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Observer-based control for fractional-order singular systems with order α $(0 < \alpha < 1)$ and input delay*

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Abstract: In this paper, observer-based control for fractional-order singular systems with order α $(0 < \alpha < 1)$ and input delay is studied. On the basis of the Smith predictor and approximation error, the system with input delay is approximately equivalent to the system without input delay. Furthermore, based on the linear matrix inequality (LMI) technique, the necessary and sufficient condition of observer-based control is proposed. Since the condition is a nonstrict LMI, including the equality constraint, it will lead to some trouble when solving problems using toolbox. Thus, the strict LMI-based condition is improved in the paper. Finally, a numerical example and a direct current motor example are given to illustrate the effectiveness of the strict LMI-based condition.

Key words: Observer-based control; Singular systems; Fractional order; Input delay; Linear matrix inequality
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1 Introduction

Recently, a large number of researchers have focused on fractional-order systems, which describe the model of the real-world phenomena with memory more concisely and precisely (Du and Lu, 2021; Hua et al., 2021; Saffarian and Mohebbi, 2021; Jiang et al., 2022). In general, stability analysis is fundamental for control systems, including fractional-order systems. The first criterion has been proposed by Matignon (1998) for fractional-order systems with $0 < \alpha < 2$. Although stability can be judged through the location of poles, it is still difficult to design controllers in practice. The linear matrix inequality (LMI) technique can be better used to solve control theory, which can be expressed by convex optimization problems. Thus, to perform the stability analysis for fractional-order systems with order α $(0 < \alpha < 1$ or $1 < \alpha < 2)$, the LMI-based conditions have been derived (Lu and Chen, 2009, 2010; Sabatier et al., 2010).

Singular systems, also referred to as descriptor systems, are a class of widespread systems (Xu et al., 2002). Many real-world systems, such as economic systems, circuit systems, and viscoelastic systems, can be more accurately described by singular systems (Xu and Lam, 2006; Guerrero et al., 2021; Li YC and Ma, 2021; Zhang L et al., 2021). Different from the normal system, admissibility analysis for singular systems, including regular, impulse-free, and stable ones, is the fundamental problem. Recently, we have noticed that, by introducing the fractional calculus, singular systems can describe phenomena more accurately. Therefore, on the basis of fractional-order and singular system results, a large number of papers have been published (N'Doye et al., 2013; Ji

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and Qiu, 2015; Marir et al., 2017; Zhang XF and Chen, 2018; Marir and Chadli, 2019; Wei et al., 2019; Wu et al., 2020). In these works, stability and stabilization have been widely studied (N'Doye et al., 2013; Wu et al., 2020), since they can be directly derived. In Marir et al. (2017) and Marir and Chadli (2019), the admissibility conditions were derived, and the strict LMI-based conditions were discussed further for fractional-order singular systems with $1 < \alpha < 2$. Moreover, the admissibility conditions were derived for $0 < \alpha < 1$ in Zhang XF and Chen (2018), and to improve the nonstrict LMI-based conditions, two kinds of strict LMI-based conditions were proposed. Lin et al. (2018) studied observer-based control for fractional-order singular systems with order α $(1 \leq \alpha < 2)$. It is noteworthy that the stability region of α $(0 < \alpha < 1)$ is nonconvex. Thus, the study of α $(0 < \alpha < 1)$ is more difficult and interesting (Zhang XF and Chen, 2018). In addition, various researchers (Li RC and Zhang, 2020; Udhayakumar et al., 2020) have reported the conditions of polytopic uncertainties and Takagi–Sugeno (T-S) fuzzy control for fractional-order singular systems. Delayed neutral networks have also been studied (Aghayan et al., 2021).

In practice, state estimation and observer-based controller should be required for fractional-order systems. Many efforts have been made for observer-based control (Lan et al., 2012; Lan and Zhou, 2013; Li C et al., 2014; Ibrir and Bettayeb, 2015; Li BX and Zhang, 2016; Geng et al., 2020). In Lan et al. (2012) and Lan and Zhou (2013), the conditions of observer-based control of $1 < \alpha < 2$ have been proposed. Furthermore, Li C et al. (2014) investigated nonlinear systems with $0 < \alpha < 2$. Observer-based control of $0 < \alpha < 1$ has been developed in Li BX and Zhang (2016) using the singular value decomposition concept. In addition, new sufficient conditions of observer-based control have been proposed for uncertain systems of $1 < \alpha < 2$ and $0 < \alpha < 1$ in Ibrir and Bettayeb (2015). Moreover, observer-based control has been studied for fractional-order systems with input delay (Geng et al., 2020) using the Smith predictor, which is a class of effective dead-time compensators (Nguyen et al., 2021). To overcome the shortcomings, methods that include sensitivity to model errors and unstable processes have been proposed in Stamova (2014). Pu and Wang (2020) proposed the conditions to deal with state and input delay for fractional-order systems. Marir et al. (2022a) studied the bounded real lemma for fractional-order singular systems with order α $(1 \leq \alpha < 2)$, and Marir et al. (2022b) derived the conditions of H_∞ static output-feedback control. Even so, H_∞ control of order α $(0 < \alpha < 1)$ was rarely studied.

Although a lot of papers have been published for input delay, it is worth mentioning that most research is mainly in the field of nonsingular fractional-order systems with input delay. For general singular systems with input delay, these were rarely studied. In short, the contributions are as follows:

1. Using the Smith predictor, observer-based control is first studied for fractional-order singular systems with order α $(0 < \alpha < 1)$ and input delay.

2. The necessary and sufficient condition based on nonstrict LMI is presented. Then, the condition based on strict LMI is improved.

The notations used throughout the paper are as follows: $\mathbb{R}$ denotes the real set, $\boldsymbol{A}^{\mathrm{T}}$ denotes the transpose of $\boldsymbol{A}$, $\mathrm{sym}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}}$, and $\mathrm{spec}(\boldsymbol{X}, \boldsymbol{Y}, \alpha)$ is the spectrum of $\det(s^\alpha \boldsymbol{X} - \boldsymbol{Y}) = 0$. Moreover, $\det(\cdot)$ and $\deg(\cdot)$ denote the determinant and degree, respectively.

