\in\Delta_{2^n}\big|\boldsymbol{z}^{[1]}\boldsymbol{z}^{[2]}\ldots\boldsymbol{z}^{[i]}=\boldsymbol{\delta}_{2^{N_i}}^l\}, \tag{16}$$

which follows from the construction of $\mathcal{S}^{1,2,\ldots,i}$ that $|S_l^{1,2,\ldots,i}|=2^{n-N_i}$ for any l. So, $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$ is an equal vertex partition. Consider Eq. (11) and fix $\boldsymbol{u}(t)=\boldsymbol{\delta}_{2^m}^j$, $j=1,2,\ldots,2^m$. For any l and any $\boldsymbol{x}(t)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{z}(t)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{\delta}_{2^{N_i}}^l\boldsymbol{\delta}_{2^{n-N_i}}^r\in S_l^{1,2,\ldots,i}$, where $\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[i]}(t)=\boldsymbol{\delta}_{2^{N_i}}^l$, we have

$$\begin{aligned}&\boldsymbol{z}^{[1]}(t+1)\boldsymbol{z}^{[2]}(t+1)\ldots\boldsymbol{z}^{[i]}(t+1)\\&\quad=\boldsymbol{G}_{1,2,\ldots,i}\boldsymbol{\delta}_{2^m}^j\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[i]}(t):=\boldsymbol{\delta}_{2^{N_i}}^{\alpha_l^j}.\end{aligned} \tag{17}$$

Hence, from Eqs. (16) and (17), we have $\boldsymbol{x}(t+1)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{z}(t+1)\in S_{\alpha_l^j}^{1,2,\ldots,i}$, where $\boldsymbol{z}(t+1)=\boldsymbol{\delta}_{2^{N_i}}^{\alpha_l^j}\boldsymbol{z}^{[i+1]}(t+1)\boldsymbol{z}^{[i+2]}(t+1)$. Let $\mathcal{N}^j(\cdot)$ represent out-neighborhood under $\boldsymbol{u}=\boldsymbol{\delta}_{2^m}^j$. Then we have $\mathcal{N}^j(S_l^{1,2,\ldots,i})\subset S_{\alpha_l^j}^{1,2,\ldots,i}$. $\mathcal{S}^{1,2,\ldots,i}$ is a PEVP under $\boldsymbol{u}=\boldsymbol{\delta}_{2^m}^j$. Considering $j=1,2,\ldots,2^m$, we have that $\mathcal{S}^{1,2,\ldots,i}$ is a common PEVP of $\mathcal{G}_j$.

Next we prove $\mathcal{S}^{1,2,\ldots,i+1}\sqsubset\mathcal{S}^{1,2,\ldots,i}$ for any $i\in[1,p-1]$. For any l and any $\boldsymbol{x}(t)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{\delta}_{2^{N_{i+1}}}^l\boldsymbol{\delta}_{2^{n-N_{i+1}}}^r\in S_l^{1,2,\ldots,i+1}$, set $\boldsymbol{\delta}_{2^{N_{i+1}}}^l=\boldsymbol{\delta}_{2^{N_i}}^{l_1}\boldsymbol{\delta}_{2^{n_{i+1}}}^{l_2}$. Then

$$\boldsymbol{x}(t)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{\delta}_{2^{N_i}}^{l_1}\boldsymbol{\delta}_{2^{n_{i+1}}}^{l_2}\boldsymbol{\delta}_{2^{n-N_{i+1}}}^r\in S_{l_1}^{1,2,\ldots,i}, \tag{18}$$

where $S_l^{1,2,\ldots,i+1}\subset S_{l_1}^{1,2,\ldots,i}$. Thus, $\mathcal{S}^{1,2,\ldots,i+1}\sqsubset\mathcal{S}^{1,2,\ldots,i}$. So, we have that $\{\mathcal{S}^1,\mathcal{S}^{1,2},\ldots,\mathcal{S}^{1,2,\ldots,p}\}$ is a set of NPEVPs. The necessity is proved.

Sufficiency: Since

$$\mathcal{S}^{1,2,\ldots p}\sqsubset\mathcal{S}^{1,2,\ldots p-1}\sqsubset\ldots\sqsubset\mathcal{S}^{1,2}\sqsubset\mathcal{S}^1, \tag{19}$$

where $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$ with $|S_l^{1,2,\ldots,i}|=2^{n-N_i}$, for any $i\in[1,p]$, we can set

$$\mathcal{S}^{1,2,\ldots,i}=\{S_{l_1,l_2,\ldots,l_i}^{1,2,\ldots,i}\big|1\le l_k\le 2^{n_k},1\le k\le i\}, \tag{20}$$

where $S_{l_1,l_2,\ldots,l_i}^{1,2,\ldots,i}=\bigcup\limits_{l_{i+1}=1}^{2^{n_{i+1}}}S_{l_1,l_2,\ldots,l_{i+1}}^{1,2,\ldots,i+1}$ and $S_{l_1,l_2,\ldots,l_{i+1}}^{1,2,\ldots,i+1}\subset S_{l_1,l_2,\ldots,l_i}^{1,2,\ldots,i}$ for $i=1,2,\ldots,p-1$. Thus, we have $\mathcal{S}^1=\{S_{l_1}^1|1\le l_1\le 2^{n_1}\}$, where

$$S_{l_1}^1=\bigcup_{l_2=1}^{2^{n_2}}\bigcup_{l_3=1}^{2^{n_3}}\cdots\bigcup_{l_p=1}^{2^{n_p}}S_{l_1,l_2,\ldots,l_p}^{1,2,\ldots,p}. \tag{21}$$

Construct a series of sets as $\mathcal{S}^i = \{S^i_{l_i} | 1 \le l_i \le 2^{n_i}\}$, where

$$S^i_{l_i} = \bigcup_{l_1=1}^{2^{n_1}} \bigcup_{l_2=1}^{2^{n_2}} \cdots \bigcup_{l_{i-1}=1}^{2^{n_i-1}} \bigcup_{l_{i+1}=1}^{2^{n_i+1}} \bigcup_{l_{i+2}=1}^{2^{n_i+2}} \cdots \bigcup_{l_p=1}^{2^{n_p}} S^{1,2,\ldots,p}_{l_1,l_2,\ldots,l_p}, \tag{22}$$

and $i = 2, 3, \ldots, p$. From the construction, we have that $\mathcal{S}^i$ is an equal vertex partition with $|S^i_{l_i}| = 2^{n-n_i}$. We claim that for any $i \in [1, p]$,

$$\mathcal{S}^{1,2,\ldots,i} = \mathcal{S}^1 \wedge \mathcal{S}^2 \wedge \ldots \wedge \mathcal{S}^i. \tag{23}$$

In fact, for any $S^1_{r_1} \in \mathcal{S}^1, S^2_{r_2} \in \mathcal{S}^2, \ldots, S^i_{r_i} \in \mathcal{S}^i$, where $r_i \in \{1, 2, \ldots, 2^{n_i}\}$, we have

$$\begin{aligned}
& S^1_{r_1} \bigcap S^2_{r_2} \bigcap \cdots \bigcap S^i_{r_i} \\
=& \left(\bigcup_{l_2=1}^{2^{n_2}} \bigcup_{l_3=1}^{2^{n_3}} \cdots \bigcup_{l_p=1}^{2^{n_p}} S^{1,2,\ldots,p}_{r_1 l_2,l_3,\ldots,l_p} \right) \bigcap \\
& \left(\bigcup_{l_1=1}^{2^{n_1}} \bigcup_{l_3=1}^{2^{n_3}} \cdots \bigcup_{l_p=1}^{2^{n_p}} S^{1,2,\ldots,p}_{l_1,r_2,\ldots,l_p} \right) \bigcap \cdots \bigcap \\
& \left(\bigcup_{l_1=1}^{2^{n_1}} \bigcup_{l_2=1}^{2^{n_2}} \cdots \bigcup_{l_{i-1}=1}^{2^{n_i-1}} \bigcup_{l_{i+1}=1}^{2^{n_i+1}} \cdots \bigcup_{l_p=1}^{2^{n_p}} S^{1,2,\ldots,p}_{l_1,l_2,\ldots,r_i,l_{i+1},\ldots,l_p} \right) \\
=& \bigcup_{l_{i+1}=1}^{2^{n_i+1}} \bigcup_{l_{i+2}=1}^{2^{n_i+2}} \cdots \bigcup_{l_p=1}^{2^{n_p}} S^{1,2,\ldots,p}_{r_1,r_2,\ldots,r_i,l_{i+1},l_{i+2},\ldots,l_p} \\
=& S^{1,2,\ldots,i}_{r_1,r_2,\ldots,r_i}.
\end{aligned}$$

From the arbitrariness of r_i, Eq. (23) holds.

Next we use these equal vertex partitions $\mathcal{S}^i$ ($i = 1, 2, \ldots, p$) to construct a logical coordinate transformation matrix $\boldsymbol{T}$ and prove that the Type-I cascading decomposition problem of Eq. (8) is solvable under $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$.

We denote the equivalence relation induced by $\mathcal{S}^i$ as $\overset{\mathcal{S}^i}{\sim}$, $i = 1, 2, \ldots, p$. For any $i \in [1, p]$ and $q \in S^i_{l_i}$, let $j^i_q = l_i$, where $l_i \in \{1, 2, \ldots, 2^{n_i}\}$. Construct logical matrices as

$$\boldsymbol{Q}_i = \boldsymbol{\delta}_{2^{n_i}}[j^i_1, j^i_2, \ldots, j^i_{2^n}], i = 1, 2, \ldots, p. \tag{24}$$

From the construction of $\boldsymbol{Q}_i$, we have $s \overset{\mathcal{S}^i}{\sim} t \Leftrightarrow j^i_s = j^i_t$. Set

$$\boldsymbol{Q}_1 * \boldsymbol{Q}_2 * \ldots * \boldsymbol{Q}_i = \boldsymbol{\delta}_{2^n}[j^{1,2,\ldots,i}_1, j^{1,2,\ldots,i}_2, \ldots, j^{1,2,\ldots,i}_{2^n}], \tag{25}$$

where $*$ is the Khatri-Rao product. It follows from Eq. (25) that

$$\begin{aligned}
j^{1,2,\ldots,i}_s = j^{1,2,\ldots,i}_t &\Leftrightarrow j^1_s = j^1_t, j^2_s = j^2_t, \ldots, j^i_s = j^i_t \\
&\Leftrightarrow s \overset{\mathcal{S}^1}{\sim} t, s \overset{\mathcal{S}^2}{\sim} t, \ldots, s \overset{\mathcal{S}^i}{\sim} t \\
&\Leftrightarrow s \overset{\mathcal{S}^{1,2,\ldots,i}}{\sim} t.
\end{aligned} \tag{26}$$

Since $\mathcal{S}^{1,2,\ldots,p}$ is an equal vertex partition of V with $|\mathcal{S}^{1,2,\ldots,p}| = 2^{N_p}$, we have

$$(\boldsymbol{Q}_1 * \boldsymbol{Q}_2 * \ldots * \boldsymbol{Q}_p)\mathbf{1}_{2^n} = 2^{n-N_p}\mathbf{1}_{2^{N_p}}. \tag{27}$$

Thus, we can rearrange the columns of $\boldsymbol{I}_{2^{N_p}} \otimes \mathbf{1}^T_{2^{n-N_P}}$ to obtain $\boldsymbol{Q}_1 * \boldsymbol{Q}_2 * \ldots * \boldsymbol{Q}_p$. There exists a permutation matrix $\boldsymbol{T}$ such that

$$(\boldsymbol{I}_{2^{N_p}} \otimes \mathbf{1}^{\mathrm{T}}_{2^{n-N_P}})\boldsymbol{T} = \boldsymbol{Q}_1 * \boldsymbol{Q}_2 * \ldots * \boldsymbol{Q}_p. \tag{28}$$