2 Preliminaries and problem formulation

2.1 Preliminaries

The Caputo definition (Du and Lu, 2021) of a fractional derivative operator is given by

$$\mathrm{D}^\alpha f(t) = \frac{1}{\Gamma(l-\alpha)} \int_0^t \frac{f^{(l)}(\tau)}{(t-\tau)^{\alpha-l+1}} \mathrm{d}\tau,$$

where $\Gamma(\cdot)$ is the Gamma function and $\alpha \in (l-1, l)$.

Consider the following system:

$$\begin{cases} \boldsymbol{E}\mathrm{D}^\alpha \boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_1\boldsymbol{u}(t) + \boldsymbol{B}_2\boldsymbol{u}(t-\tau), \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \end{cases} \tag{1}$$

where $0 < \alpha < 1$, D^α denotes the Caputo derivative, and $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, and $\boldsymbol{y}(t) \in \mathbb{R}^p$ denote the state vector, input, and output, respectively. $\boldsymbol{E} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m}$, $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m}$, and $\boldsymbol{C} \in \mathbb{R}^{p\times n}$ are constant matrices, and $\mathrm{rank}(\boldsymbol{E}) < n$. $\tau \in [0, \infty)$ denotes the input delay constant.

Before proceeding, one definition and one lemma for system (1) are introduced as follows:

Definition 1 (Zhang XF and Chen, 2018) The fractional-order linear singular system $\boldsymbol{E}\mathrm{D}^{\alpha}\boldsymbol{x}(t)=\boldsymbol{A}\boldsymbol{x}(t)$ with $\mathrm{rank}(\boldsymbol{E})<n$ is admissible if $\det(s^{\alpha}\boldsymbol{E}-\boldsymbol{A})\neq 0$ (regular), $\deg(\det(s^{\alpha}\boldsymbol{E}-\boldsymbol{A}))=\mathrm{rank}(\boldsymbol{E})$ (impulse-free), and $|\arg(\mathrm{spec}(\boldsymbol{E},\boldsymbol{A},\alpha))|>\pi\alpha/2$ (stable).

Lemma 1 (Zhang XF and Chen, 2018) The fractional-order linear singular system $\boldsymbol{E}\mathrm{D}^{\alpha}\boldsymbol{x}(t)=\boldsymbol{A}\boldsymbol{x}(t)$ with $\mathrm{rank}(\boldsymbol{E})<n$ is asymptotically admissible if and only if the following inequalities hold with two matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$:

$$\begin{bmatrix}\boldsymbol{EX} & \boldsymbol{EY}\\ -\boldsymbol{EY} & \boldsymbol{EX}\end{bmatrix}=\begin{bmatrix}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}} & -\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\\ \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}} & \boldsymbol{X}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\end{bmatrix}\geq 0,$$
$$a\boldsymbol{AX}+a\boldsymbol{X}\boldsymbol{A}^{\mathrm{T}}+b\boldsymbol{AY}-b\boldsymbol{Y}\boldsymbol{A}^{\mathrm{T}}<0,$$

where $a=\sin(\pi\alpha/2)$ and $b=\cos(\pi\alpha/2)$.

2.2 Problem formulation

For system (1), define the following state transformation:

$$\boldsymbol{Ez}(t)=\boldsymbol{Ex}(t)+\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s. \tag{2}$$

In particular, if $\boldsymbol{E}$ is the identity matrix, the following equation holds according to the results in Si-Ammour et al. (2009):

$$\boldsymbol{z}(t)=\boldsymbol{x}(t)+\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s. \tag{3}$$

According to Geng et al. (2020), we have

$$\begin{aligned}&\mathrm{D}^{\alpha}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s\\
=&\mathrm{D}^{\alpha}\int_{0}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s\\
&-\mathrm{D}^{\alpha}\int_{0}^{t-\tau}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s\\
=&\boldsymbol{A}\int_{0}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s+e_{\alpha}^{-\boldsymbol{A}\tau}\boldsymbol{B}_2\boldsymbol{u}(t)\\
&-\boldsymbol{A}\int_{0}^{t-\tau}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s-\boldsymbol{B}_2\boldsymbol{u}(t-\tau),\end{aligned} \tag{4}$$

where $e_{\alpha}^{-\boldsymbol{A}\tau}=\tau^{\alpha-1}\sum\limits_{k=0}^{\infty}\dfrac{(-\boldsymbol{A})^k\tau^{k\alpha}}{\Gamma(\alpha k+\alpha)}$.

From Eqs. (2) and (4), we obtain

$$\begin{aligned}\boldsymbol{E}\mathrm{D}^{\alpha}\boldsymbol{z}(t)=&\boldsymbol{E}\mathrm{D}^{\alpha}\boldsymbol{x}(t)+\mathrm{D}^{\alpha}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s\\
=&\boldsymbol{Ax}(t)+\boldsymbol{B}_1\boldsymbol{u}(t)+\boldsymbol{B}_2\boldsymbol{u}(t-\tau)\\
&+\mathrm{D}^{\alpha}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s\\
=&\boldsymbol{Ax}(t)+\boldsymbol{B}_1\boldsymbol{u}(t)\\
&+\boldsymbol{A}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s.\end{aligned}$$

According to Eq. (3), system (1) can be rewritten as follows:

$$\begin{cases}\boldsymbol{E}\mathrm{D}^{\alpha}\boldsymbol{z}(t)=\boldsymbol{Az}(t)+\bar{\boldsymbol{B}}\boldsymbol{u}(t),\\ \boldsymbol{y}(t)=\boldsymbol{Cz}(t)-\boldsymbol{C}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s,\end{cases} \tag{5}$$

where $\bar{\boldsymbol{B}}=\boldsymbol{B}_1+e_{\alpha}^{-\boldsymbol{A}\tau}\boldsymbol{B}_2$.