Furthermore, for any $i \in [1, p]$, we have

$$\begin{aligned}
&(\boldsymbol{I}_{2^{N_i}} \otimes \mathbf{1}^{\mathrm{T}}_{2^{n-N_i}})\boldsymbol{T} \\
=&(\boldsymbol{I}_{2^{N_i}} \otimes \mathbf{1}^{\mathrm{T}}_2)(\boldsymbol{I}_{2^{N_{i+1}}} \otimes \mathbf{1}^{\mathrm{T}}_{2^{n-N_{i+1}}})\boldsymbol{T} \\
=&(\boldsymbol{I}_{2^{N_i}} \otimes \mathbf{1}^{\mathrm{T}}_2)(\boldsymbol{I}_{2^{N_{i+1}}} \otimes \mathbf{1}^{\mathrm{T}}_2)(\boldsymbol{I}_{2^{N_{i+2}}} \otimes \mathbf{1}^{T}_{2^{n-N_{i+2}}})\boldsymbol{T} \\
&\vdots \\
=&(\boldsymbol{I}_{2^{N_i}} \otimes \mathbf{1}^{\mathrm{T}}_2)(\boldsymbol{I}_{2^{N_{i+1}}} \otimes \mathbf{1}^{\mathrm{T}}_2) \ldots (\boldsymbol{I}_{2^{N_p}} \otimes \mathbf{1}^{\mathrm{T}}_{2^{n-N_p}})\boldsymbol{T} \\
=&(\boldsymbol{I}_{2^{N_i}} \otimes \mathbf{1}^{\mathrm{T}}_2)(\boldsymbol{I}_{2^{N_{i+1}}} \otimes \mathbf{1}^{\mathrm{T}}_2) \ldots (\boldsymbol{I}_{2^{N_{p-1}}} \otimes \mathbf{1}^{\mathrm{T}}_2) \cdot \boldsymbol{Q}_1 \\
&* \boldsymbol{Q}_2 * \ldots * \boldsymbol{Q}_p \\
=&\boldsymbol{Q}_1 * \boldsymbol{Q}_2 * \ldots * \boldsymbol{Q}_i.
\end{aligned} \tag{29}$$

For any $i \in [1, p]$, it follows from Eqs. (25), (26), and (29) and Lemma 1 that Eq. (8) can be converted into Eq. (11) under the transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$. Thus, from Proposition 1, we have that the Type-I cascading decomposition problem of Eq. (8) is solved by $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$.

Remark 3 Theorem 1 presents a necessary and sufficient graphic condition for the solvability of Type-I cascading decomposition problem for the first time. The graphic condition gives a simple and clear description for Type-I cascading decomposition. Compared with the algebraic condition related to the existence of solutions to some algebraic equations (Cheng et al., 2011), the graphic condition is related only to the state transition diagram, i.e., the structure matrix of BCNs. In addition, based on Theorem 1, we can directly design an algorithm to construct the logical coordinate transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$, while in Zou et al. (2019), how to construct the logical coordinate transformation is still an open problem. In this study, we do not give an algorithm to search for a set of NPEVPs shown in Theorem 1. How to search for a set of NPEVPs is a new issue in graph theory. Based on Theorem 1, we give Algorithm 1 to compute logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$.

Remark 4 Since the construction of $\mathcal{S}^i$ and $\boldsymbol{Q}_i$ is related to 2^{n_i}, $i=1,2,\ldots,p$, and the computation of $\boldsymbol{T}$ from Eq. (28) is related to 2^n, Algorithm 1 has at least exponential time complexity as n is very large.

Algorithm 1 Computing logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$

1: Compute Eq. (8) from Eq. (7). Let $\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2,\ldots,\boldsymbol{L}_{2^m}]$ and $\mathcal{G}_j$ be the state transition diagram of $\boldsymbol{L}_j$, $j=1,2,\ldots,2^m$

2: Search for an NPEVP $\mathcal{S}^{1,2,\ldots,p}\sqsubset\mathcal{S}^{1,2,\ldots,p-1}\sqsubset\ldots\sqsubset\mathcal{S}^{1,2}\sqsubset\mathcal{S}^1$, where for any $i=1,2,\ldots,p$, $\mathcal{S}^{1,2,\ldots,i}$ is a common PEVP of $\mathcal{G}_j$, $j=1,2\ldots,2^m$

3: Construct a series of sets $\mathcal{S}^i$ in Eq. (22) and $\boldsymbol{Q}_i$ in Eq. (24)

4: Compute $\boldsymbol{T}$ from Eq. (28)

We consider an example in Cheng et al. (2011) to illustrate how to find the logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$.

Example 3 Consider the following BCN described by Cheng et al. (2011):

$$\begin{cases}\boldsymbol{x}_1(t+1)=\neg\boldsymbol{x}_4(t)\bar{\vee}\big(\boldsymbol{x}_1(t)\rightarrow\boldsymbol{u}_1(t)\big),\\ \boldsymbol{x}_2(t+1)=\Big(\big(\boldsymbol{x}_1(t)\bar{\vee}\boldsymbol{x}_4(t)\big)\leftrightarrow\boldsymbol{x}_2(t)\Big)\vee\boldsymbol{u}_2(t),\\ \boldsymbol{x}_3(t+1)=\neg\,\boldsymbol{x}_4(t)\\ \quad\leftrightarrow\Big(\boldsymbol{x}_4(t)\wedge\big(\boldsymbol{x}_3(t)\leftrightarrow(\boldsymbol{x}_1(t)\bar{\vee}\boldsymbol{x}_4(t))\big)\Big),\\ \boldsymbol{x}_4(t+1)=\boldsymbol{x}_1(t)\rightarrow\boldsymbol{u}_1(t),\end{cases}\tag{30}$$

where $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{u}_1$, and $\boldsymbol{u}_2\in\Delta_2$.