Thus, the following observer-based controller is obtained:

$$\begin{cases}\boldsymbol{E}\mathrm{D}^{\alpha}\hat{\boldsymbol{z}}(t)=\boldsymbol{A}\hat{\boldsymbol{z}}(t)+\bar{\boldsymbol{B}}\boldsymbol{u}(t)+\boldsymbol{L}(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)),\\ \boldsymbol{u}(t)=\boldsymbol{K}\hat{\boldsymbol{z}}(t),\end{cases} \tag{6}$$

where $\hat{\boldsymbol{z}}(t)$ denotes the state estimate, $\boldsymbol{K}$ is the observer, and $\boldsymbol{L}$ is the controller gain to be determined. From $\boldsymbol{y}(t)=\boldsymbol{Cx}(t)$, we obtain

$$\begin{cases}\boldsymbol{E}\mathrm{D}^{\alpha}\hat{\boldsymbol{z}}(t)=(\boldsymbol{A}-\boldsymbol{LC})\hat{\boldsymbol{z}}(t)+\bar{\boldsymbol{B}}\boldsymbol{u}(t)+\boldsymbol{Ly}(t)\\ \qquad\qquad +\boldsymbol{LC}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s,\\ \boldsymbol{u}(t)=\boldsymbol{K}\hat{\boldsymbol{z}}(t).\end{cases} \tag{7}$$

We define the error $\boldsymbol{e}(t)=\boldsymbol{z}(t)-\hat{\boldsymbol{z}}(t)$. Hence, from Eqs. (5) and (6), we have

$$\boldsymbol{E}\mathrm{D}^{\alpha}\boldsymbol{e}(t)=(\boldsymbol{A}-\boldsymbol{LC})\boldsymbol{e}(t). \tag{8}$$

Then, we obtain

$$\bar{\boldsymbol{E}}\mathrm{D}^{\alpha}\bar{\boldsymbol{z}}(t)=\bar{\boldsymbol{A}}\bar{\boldsymbol{z}}(t), \tag{9}$$

where

$$\begin{cases}\bar{\boldsymbol{E}}=\begin{bmatrix}\boldsymbol{E} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{E}\end{bmatrix},\ \bar{\boldsymbol{z}}(t)=\begin{bmatrix}\boldsymbol{z}(t)\\ \boldsymbol{z}(t)-\hat{\boldsymbol{z}}(t)\end{bmatrix},\\ \bar{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{A}+\bar{\boldsymbol{B}}\boldsymbol{K} & -\bar{\boldsymbol{B}}\boldsymbol{K}\\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{LC}\end{bmatrix}.\end{cases} \tag{10}$$

Remark 1 Our aim is to design the controller in Eq. (7), and to ensure that system (9) is asymptotically admissible. According to Lemma 2.1 in Geng et al. (2020), $\boldsymbol{LC}\int_{t-\tau}^{t}e_{\alpha}^{\boldsymbol{A}(t-s-\tau)}\boldsymbol{B}_2\boldsymbol{u}(s)\mathrm{d}s$ does not affect the admissibility for systems (1) and (9).

3 Main results

In this section, the LMI-based necessary and sufficient conditions of system (9) are proposed. First, the nonstrict LMI-based condition is given in the following theorem:

Theorem 1 System (9) with matrices $\boldsymbol{K}$ and $\boldsymbol{L}$ is asymptotically admissible, if and only if there exist matrices $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Z}$, and $\boldsymbol{R}$ such that the following inequalities hold:

$$\begin{bmatrix}\boldsymbol{EX}_1 & \boldsymbol{EY}_1\\ -\boldsymbol{EY}_1 & \boldsymbol{EX}_1\end{bmatrix}=\begin{bmatrix}\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}} & -\boldsymbol{Y}_1^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\\ \boldsymbol{Y}_1^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}} & \boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\end{bmatrix}\geq 0, \tag{11}$$

$$\begin{bmatrix}\boldsymbol{EX}_2 & \boldsymbol{EY}_2\\ -\boldsymbol{EY}_2 & \boldsymbol{EX}_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}} & -\boldsymbol{Y}_2^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\\ \boldsymbol{Y}_2^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}} & \boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\end{bmatrix}\geq 0, \tag{12}$$

$$\boldsymbol{P}_1=a\boldsymbol{X}_1+b\boldsymbol{Y}_1,\ \mathrm{sym}(\boldsymbol{P}_1\boldsymbol{A}-\boldsymbol{ZC})<0, \tag{13}$$

$$\boldsymbol{P}_2=a\boldsymbol{X}_2+b\boldsymbol{Y}_2,\ \mathrm{sym}(\boldsymbol{AP}_2+\bar{\boldsymbol{B}}\boldsymbol{R})<0, \tag{14}$$

where $a=\sin(\pi\alpha/2)$ and $b=\cos(\pi\alpha/2)$. Furthermore, $\boldsymbol{L}$ and $\boldsymbol{K}$ are given by

$$\boldsymbol{L}=\boldsymbol{P}_1^{-1}\boldsymbol{Z},\ \boldsymbol{K}=\boldsymbol{RP}_2^{-1}. \tag{15}$$

Proof (Sufficiency) Let $\boldsymbol{Z}=\boldsymbol{P}_1\boldsymbol{L}$ and $\boldsymbol{R}=\boldsymbol{KP}_2$. Then, according to inequalities (13) and (14), we obtain

$$\mathrm{sym}(\boldsymbol{P}_1(\boldsymbol{A}-\boldsymbol{LC}))<0, \tag{16}$$

$$\mathrm{sym}((\boldsymbol{A}+\bar{\boldsymbol{B}}\boldsymbol{K})\boldsymbol{P}_2)<0. \tag{17}$$

Pre- and post-multiplying inequality (14) by $\boldsymbol{P}_1^{-1}$ and its transpose, we obtain

$$\mathrm{sym}\left((\boldsymbol{A}-\boldsymbol{LC})\boldsymbol{P}_1^{-\mathrm{T}}\right)<0. \tag{18}$$

So, we can easily find a scalar μ satisfying

$$\mathrm{sym}\left(\begin{bmatrix}\boldsymbol{A}+\bar{\boldsymbol{B}}\boldsymbol{K} & -\bar{\boldsymbol{B}}\boldsymbol{K}\\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{LC}\end{bmatrix}\begin{bmatrix}\mu\boldsymbol{P}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{P}_1^{-\mathrm{T}}\end{bmatrix}\right)<0. \tag{19}$$

Choose $\boldsymbol{P}=\begin{bmatrix}\mu\boldsymbol{P}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{P}_1^{-\mathrm{T}}\end{bmatrix}=a\boldsymbol{X}+b\boldsymbol{Y}$; from Lemma 1, system (9) is asymptotically admissible.

(Necessity) Suppose that system (9) is asymptotically admissible. Choose a nonsingular matrix $\bar{\boldsymbol{P}}$:

$$\bar{\boldsymbol{P}}=\begin{bmatrix}? & ?\\ ? & \bar{\boldsymbol{P}}_1\end{bmatrix},$$

where "?" is the unused part in the following proof.