Let $\boldsymbol{x}=\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4$, $\boldsymbol{u}=\boldsymbol{u}_1\boldsymbol{u}_2$. Then the algebraic form of Eq. (30) is

$$\boldsymbol{x}(t+1)=\boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{x}(t),\tag{31}$$

with

$$\begin{aligned}\boldsymbol{L}=\boldsymbol{\delta}_{2^4}[&1,11,3,11,1,11,3,11,3,11,1,11,3,11,1,11,\\&5,11,7,11,1,15,3,15,3,15,1,15,7,11,5,11,\\&10,4,12,4,10,4,12,4,3,11,1,11,3,11,1,11,\\&14,4,16,4,10,8,12,8,3,15,1,15,7,11,5,11].\end{aligned}$$

Let $\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2,\boldsymbol{L}_3,\boldsymbol{L}_4]$. As shown in Figs. 2–5, the state transition diagram $\mathcal{G}_i$ of $\boldsymbol{L}_i$ has an NPEVP $\mathcal{S}^{12}\sqsubset\mathcal{S}^1$, where $\mathcal{S}^1=\{S_i^1\}_{i=1}^4$ and $\mathcal{S}^{1,2}=\{S_{l_1,l_2}^{12}|1\le l_1\le4,1\le l_2\le2\}$ are common PEVP of $\mathcal{G}_i$ with $S_1^1=\{9,11,13,15\}$, $S_2^1=\{1,3,5,7\}$, $S_3^1=\{2,4,6,8\}$, $S_4^1=\{10,12,14,16\}$, $S_{1,1}^{1,2}=\{9,11\}$, $S_{1,2}^{1,2}=\{13,15\}$, $S_{2,1}^{1,2}=\{1,3\}$, $S_{2,2}^{1,2}=\{5,7\}$, $S_{3,1}^{1,2}=\{2,4\}$, $S_{3,2}^{1,2}=\{6,8\}$, $S_{4,1}^{1,2}=\{10,12\}$, and $S_{4,2}^{1,2}=\{14,16\}$.

From Eq. (22), we construct $\mathcal{S}^2=\{S_1^2,S_2^2\}$ as

$$S_1^2=\bigcup_{l_1=1}^4 S_{l_1,1}^{1,2}=\{9,11,1,3,2,4,10,12\},$$

$$S_2^2=\bigcup_{l_1=1}^4 S_{l_1,2}^{1,2}=\{13,15,5,7,6,8,14,16\}.$$

From the equal vertex partitions $\mathcal{S}^1$ and $\mathcal{S}^2$, we have

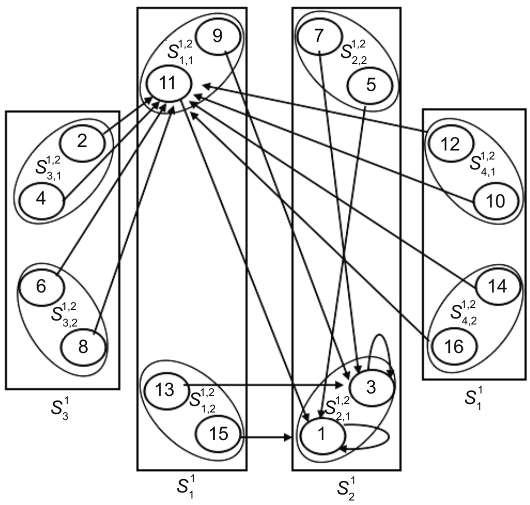

Fig. 2 State transition diagram of $\mathcal{G}_1$

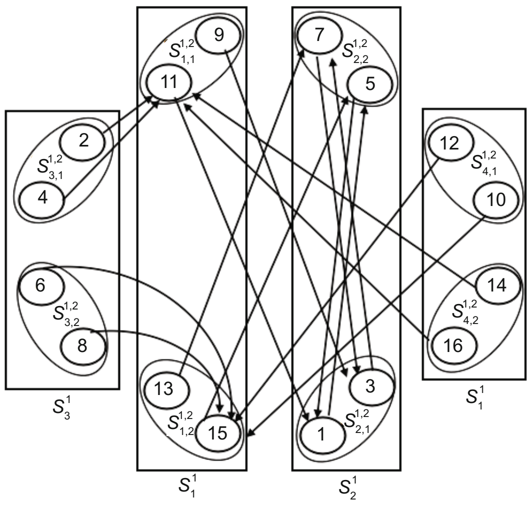

Fig. 3 State transition diagram of $\mathcal{G}_2$

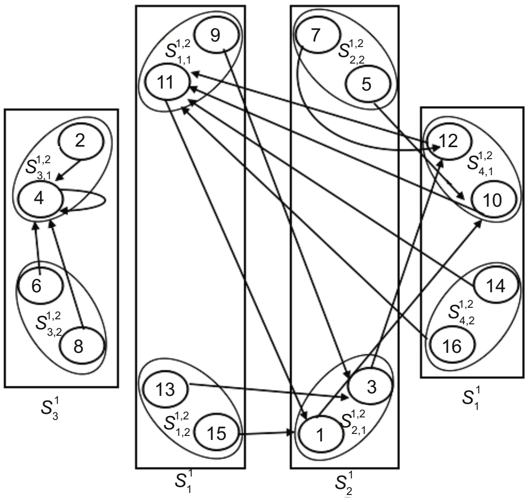

Fig. 4 State transition diagram of $\mathcal{G}_3$

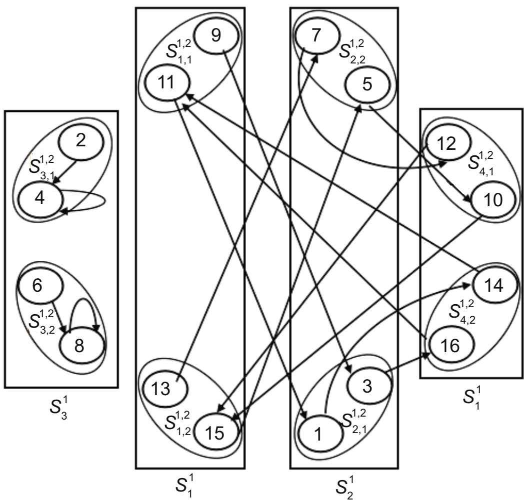

Fig. 5 State transition diagram of $\mathcal{G}_4$

$$\boldsymbol{Q}_1 = \boldsymbol{\delta}_4[2,3,2,3,2,3,2,3,1,4,1,4,1,4,1,4],$$
$$\boldsymbol{Q}_2 = \boldsymbol{\delta}_2[1,1,1,1,2,2,2,2,1,1,1,1,2,2,2,2].$$

Then we have

$$\boldsymbol{Q}_1 * \boldsymbol{Q}_2 = \boldsymbol{\delta}_8[3,5,3,5,4,6,4,6,1,7,1,7,2,8,2,8].$$

Considering Eq. (28), let $(\boldsymbol{I}_{2^3} \otimes \boldsymbol{1}_2^{\mathrm{T}})\boldsymbol{T} = \boldsymbol{Q}_1 * \boldsymbol{Q}_2$. A permutation matrix $\boldsymbol{T}$ is obtained as

$$\boldsymbol{T} = \boldsymbol{\delta}_{2^4}[5,9,6,10,7,11,8,12,1,13,2,14,3,15,4,16]. \tag{32}$$

From the sufficiency proof of Theorem 1, under the logical coordinate transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$, Eq. (30) can be converted into Eq. (10) as

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1) = \boldsymbol{G}_1\boldsymbol{u}(t)\boldsymbol{z}^{[1]}(t), \\ \boldsymbol{z}^{[2]}(t+1) = \boldsymbol{G}_2\boldsymbol{u}(t)\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t), \\ \boldsymbol{z}^{[3]}(t+1) = \boldsymbol{G}_3\boldsymbol{u}(t)\boldsymbol{z}(t), \end{cases} \tag{33}$$

where $\boldsymbol{z}^{[1]} = \boldsymbol{z}_1\boldsymbol{z}_2$, $\boldsymbol{z}^{[2]} = \boldsymbol{z}_3$, $\boldsymbol{z}^{[3]} = \boldsymbol{z}_4$, and

$$\begin{aligned} \boldsymbol{G}_1 &= \boldsymbol{\delta}_4[2,2,1,1,2,2,1,1,2,4,3,1,2,4,3,1], \\ \boldsymbol{G}_2 &= \boldsymbol{\delta}_2[1,1,1,1,1,1,1,1,2,2,1,1,2,2,1, \\ &\qquad 1,1,1,1,1,1,1,1,2,2,1,1,2,2,1], \\ \boldsymbol{G}_3 &= \boldsymbol{\delta}_2[2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2, \\ &\qquad 2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2, \\ &\qquad 2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2, \\ &\qquad 2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2]. \end{aligned}$$

Using the model construction method in Cheng et al. (2011), we obtain the following dynamic logical equations:

$$\begin{cases} \boldsymbol{z}_1(t+1) = \boldsymbol{u}_1(t) \vee \Big(\neg\, \boldsymbol{u}_1(t) \wedge \big(\boldsymbol{z}_1(t) \leftrightarrow \boldsymbol{z}_2(t)\big)\Big), \\ \boldsymbol{z}_2(t+1) = \neg\, \boldsymbol{z}_1(t), \\ \boldsymbol{z}_3(t+1) = \boldsymbol{u}_2(t) \vee \Big(\neg\, \boldsymbol{u}_2 \wedge \big(\boldsymbol{z}_2(t) \leftrightarrow \boldsymbol{z}_3(t)\big)\Big), \\ \boldsymbol{z}_4(t+1) = \big(\boldsymbol{z}_1(t) \wedge \boldsymbol{z}_2(t) \wedge \boldsymbol{z}_4(t)\big) \\ \qquad\qquad \vee \big(\boldsymbol{z}_1(t) \wedge \neg\, \boldsymbol{z}_2(t) \wedge \neg\, \boldsymbol{z}_4(t)\big) \vee \neg\, \boldsymbol{z}_1. \end{cases} \tag{34}$$

The Type-I cascading decomposition problem discussed above does not involve input decomposition. As mentioned in Section 1, the cascading state-space decomposition problem with cascading inputs called Type-II cascading decomposition is also a theoretically interesting and practically useful problem, which we investigate in the next section.

4 Type-II cascading decomposition

The Type-II cascading decomposition problem is defined as follows:

Definition 8 Considering Eq. (7) with Eq. (8), the Type-II cascading decomposition problem is solvable if there exists a logical coordinate transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ such that under the $\boldsymbol{z}$ coordinate frame the system becomes

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1)=\boldsymbol{G}_1\boldsymbol{u}^1(t)\boldsymbol{z}^{[1]}(t), \\ \boldsymbol{z}^{[2]}(t+1)=\boldsymbol{G}_2\boldsymbol{u}^1(t)\boldsymbol{u}^2(t)\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t), \\ \quad\vdots \\ \boldsymbol{z}^{[p]}(t+1)=\boldsymbol{G}_p\boldsymbol{u}^1(t)\boldsymbol{u}^2(t)\ldots\boldsymbol{u}^p(t) \\ \qquad\qquad \cdot\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[p]}(t), \\ \boldsymbol{z}^{[p+1]}(t+1)=\boldsymbol{G}_{p+1}\boldsymbol{u}(t)\boldsymbol{z}(t), \end{cases} \tag{35}$$

where $\boldsymbol{z}=\ltimes_{i=1}^{p+1}\boldsymbol{z}^{[i]}$, $\boldsymbol{u}=\ltimes_{i=1}^{p+1}\boldsymbol{u}^i$, $\boldsymbol{z}^{[i]}=\ltimes_{l=N_{i-1}+1}^{N_i}\boldsymbol{z}_l\in\Delta_{2^{n_i}}$, $\boldsymbol{u}^i=\ltimes_{l=M_{i-1}+1}^{M_i}\boldsymbol{u}_l\in\Delta_{2^{m_i}}$, $\boldsymbol{G}_i\in\mathcal{L}_{2^{n_i}\times 2^{N_i+M_i}}$ for any $1\le i\le p+1$, $N_0=0$, $N_i:=\sum_{s=1}^{i}n_s$, $n_1+n_2+\ldots+n_{p+1}=n$, $M_0=0$, $M_i:=\sum_{s=1}^{i}m_s$, and $m_1+m_2+\ldots+m_{p+1}=m$.