Substituting the above equation into Lemma 1, we have

$$\mathrm{sym}\left(\begin{bmatrix}? & ?\\ ? & (\boldsymbol{A}-\boldsymbol{LC})\bar{\boldsymbol{P}}_1\end{bmatrix}\right)<0. \tag{20}$$

Therefore, we can obtain

$$\mathrm{sym}((\boldsymbol{A}-\boldsymbol{LC})\bar{\boldsymbol{P}}_1)<0. \tag{21}$$

Pre- and post-multiplying inequality (19) by $\bar{\boldsymbol{P}}_1^{-\mathrm{T}}$ and its transpose, we obtain

$$\mathrm{sym}(\bar{\boldsymbol{P}}_1^{-\mathrm{T}}(\boldsymbol{A}-\boldsymbol{LC}))<0.$$

Then, set $\boldsymbol{Z}=\bar{\boldsymbol{P}}_1^{-\mathrm{T}}\boldsymbol{L}$, $\boldsymbol{P}_1=\bar{\boldsymbol{P}}_1^{-\mathrm{T}}$; thus, inequality (11) can be obtained.

Similarly, when choosing $\hat{\boldsymbol{P}}^{-\mathrm{T}}=\begin{bmatrix}\hat{\boldsymbol{P}}_2 & ?\\ ? & ?\end{bmatrix}$, we can obtain

$$\mathrm{sym}\left(\begin{bmatrix}(\boldsymbol{A}+\bar{\boldsymbol{B}}\boldsymbol{K})\hat{\boldsymbol{P}}_2 & ?\\ ? & ?\end{bmatrix}\right)<0. \tag{22}$$

Thus, we have

$$\mathrm{sym}((\boldsymbol{A}+\bar{\boldsymbol{B}}\boldsymbol{K})\hat{\boldsymbol{P}}_2)<0. \tag{23}$$

Letting $\boldsymbol{R}=\boldsymbol{K}\hat{\boldsymbol{P}}_2$ and $\hat{\boldsymbol{P}}_2=\boldsymbol{P}_2$, inequality (12) can be derived. The proof is completed.

Remark 2 Theorem 1 in this study is equivalent to Theorem 3.1 in Geng et al. (2020), when $\boldsymbol{E}=\boldsymbol{I}$; i.e., Theorem 1 can be regarded as the extension of results of normal fractional-order systems. Inequalities (11) and (12) are nonstrict LMIs, which contain equality constraints. Due to round-off errors in numerical calculations, equality constraints are fragile and usually cannot be well satisfied. As a result, the strict LMI-based condition is proposed in the following theorem:

Theorem 2 System (9) with $\boldsymbol{K}$ and $\boldsymbol{L}$ is asymptotically admissible, if and only if there exist matrices $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Q}$, $\boldsymbol{Z}$, and $\boldsymbol{R}$ such that the following inequalities hold:

$$\begin{bmatrix}\boldsymbol{X}_1 & \boldsymbol{Y}_1\\ -\boldsymbol{Y}_1 & \boldsymbol{X}_1\end{bmatrix}>0, \tag{24}$$

$$\begin{bmatrix}\boldsymbol{X}_2 & \boldsymbol{Y}_2\\ -\boldsymbol{Y}_2 & \boldsymbol{X}_2\end{bmatrix}>0, \tag{25}$$

$$\boldsymbol{P}_1=a\boldsymbol{X}_1+b\boldsymbol{Y}_1,\ \mathrm{sym}(\boldsymbol{P}_1\boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}+\boldsymbol{SQA}-\boldsymbol{ZC})<0, \tag{26}$$

$$\boldsymbol{P}_2=a\boldsymbol{X}_2+b\boldsymbol{Y}_2,\ \mathrm{sym}(\boldsymbol{AP}_2\boldsymbol{E}^{\mathrm{T}}+\boldsymbol{ASQ}+\bar{\boldsymbol{B}}\boldsymbol{R})<0, \tag{27}$$

where $a=\sin(\pi\alpha/2)$, $b=\cos(\pi\alpha/2)$, and $S\in\mathbb{R}^{n\times(n-r)}$ satisfying $\boldsymbol{ES}=0$. Here, $\boldsymbol{L}$ and $\boldsymbol{K}$ are given by

$$\boldsymbol{L}=\boldsymbol{P}_1^{-1}\boldsymbol{Z},\ \boldsymbol{K}=\boldsymbol{RP}_2^{-1}. \tag{28}$$

Proof Suppose that inequalities (24) and (25) hold with matrices $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$, and that $\boldsymbol{Q} \in \mathbb{R}^{(n-r)\times n}$ should satisfy $\boldsymbol{SQ} \in \mathbb{R}^{n\times n}$. Let

$$\bar{\boldsymbol{X}}_1 = \boldsymbol{X}_1\boldsymbol{E}^{\mathrm{T}} + a^{-1}\boldsymbol{SQ},\ \bar{\boldsymbol{Y}}_1 = \boldsymbol{Y}_1\boldsymbol{E}^{\mathrm{T}},$$

$$\bar{\boldsymbol{X}}_2 = \boldsymbol{X}_2\boldsymbol{E}^{\mathrm{T}} + a^{-1}\boldsymbol{SQ},\ \bar{\boldsymbol{Y}}_2 = \boldsymbol{Y}_2\boldsymbol{E}^{\mathrm{T}}.$$

Then, substituting the above equations into inequalities (26) and (27), we can obtain

$$\mathrm{sym}(a\bar{\boldsymbol{X}}_1\boldsymbol{A} + b\bar{\boldsymbol{Y}}_1\boldsymbol{A} - \boldsymbol{ZC}) < 0, \tag{29}$$

$$\mathrm{sym}(a\boldsymbol{A}\bar{\boldsymbol{X}}_2 + b\boldsymbol{A}\bar{\boldsymbol{Y}}_2 + \bar{\boldsymbol{B}}\boldsymbol{R}) < 0. \tag{30}$$

According to Theorem 1, Theorem 2 can be proved directly.

Remark 3 The condition of observer-based control for system (9) is presented based on the strict LMI in Theorem 2, which eliminates the equality constraints. Thus, the condition is less conservative and easier to solve using the LMI toolbox.

4 Simulations

In this section, two examples are given to illustrate the effectiveness of our condition. The first one is a numerical example, and the second one is a practical system of direct current (DC) motor.

4.1 Example 1

System (9) with parameters as follows is considered:

$$\boldsymbol{E} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ -2 & -2 & 1 & -2 \\ -1 & -1 & 0 & 0 \end{bmatrix},\ \boldsymbol{A} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 3 & 2 & 0 \\ 2 & -5 & -1 & -5 \\ 1 & -2 & -1 & -1 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},\ \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 2 & 1 & -1 \end{bmatrix},\ \alpha = \frac{1}{3},\ \tau = 0.5.$$

Choose $\boldsymbol{S} = \begin{bmatrix} -1 & 1 & -2 & -1 \end{bmatrix}^{\mathrm{T}}$, which can satisfy $\boldsymbol{ES} = 0$. Then, solving inequalities (24)–(27) and Eq. (28) in Theorem 2, we can obtain the feasible solutions as follows:

$$\boldsymbol{K} = [0.3225\ \ -0.6320\ \ -0.1244\ \ 0.0123],$$

$$\boldsymbol{L} = [-6.3313\ \ -8.6988\ \ 1.6989\ \ -7.6680]^{\mathrm{T}}.$$

Fig. 1 shows that system (9) with the above $\boldsymbol{K}$ and $\boldsymbol{L}$ is asymptotically admissible. Fig. 2 shows that the observation errors can converge to zero, and the effectiveness of the observer-based controller is verified.

Remark 4 Theorem 2 is based on the strict LMI, which avoids the computational complexity and it can be viewed as a generalization of the results of integer-order systems. Compared with the results in

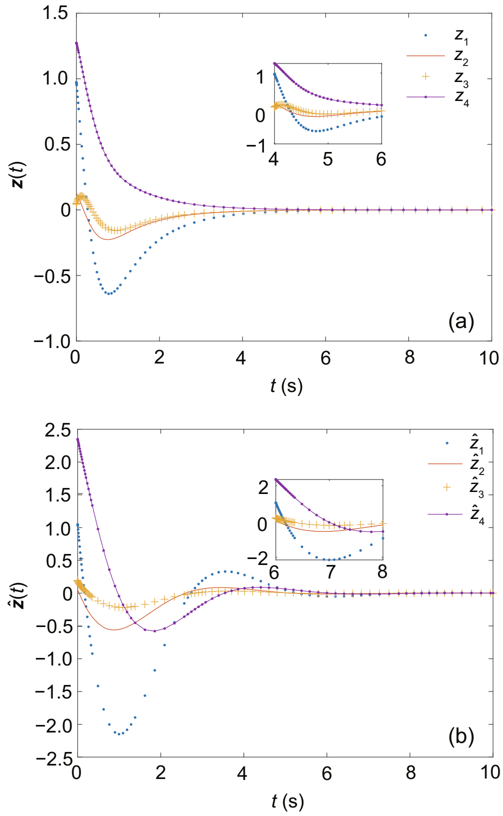

Fig. 1 System states $z(t)$ (a) and observer states $\hat{z}(t)$ (b)

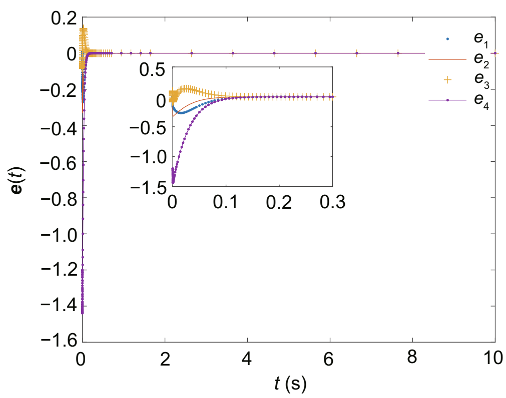

Fig. 2 Observer errors $e(t)$

Wu et al. (2020), Theorem 2 overcomes the problem of solving different orders $\alpha \in (0,1)$ and reduces the computational cost.

4.2 Example 2

DC motor, as a kind of actuator, is used widely. In this study, the DC motor model is used as the example shown in Fig. 3.

The variables and inputs of the system are shown in Tables 1 and 2, respectively.

In fact, the DC motor with delayed inputs actually exists (Léchappé et al., 2016), and thus $u(t-\tau)$ is introduced to denote the delay voltage of the source. Based on the mechanical and electrical laws (Li H and Yang, 2019; Lee, 2020), we can obtain the following system with $0<\alpha<1$:

$$\begin{cases} L\mathrm{D}^{\alpha} i(t) = u_{\mathrm{L}}(t), \\ J\mathrm{D}^{\alpha} w(t) = K_{\mathrm{t}} i(t) - b w(t), \\ u(t) + u(t-\tau) = u_{\mathrm{L}}(t) - i(t)R + K_{\mathrm{w}} w(t), \\ y(t) = w(t), \end{cases} \tag{31}$$

where $y(t)$ is the output, and J and b are as follows:

$$J = J_{\mathrm{m}} + \frac{J_{\mathrm{c}}}{n_0^2},\ b = b_{\mathrm{m}} + \frac{b_{\mathrm{c}}}{n_0^2}. \tag{32}$$

The parameters in Eqs. (31) and (32) are given in Table 3.