The following proposition can be derived in a similar way to Proposition 1. The specific process of proof is omitted here:

Proposition 2 The Type-II cascading decomposition problem is solvable if and only if there exists a logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$ such that under the $\boldsymbol{z}$ coordinate frame. For any $i\in[1,p]$, Eq. (8) can be converted into

$$\begin{cases} \boldsymbol{z}^{[1]}(t+1)\boldsymbol{z}^{[2]}(t+1)\ldots\boldsymbol{z}^{[i]}(t+1) \\ =\boldsymbol{G}_{1,2,\ldots,i}\boldsymbol{u}^1(t)\boldsymbol{u}^2(t)\ldots\boldsymbol{u}^i(t)\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[i]}(t), \\ \boldsymbol{z}^{[i+1]}(t+1)\boldsymbol{z}^{[i+2]}(t+1)\ldots\boldsymbol{z}^{[p+1]}(t+1) \\ =\boldsymbol{G}_{i+1,i+2,\ldots,p+1}\boldsymbol{u}(t)\boldsymbol{z}(t), \end{cases} \tag{36}$$

where $\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t)\ldots\boldsymbol{z}^{[i]}(t)\in\Delta_{2^{N_i}}$, $\boldsymbol{G}_{1,2,\ldots,i}\in\mathcal{L}_{2^{N_i}\times 2^{N_i+M_i}}$, and $\boldsymbol{G}_{i+1,i+2,\ldots,p+1}\in\mathcal{L}_{2^{n-N_i}\times 2^{n+m}}$.

Let $\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2,\ldots,\boldsymbol{L}_{2^m}]$. For convenience of description, two sets are defined here. For any $i=1,2,\ldots,p+1$ and $j=1,2,\ldots,2^{M_i}$, we define

$$\begin{cases} \boldsymbol{I}_{\boldsymbol{u}^1\boldsymbol{u}^2\ldots\boldsymbol{u}^i=\boldsymbol{\delta}_{2^{M_i}}^j}:=\Big\{r\in[1,2^m]\,\Big|\,\boldsymbol{u}=\boldsymbol{\delta}_{2^m}^r, \\ \qquad\qquad \boldsymbol{u}^1\boldsymbol{u}^2\ldots\boldsymbol{u}^i=\boldsymbol{\delta}_{2^{M_i}}^j\Big\}, \\ \boldsymbol{M}_j^{1,2,\ldots,i}:=\sum\limits_{r\in\boldsymbol{I}_{\boldsymbol{u}^1\boldsymbol{u}^2\ldots\boldsymbol{u}^i=\boldsymbol{\delta}_{2^{M_i}}^j}}\boldsymbol{L}_r. \end{cases} \tag{37}$$

The state transition diagram of $\boldsymbol{M}_j^{1,2,\ldots,i}$ is denoted by $\mathcal{G}_j^{1,2,\ldots,i}$.

Theorem 2 For any $i\in[1,p]$, there exists a logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}_i\boldsymbol{x}$ such that under the $\boldsymbol{z}$ coordinate frame, BCN becomes Eq. (36) if and only if the diagram $\mathcal{G}_j^{1,2,\ldots,i}$ has a common PEVP $\mathcal{S}^i:=\{S_l^i\}_{l=1}^{2^{N_i}}$ with $|S_l^i|=2^{n-N_i}$.

Proof Necessity: Set $\boldsymbol{z}^1\boldsymbol{z}^2\ldots\boldsymbol{z}^i=\boldsymbol{\delta}_{2^{N_i}}^l$ and define $S_l^i=\{\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{z}\mid\boldsymbol{z}^1\boldsymbol{z}^2\ldots\boldsymbol{z}^i=\boldsymbol{\delta}_{2^{N_i}}^l\}$. Then we have $|S_l^i|=2^{n-N_i}$, and $\mathcal{S}^i=\{S_l^i\}_{l=1}^{2^{N_i}}$ is an equal vertex partition.

Considering Eq. (36), we set $\boldsymbol{u}_1\boldsymbol{u}_2\ldots\boldsymbol{u}_i=\boldsymbol{\delta}_{2^{M_i}}^j$ for any $j=1,2,\ldots,2^{M_i}$. For any

$$\boldsymbol{x}(t)=\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{z}(t)=\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{\delta}_{2^{N_i}}^l\boldsymbol{\delta}_{2^{n-N_i}}^i\in S_l^i, \tag{38}$$

where $\boldsymbol{z}^1(t)\boldsymbol{z}^2(t)\ldots\boldsymbol{z}^i(t)=\boldsymbol{\delta}_{2^{N_i}}^l$, we have

$$\begin{aligned} &\boldsymbol{z}^1(t+1)\boldsymbol{z}^2(t+1)\ldots\boldsymbol{z}^i(t+1) \\ &=\boldsymbol{G}_{1,2,\ldots,i}\boldsymbol{\delta}_{2^{M_i}}^j\boldsymbol{z}^1(t)\boldsymbol{z}^2(t)\ldots\boldsymbol{z}^i(t):=\boldsymbol{\delta}_{2^{N_i}}^{\alpha_l^j}. \end{aligned} \tag{39}$$

Then

$$\boldsymbol{x}(t+1)=\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{z}(t+1)\in S_{\alpha_l^j}^i. \tag{40}$$

This implies that $N^j(S_l^i)\subset S_{\alpha_l^j}^i$, and $\mathcal{S}^i$ is a PEVP under $\boldsymbol{u}=\boldsymbol{\delta}_{2^{M_i}}^j$. From the arbitrariness of j, we have that $\mathcal{S}^i$ is a common PEVP of $\mathcal{G}_j$, $j=1,2\ldots,2^{M_i}$.

Sufficiency: Since $\mathcal{S}^i=\{S_l^i\}_{l=1}^{2^{N_i}}$ is an equal vertex partition with $|S_l^i|=2^{n-N_i}$, for any $\boldsymbol{x}=\boldsymbol{\delta}_{2^n}^q\in S_l^i$, $l=1,2,\ldots,2^{N_i}$, let $\boldsymbol{Q}_i\boldsymbol{\delta}_{2^n}^q=\boldsymbol{\delta}_{2^{N_i}}^l$. Then there exists a permutation matrix $\boldsymbol{T}_i\in\mathcal{L}_{2^n\times 2^n}$ such that $\boldsymbol{Q}_i=(\boldsymbol{I}_{2^{N_i}}\otimes\boldsymbol{1}_{2^{n-N_i}}^{\mathrm{T}})\boldsymbol{T}_i$. Let $\boldsymbol{z}=\boldsymbol{T}_i\boldsymbol{x}$. Then

$$\boldsymbol{Q}_i\boldsymbol{x}=(\boldsymbol{I}_{2^{N_i}}\otimes\boldsymbol{1}_{2^{n-N_i}}^{\mathrm{T}})\boldsymbol{T}_i\boldsymbol{x}=\boldsymbol{z}^1\boldsymbol{z}^2\ldots\boldsymbol{z}^i. \tag{41}$$

It follows from $\mathcal{S}^i$ being a common PEVP of $\mathcal{G}_j^{1,2,\ldots,i}$ ($j=1,2\ldots,2^{M_i}$) that for any l and j, there exists an α_l^j such that $N^j(S_l^i)\subset S_{\alpha_l^j}^i$. Thus, for any $\boldsymbol{x}(t)=\boldsymbol{\delta}_{2^n}^q\in S_l^i$, we have $\boldsymbol{x}(t+1)\in S_{\alpha_l^j}^i$ and

$$\boldsymbol{z}^1(t+1)\boldsymbol{z}^2(t+1)\ldots\boldsymbol{z}^i(t+1)=\boldsymbol{Q}_i\boldsymbol{x}(t+1)=\boldsymbol{\delta}_{2^{N_i}}^{\alpha_l^j}. \tag{42}$$

So, there exists a $\boldsymbol{G}_{1,2,\ldots,i}^j\in\mathcal{L}_{2^{N_i}\times 2^{N_i}}$ such that $\boldsymbol{z}^1(t+1)\boldsymbol{z}^2(t+1)\ldots\boldsymbol{z}^i(t+1)=\boldsymbol{G}_{1,2,\ldots,i}^j\boldsymbol{z}^1(t)\boldsymbol{z}^2(t)\ldots\boldsymbol{z}^i(t)$. Given the arbitrariness of j, we have

$$\begin{aligned} &\boldsymbol{z}^1(t+1)\boldsymbol{z}^2(t+1)\ldots\boldsymbol{z}^i(t+1) \\ &=\boldsymbol{G}_{1,2,\ldots,i}\boldsymbol{u}^1(t)\boldsymbol{u}^2(t)\ldots\boldsymbol{u}^i(t)\boldsymbol{z}^1(t)\boldsymbol{z}^2(t)\ldots\boldsymbol{z}^i(t), \end{aligned}$$

where $\boldsymbol{G}_{1,2,\ldots,i}\in\mathcal{L}_{2^{N_i}\times 2^{N_i+M_i}}$. Therefore, Eq. (36) holds.

Similar to the Type-I cascading decomposition problem of BCNs, a graphic condition for Type-II cascading decomposition is provided here.

Theorem 3 Considering Eq. (8), the Type-II cascading decomposition problem is solvable by a coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$ if and only if there exists an NPEVP $\mathcal{S}^{1,2,\ldots,p}\sqsubset\mathcal{S}^{1,2,\ldots,p-1}\sqsubset\ldots\sqsubset\mathcal{S}^{1,2}\sqsubset\mathcal{S}^{1}$, where for any i, $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$ with $|S_l^{1,2,\ldots,i}|=2^{n-N_i}$ is a common PEVP of $\mathcal{G}_j^{1,2,\ldots,i}$, $j=1,2,\ldots,2^{M_i}$.

Proof Necessity: For any $i=1,2,\ldots,p$, set $\boldsymbol{z}^1\boldsymbol{z}^2\ldots\boldsymbol{z}^i=\boldsymbol{\delta}_{2^{N_i}}^l$ and define $S_l^{1,2,\ldots,i}=\{\boldsymbol{T}^{\mathrm{T}}\boldsymbol{z}\in\Delta_{2^n}\big|\boldsymbol{z}^1\boldsymbol{z}^2\ldots\boldsymbol{z}^i=\boldsymbol{\delta}_{2^{N_i}}^l\}$. Let $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$. By the same process as the proof of necessity of Theorem 2, for each i, $\mathcal{S}^{1,2,\ldots,i}=\{S_l^{1,2,\ldots,i}\}_{l=1}^{2^{N_i}}$ is a common PEVP of $\mathcal{G}_1^{1,2,\ldots,i},\mathcal{G}_2^{1,2,\ldots,i}\ldots,\mathcal{G}_{2^{M_i}}^{1,2,\ldots,i}$.

Next we prove $\mathcal{S}^{1\ldots i+1}\sqsubset\mathcal{S}^{1,2,\ldots,i}$ for any $i=1,2,\ldots,p-1$. For any l and any $\boldsymbol{x}(t)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{\delta}_{2^{N_{i+1}}}^l\boldsymbol{\delta}_{2^{n-N_{i+1}}}^r\in S_l^{1,2,\ldots,i+1}$, we set $\boldsymbol{\delta}_{2^{N_{i+1}}}^l=\boldsymbol{\delta}_{2^{N_i}}^{l_1}\boldsymbol{\delta}_{2^{n_{i+1}}}^{l_2}$. Then we have

$$\boldsymbol{x}(t)=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{\delta}_{2^{N_i}}^{l_1}\boldsymbol{\delta}_{2^{n_{i+1}}}^{l_2}\boldsymbol{\delta}_{2^{n-N_{i+1}}}^{r}\in S_{l_i}^{1,2,\ldots,i}. \tag{43}$$

Thus, $S_l^{1,2,\ldots,i+1}\subset S_{l_1}^{1,2,\ldots,i}$ and $\mathcal{S}^{1,2,\ldots,i+1}\sqsubset\mathcal{S}^{1,2,\ldots,i}$. Then the necessity is proved.