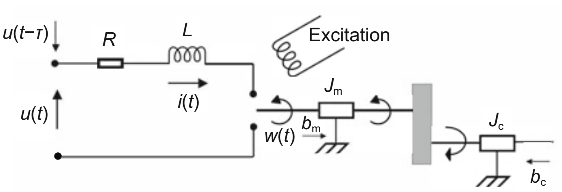

Fig. 3 Block diagram of the direct current (DC) motor

Table 1 Variables of the system

Variable	Symbol	Unit
Voltage of inductance	$u_{\mathrm{L}}(t)$	V
Current	$i(t)$	A
Speed of shaft	$w(t)$	r/min

Table 2 Inputs of the system

Input	Symbol	Unit
Voltage of source	$u(t)$	V
Delay voltage of source	$u(t-\tau)$	V

Suppose $i(t)$ and $u_{\mathrm{L}}(t)$ are measurable. Then $i(t)$, $w(t)$, and $u_{\mathrm{L}}(t)$ can be expressed as $x_1(t)$, $x_2(t)$, and $x_3(t)$, respectively. From Eq. (31), we have

$$\begin{cases} \begin{bmatrix} L & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{D}^{\alpha} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ K_{\mathrm{t}} & -b & 0 \\ R & -K_{\mathrm{w}} & -1 \end{bmatrix} \\ \quad \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t-\tau), \\ y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}. \end{cases} \tag{33}$$

Choose the parameter settings shown in Table 4. Hence, system (33) can be expressed as follows:

$$\begin{cases} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{D}^{\alpha} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2.5 & -2 & 0 \\ 4 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \\ \quad + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t-\tau), \\ y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}. \end{cases} \tag{34}$$

We choose $\boldsymbol{S} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$, which satisfies $\boldsymbol{ES} = 0$; different combinations of α and τ can be

Table 3 Parameters of the system

Parameter	Symbol	Unit
Input delay constant	τ	s
Inductance	L	mH
Armature winding resistance	R	Ω
Electromotive force constant	K_{t}	$\mathrm{kg \cdot cm^2/s^2}$
Torque constant	K_{w}	V/(rad·s)
Moment of inertia of rotor	J_{c}	$\mathrm{kg \cdot cm^2}$
Moment of inertia of load	J_{m}	$\mathrm{kg \cdot cm^2}$
Damping ratio of the motor	b_{m}	–
Damping ratio of the load	b_{c}	–
Gear ratio	n_0	–

Table 4 Parameter settings

Parameter	Value	Parameter	Value
L	2 mH	J_{m}	$1\ \mathrm{kg \cdot cm^2}$
R	4 Ω	b_{m}	1.5
K_{t}	$2.5\ \mathrm{kg \cdot cm^2/s^2}$	b_{c}	2
K_{w}	4 V/(rad·s)	n_0	2
J_{c}	$8\ \mathrm{kg \cdot cm^2}$		

used to verify the applicability of Theorem 2 in this study, as shown in Tables 5 and 6.

Table 5 Simulation results for $\boldsymbol{K}$

α	τ	$\boldsymbol{K}$
0.21	0.37	$[1.3729\quad 2.2329\quad -0.8073]$
	1.21	$[1.0350\quad 2.7757\quad -0.3342]$
	2.35	$[0.4279\quad 3.0690\quad -0.2465]$
0.64	0.37	$[4.1368\quad -2.6281\quad 0.4634]$
	1.21	$[2.5584\quad -1.3251\quad -0.1366]$
	2.35	$[2.0506\quad -0.8866\quad -0.3650]$
0.98	0.37	$[2.1249\quad -1.2776\quad 0.6772]$
	1.21	$[2.1000\quad -1.2840\quad 0.3802]$
	2.35	$[2.1017\quad -1.2961\quad 0.2318]$

Table 6 Simulation results for $\boldsymbol{L}$

α	τ	$\boldsymbol{L}$
0.21	0.37	$[12.7852\quad 11.7590\quad 5.2006]^{\mathrm{T}}$
	1.21	$[8.7194\quad 12.7752\quad 13.3779]^{\mathrm{T}}$
	2.35	$[5.0822\quad 10.5641\quad 16.8711]^{\mathrm{T}}$
0.64	0.37	$[-0.3521\quad -15.2569\quad -41.2119]^{\mathrm{T}}$
	1.21	$[-2.0668\quad -14.5905\quad -45.0673]^{\mathrm{T}}$
	2.35	$[-4.1104\quad -14.1993\quad -49.4751]^{\mathrm{T}}$
0.98	0.37	$[2.8932\quad -7.8294\quad -10.9265]^{\mathrm{T}}$
	1.21	$[2.1371\quad -8.2418\quad -14.6744]^{\mathrm{T}}$
	2.35	$[1.2399\quad -8.6320\quad -19.3545]^{\mathrm{T}}$

The states and observer errors of system (9) are given in Figs. 4–6, using three different combinations of α and τ. The system with the state-feedback controller can be stabilized, and observer errors of the system can fluctuate in a small range, which can illustrate that the controller design and observer design are effective.

Remark 5 According to the results of Example 2, it is shown that we can always find a set of $\boldsymbol{K}$ and $\boldsymbol{L}$ which is suitable to make the system stable, no matter what values of α and τ we take. The effectiveness of Theorem 2 proposed in this study can be verified.

5 Conclusions

The paper deals with observer-based controller design for fractional-order singular systems with $0 < \alpha < 1$ and input delay. Using the linear matrix inequality (LMI) technique, the necessary and

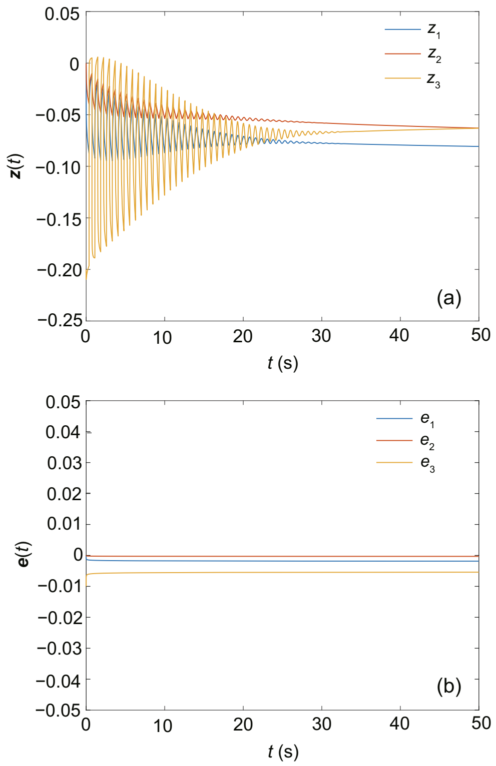

Fig. 4 States $z(t)$ (a) and errors $e(t)$ (b) with $\alpha = 0.21$ and $\tau = 0.37$

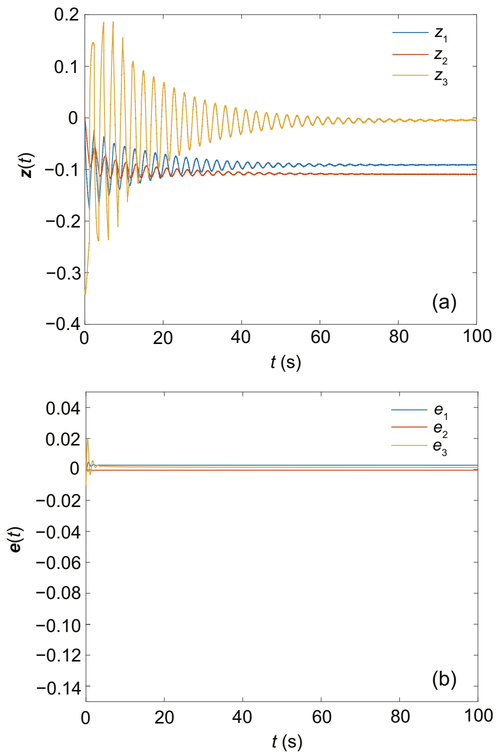

Fig. 5 States $z(t)$ (a) and errors $e(t)$ (b) with $\alpha = 0.64$ and $\tau = 1.21$

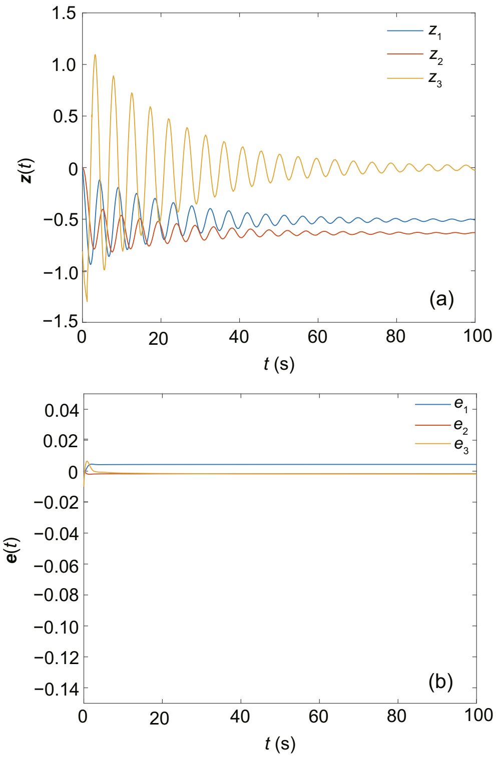

Fig. 6 States $z(t)$ (a) and errors $e(t)$ (b) with $\alpha = 0.98$ and $\tau = 2.35$

sufficient condition based on the nonstrict LMI is obtained. In the case of random error, the nonstrict LMI-based condition will cause trouble. When we improve the condition based on the strict LMI, the condition is easier to handle. Finally, the numerical example and the DC motor example are given to illustrate the effectiveness of the proposed condition.

In the future, observer-based robust control and observer-based H_∞ control for fractional-order singular systems will be studied.

Contributors

Bingxin LI designed the research. Bingxin LI and Xiangfei ZHAO processed the data. Bingxin LI drafted the paper. Xuefeng ZHANG helped organize the paper. Xuefeng ZHANG and Xin ZHAO revised and finalized the paper.

Compliance with ethics guidelines

Bingxin LI, Xiangfei ZHAO, Xuefeng ZHANG, and Xin ZHAO declare that they have no conflict of interest.

References

Aghayan ZS, Alfi A, Tenreiro Machado JA, 2021. Stability analysis of uncertain fractional-order neutral-type delay systems with actuator saturation. *Front Inform Technol Electron Eng*, 22(10):1402-1412. https://doi.org/10.1631/FITEE.2000438

Du FF, Lu JG, 2021. Explicit solutions and asymptotic behaviors of Caputo discrete fractional-order equations with variable coefficients. *Chaos Sol Fract*, 153:111490. https://doi.org/10.1016/J.CHAOS.2021.111490

Geng WT, Lin C, Chen B, 2020. Observer-based stabilizing control for fractional-order systems with input delay. *ISA Trans*, 100:103-108. https://doi.org/10.1016/j.isatra.2019.11.026

Guerrero JC, Chávez-Fuentes JR, Casavilca JE, et al., 2021. Stability analysis of discrete-time Markov jump linear singular systems with partially known transition probabilities. *Syst Contr Lett*, 158:105057. https://doi.org/10.1016/j.sysconle.2021.105057

Hua CC, Ning JH, Guan XP, 2021. Controller design for fractional-order interconnected systems with unmodeled dynamics. *Nonl Dynam*, 103(2):1599-1610. https://doi.org/10.1007/s11071-020-06177-2

Ibrir S, Bettayeb M, 2015. New sufficient conditions for observer-based control of fractional-order uncertain systems. *Automatica*, 59:216-223. https://doi.org/10.1016/j.automatica.2015.06.002

Ji YD, Qiu JQ, 2015. Stabilization of fractional-order singular uncertain systems. *ISA Trans*, 56:53-64. https://doi.org/10.1016/j.isatra.2014.11.016

Jiang LQ, Wang ST, Xie YL, et al., 2022. Fractional robust finite time control of four-wheel-steering mobile robots subject to serious time-varying perturbations. *Mech Mach Theory*, 169:104634. https://doi.org/10.1016/J.MECHMACHTHEORY.2021.104634

Lan YH, Zhou Y, 2013. Non-fragile observer-based robust control for a class of fractional-order nonlinear systems. *Syst Contr Lett*, 62(12):1143-1150. https://doi.org/10.1016/j.sysconle.2013.09.007

Lan YH, Huang HX, Zhou Y, 2012. Observer-based robust control of α ($1 \le \alpha < 2$) fractional-order uncertain systems: a linear matrix inequality approach. *IET Contr Theory Appl*, 6(2):229-234. https://doi.org/10.1049/iet-cta.2010.0484

Léchappé V, Rouquet S, González A, et al., 2016. Delay estimation and predictive control of uncertain systems with input delay: application to a DC motor. *IEEE Trans Ind Electron*, 63(9):5849-5857. https://doi.org/10.1109/TIE.2016.2527692

Lee DH, 2020. Balanced parallel instantaneous position control of PMDC motors with low-cost position sensors. *J Power Electron*, 20(3):834-843. https://doi.org/10.1007/s43236-020-00069-9