Sufficiency: The sufficiency proof is similar to that of Theorem 1, and is omitted here.

Remark 5 In Zou and Zhu (2014), Eq. (8) is said to be decomposable w.r.t. inputs if Eq. (8) under a logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$ can be converted into

$$\begin{cases}\boldsymbol{z}^{[1]}(t+1)=\boldsymbol{G}_1\boldsymbol{z}^{[1]}(t),\\ \boldsymbol{z}^{[2]}(t+1)=\boldsymbol{G}_2\boldsymbol{u}(t)\boldsymbol{z}(t).\end{cases} \tag{44}$$

The form of Eq. (44) is also a controllability decomposition in Cheng et al. (2010). Comparing Eq. (44) with Eq. (35), we have that decomposition w.r.t. inputs or controllability decomposition of BCNs is Type-II cascading decomposition when $p=1$ and $m_1=0$ in Eq. (35). Thus, Type-II cascading decomposition is the generalization of decomposition w.r.t. inputs in Zou and Zhu (2014) and controllability decomposition in Cheng et al. (2010).

Similar to Type-I cascading decomposition, we can compute the logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$ in Theorem 3.

Example 4 Considering Eq. (30) whose algebraic form is Eq. (31), let $\boldsymbol{M}_1^1=\boldsymbol{L}_1+\boldsymbol{L}_2$, $\boldsymbol{M}_2^1=\boldsymbol{L}_3+\boldsymbol{L}_4$, $\boldsymbol{M}_1^{1,2}=\boldsymbol{L}_1$, $\boldsymbol{M}_2^{1,2}=\boldsymbol{L}_2$, $\boldsymbol{M}_3^{1,2}=\boldsymbol{L}_3$, and $\boldsymbol{M}_4^{1,2}=\boldsymbol{L}_4$. The state transition diagrams $\mathcal{G}_1^1$ of $\boldsymbol{M}_1^1$ and $\mathcal{G}_2^1$ of $\boldsymbol{M}_2^1$ have a common PEVP $\mathcal{S}^1=\{S_i^1\}_{i=1}^4$, as shown in Example 3. The state transition digraphs $\mathcal{G}_1^{1,2}$ of $\boldsymbol{M}_1^{1,2}$, $\mathcal{G}_2^{1,2}$ of $\boldsymbol{M}_2^{1,2}$, $\mathcal{G}_3^{1,2}$ of $\boldsymbol{M}_2^{1,3}$, and $\mathcal{G}_4^{1,2}$ of $\boldsymbol{M}_4^{1,2}$ have a common PEVP $\mathcal{S}^{1,2}=\{S_{l_1,l_2}^{1,2}|1\le l_1\le 4,1\le l_2\le 2\}$, as shown in Example 3. From Example 3, we have $\mathcal{S}^{1,2}\sqsubset\mathcal{S}^1$.

Based on the above discussion, we have that $\mathcal{S}^{1,2}$ and $\mathcal{S}^1$ satisfy conditions in Theorem 3. So, there exists a logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$ such that Eq. (30) can be converted into Eq. (35). Omitting the same processes as shown in Example 3, we have

$$\boldsymbol{T}=\boldsymbol{\delta}_{2^4}[5,9,6,10,7,11,8,12,1,13,2,14,3,15,4,16], \tag{45}$$

and Eq. (30) is converted into

$$\begin{cases}\boldsymbol{z}^{[1]}(t+1)=\boldsymbol{G}_1\boldsymbol{u}_1(t)\boldsymbol{z}^{[1]}(t),\\ \boldsymbol{z}^{[2]}(t+1)=\boldsymbol{G}_2\boldsymbol{u}_1(t)\boldsymbol{u}_2(t)\boldsymbol{z}^{[1]}(t)\boldsymbol{z}^{[2]}(t),\\ \boldsymbol{z}^{[3]}(t+1)=\boldsymbol{G}_3\boldsymbol{u}(t)\boldsymbol{z}(t),\end{cases} \tag{46}$$

where $\boldsymbol{z}^{[1]}=\boldsymbol{z}_1\boldsymbol{z}_2$, $\boldsymbol{z}^{[2]}=\boldsymbol{z}_3$, $\boldsymbol{z}^{[3]}=\boldsymbol{z}_4$, and

$$\begin{aligned}\boldsymbol{G}_1=&\boldsymbol{\delta}_4[2,2,1,1,2,4,3,1],\\ \boldsymbol{G}_2=&\boldsymbol{\delta}_2[1,1,1,1,1,1,1,1,1,2,2,1,1,2,2,1,\\ &1,1,1,1,1,1,1,1,1,2,2,1,1,2,2,1],\\ \boldsymbol{G}_3=&\boldsymbol{\delta}_2[2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2,\\ &2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2,\\ &2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2,\\ &2,1,2,1,1,2,1,2,2,2,2,2,2,2,2,2].\end{aligned}$$

5 Conclusions

The Type-I and Type-II cascading decomposition problems for BCNs have been investigated using a graph-theoretical method in this study. Two necessary and sufficient conditions have been obtained for the solvability of Type-I cascading decomposition of BCNs. An algorithm to compute the logical coordinate transformation $\boldsymbol{z}=\boldsymbol{T}\boldsymbol{x}$ has been designed. For Type-II cascading decomposition, similar results have been presented. In future work, we will investigate some related decoupling problems of BCNs.

Compliance with ethics guidelines

Yi-feng LI and Jian-dong ZHU declare that they have no conflict of interest.

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