Li BX, Zhang XF, 2016. Observer-based robust control of fractional-order linear uncertain control systems. *IET Contr Theory Appl*, 10(14):1724-1731. https://doi.org/10.1049/iet-cta.2015.0453

Li C, Wang JC, Lu JG, et al., 2014. Observer-based stabilisation of a class of fractional order non-linear systems for $0 < \alpha < 2$ case. *IET Contr Theory Appl*, 8(13):1238-1246. https://doi.org/10.1049/iet-cta.2013.1082

Li H, Yang GH, 2019. Dynamic output feedback H_∞ control for fractional-order linear uncertain systems with actuator faults. *J Frankl Inst*, 356(8):4442-4466. https://doi.org/10.1016/j.jfranklin.2019.04.004

Li RC, Zhang XF, 2020. Adaptive sliding mode observer design for a class of T-S fuzzy descriptor fractional order systems. *IEEE Trans Fuzzy Syst*, 28(9):1951-1960. https://doi.org/10.1109/TFUZZ.2019.2928511

Li YC, Ma SP, 2021. Finite and infinite horizon indefinite linear quadratic optimal control for discrete-time singular Markov jump systems. *J Frankl Inst*, 358(17):8993-9022. https://doi.org/10.1016/j.jfranklin.2021.09.013

Lin C, Chen B, Shi P, et al., 2018. Necessary and sufficient conditions of observer-based stabilization for a class of fractional-order descriptor systems. *Syst Contr Lett*, 112:31-35. https://doi.org/10.1016/j.sysconle.2017.12.004

Lu JG, Chen GR, 2009. Robust stability and stabilization of fractional-order interval systems: an LMI approach. *IEEE Trans Autom Contr*, 54(6):1294-1299. https://doi.org/10.1109/TAC.2009.2013056

Lu JG, Chen YQ, 2010. Robust stability and stabilization of fractional-order interval systems with the fractional order α: the $0 < \alpha < 1$ case. *IEEE Trans Autom Contr*, 55(1):152-158. https://doi.org/10.1109/TAC.2009.2033738

Marir S, Chadli M, 2019. Robust admissibility and stabilization of uncertain singular fractional-order linear time-invariant systems. *IEEE/CAA J Autom Sin*, 6(3):685-692. https://doi.org/10.1109/JAS.2019.1911480

Marir S, Chadli M, Bouagada D, 2017. New admissibility conditions for singular linear continuous-time fractional-order systems. *J Frankl Inst*, 354(2):752-766. https://doi.org/10.1016/j.jfranklin.2016.10.022

Marir S, Chadli M, Basin MV, 2022a. Bounded real lemma for singular linear continuous-time fractional-order systems. *Automatica*, 135:109962. https://doi.org/10.1016/j.automatica.2021.109962

Marir S, Chadli M, Basin MV, 2022b. H_∞ static output feedback controller design for singular fractional-order systems. Proc European Control Conf, p.1-6. https://doi.org/10.23919/ECC55457.2022.9838112

Matignon D, 1998. Stability properties for generalized fractional differential systems. *ESAIM Proc*, 5:145-158. https://doi.org/10.1051/proc:1998004

N'Doye I, Darouach M, Zasadzinski M, et al., 2013. Robust stabilization of uncertain descriptor fractional-order systems. *Automatica*, 49(6):1907-1913. https://doi.org/10.1016/j.automatica.2013.02.066

Nguyen CM, Tan CP, Trinh H, 2021. State and delay reconstruction for nonlinear systems with input delays. *Appl Math Comput*, 390:125609. https://doi.org/10.1016/j.amc.2020.125609

Pu YF, Wang J, 2020. Fractional-order global optimal backpropagation machine trained by an improved fractional-order steepest descent method. *Front Inform Technol Electron Eng*, 21(6):809-833. https://doi.org/10.1631/FITEE.1900593

Sabatier J, Moze M, Farges C, 2010. LMI stability conditions for fractional order systems. *Comput Math Appl*, 59(5):1594-1609. https://doi.org/10.1016/j.camwa.2009.08.003

Saffarian M, Mohebbi A, 2021. Numerical solution of two and three dimensional time fractional damped nonlinear Klein–Gordon equation using ADI spectral element method. *Appl Math Comput*, 405:126182. https://doi.org/10.1016/j.amc.2021.126182

Si-Ammour A, Djennoune S, Bettayeb M, 2009. A sliding mode control for linear fractional systems with input and state delays. *Commun Nonl Sci Numer Simul*, 14(5):2310-2318. https://doi.org/10.1016/j.cnsns.2008.05.011

Stamova I, 2014. Global stability of impulsive fractional differential equations. *Appl Math Comput*, 237:605-612. https://doi.org/10.1016/j.amc.2014.03.067

Udhayakumar K, Rakkiyappan R, Cao JD, et al., 2020. Mittag–Leffler stability analysis of multiple equilibrium points in impulsive fractional-order quaternion-valued neural networks. *Front Inform Technol Electron Eng*, 21(2):234-246. https://doi.org/10.1631/FITEE.1900409

Wei YH, Wang JC, Liu TY, et al., 2019. Sufficient and necessary conditions for stabilizing singular fractional order systems with partially measurable state. *J Frankl Inst*, 356(4):1975-1990. https://doi.org/10.1016/j.jfranklin.2019.01.022

Wu Q, Song QK, Hu BX, et al., 2020. Robust stability of uncertain fractional order singular systems with neutral and time-varying delays. *Neurocomputing*, 401:145-152. https://doi.org/10.1016/j.neucom.2020.03.015

Xu SY, Lam J, 2006. Robust Control and Filtering of Singular Systems. Springer, Berlin, Germany.

Xu SY, van Dooren P, Stefan R, et al., 2002. Robust stability and stabilization for singular systems with state delay and parameter uncertainty. *IEEE Trans Autom Contr*, 47(7):1122-1128. https://doi.org/10.1109/TAC.2002.800651

Zhang L, Niu B, Zhao N, et al., 2021. Reachable set estimation of singular semi-Markov jump systems. *J Frankl Inst*, in press. https://doi.org/10.1016/j.jfranklin.2021.07.053

Zhang XF, Chen YQ, 2018. Admissibility and robust stabilization of continuous linear singular fractional order systems with the fractional order α: the $0 < \alpha < 1$ case. *ISA Trans*, 82:42-50. https://doi.org/10.1016/j.isatra.2017.03.008