ISSN 1009 - 3095 *Journal of Zhejiang University(SCIENCE)* V.2, No.2, P.157 - 160, Apr. - June, 2001
http://www.chinainfo.gov.cn/periodical; http://www.zju.edu.cn;
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EDGE SINGULARITY OF BONDED PIEZOELECTRIC MATERIALS WITH REPEATED EIGENVALUES*

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Received Feb. 18, 2000; revision accepted June 18, 2000

Abstract: In piezoelectric problems, the form of the general solution is dependent on the eigenvalues of the material. The singular stress field and electrical displacement field near the interface edge were deduced in this study. The results showed that the stress field and the electrical displacement field have the same singularity; and that the singularity depends not only on the mechanical properties and shape of the interface edge, but also on the piezoelectric properties of the composite material.

Key words: piezoelectric material, axisymmetric interface edge, singularity, electrical displacement
Document code: A **CLC number:** O344.3

INTRODUCTION

It is well known that stress singularity may arise at the interface edge of bonded dissimilar materials. Due to the intrinsic electroelastic coupling behavior, the stress singularity will lead to the singular behavior of the electric displacement. Ding et al. (1996) deduced the general solution for transversely isotropic piezoelectric material. There are many literatures on interface crack problems (e.g., Parton, 1976; Sosa et al., 1990; Suo et al., 1992; Beom et al., 1996), but few on general interface edge problems with arbitrary bonding angles, except for example, one on the study of the singularity for the axisymmetric interface edge under torsion (Liu et al., 1999). In this paper, the singular stress and electric displacement field near the axisymmetric interface edge of transversely isotropic piezoelectric material with repeated eigenevalues is first analyzed, and then the characteristic equation to determine the singularity is deduced.

BASIC EQUATIONS

The characteristic equation of a transversely isotropic piezoelectric material is

$$as^6 - bs^4 + cs^2 - d = 0. \tag{1}$$

where the coefficients a, b, c and d are material constants. According to the eigenvalue s_j, the general solution for a transversely isotropic piezoelectric material can be separated into the following three cases:

1. three distinct eigenvalues ($s_1^2 \neq s_2^2 \neq s_3^2$);
2. two repeated eigenvalues ($s_1^2 \neq s_2^2 = s_3^2$);
3. three repeated eigenvalues ($s_1^2 = s_2^2 = s_3^2$).

Based on the general solution and using a method similar to that described by Liu et al. (1999), we can obtain the following nontrivial displacement, stress and electrical displacement near the interface edge O' for the materials with repeated eigenvalues, expressed in the coordinate system (r, ϑ, θ) as shown in Fig. 1

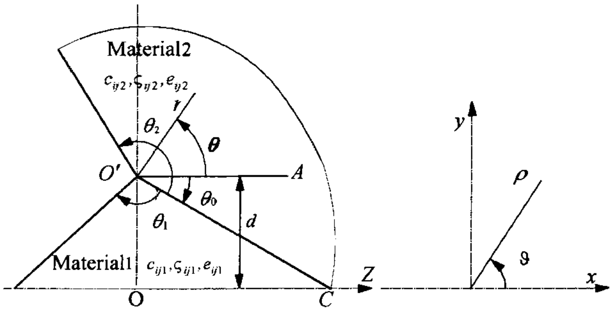

Fig. 1 Analysis model of the axisymmetric interface edge

* Project (19972060) supported by Natural Science Foundation of China.

1. The case of $s_1^2 \neq s_2^2 = s_3^2$

$$\left.\begin{aligned}
u_r &= r^{\lambda}[u_{k1}^1(\theta)A_k + u_{k2}^1(\theta)B_k],\\
v_\theta &= r^{\lambda}[v_{k1}^1(\theta)A_k + v_{k2}^1(\theta)B_k]\\
\phi &= r^{\lambda}[\phi_{k1}^1(\theta)A_k + \phi_{k2}^1(\theta)B_k],\\
\sigma_\theta &= r^{\lambda-1}[\sigma_{k1}^1(\theta)A_k + \sigma_{k2}^1(\theta)B_k]\\
\tau_{r\theta} &= r^{\lambda-1}[\tau_{k1}^1(\theta)A_k + \tau_{k2}^1(\theta)B_k],\\
D_\theta &= r^{\lambda-1}[D_{k1}^1(\theta)A_k + D_{k2}^1(\theta)B_k]
\end{aligned}\right\}\tag{2}$$

where $(i, j = 1, 2)$

$$\begin{aligned}
u_{ij}^1(\theta) &= [\alpha_{i1}\Theta_{5j}(\lambda,\theta,i)\cos\theta + s_i\Theta_{4j}\cdot(\lambda,\theta,i)\sin\theta]/s_i,\\
u_{3j}^1(\theta) &= [s_2\Theta_{4j}(\lambda-1,\theta,2)\sin\theta + \alpha_{21}\Theta_{5j}\cdot(\lambda-1,\theta,2)\cos\theta + \alpha_{41}P_{5j}(\lambda-1,\theta,2)]\cos\theta,\\
v_{ij}^1(\theta) &= [s_i\Theta_{4j}(\lambda,\theta,i)\cos\theta - \alpha_{i1}\Theta_{5j}(\lambda,\theta,i)\sin\theta]/s_i,\\
v_{3j}^1(\theta) &= s_2\Theta_{4j}(\lambda-1,\theta,2)\cos^2\theta - [\alpha_{21}\Theta_{5j}(\lambda-1,\theta,2)\cos\theta + \alpha_{41}P_{5j}(\lambda-1,\theta,2)]\sin\theta,\\
\phi_{ij}^1(\theta) &= \alpha_{i1}\Theta_{5j}(\lambda,\theta,i)/s_i,\\
\phi_{3j}^1(\theta) &= \alpha_{22}\Theta_{5j}(\lambda,\theta,2)\cos\theta + \alpha_{42}P_{5j}\cdot(\lambda-1,\theta,2),\\
\sigma_{ij}^1(\theta) &= (c_{11}-c_{12})\Theta_{1j}(\lambda,\theta,i)\cos^2\theta + (h_{1i}\cos^2\theta + h_{2i}\sin^2\theta)\Theta_{2j}\cdot(\lambda,\theta,i) - h_{3i}\Theta_{3j}(\lambda,\theta,i)\sin2\theta,\\
\sigma_{3j}^1(\theta) &= (c_{11}-c_{12})s_2\Theta_{1j}(\lambda-1,\theta,2)\cdot\cos^3\theta + (h_{13}\cos^2\theta + h_{23}\sin^2\theta)\Theta_{5j}\cdot(\lambda-1,\theta,2) + s_2\cos\theta(h_{12}\cos^2\theta +\\
&\quad h_{22}\sin^2\theta)\Theta_{2j}(\lambda-1,\theta,2) - [s_2h_{32}\Theta_{3j}(\lambda-1,\theta,2)\cos\theta + h_{33}\Theta_{4j}(\lambda-1,\theta,2)]\sin2\theta\\
\tau_{ij}^1(\theta) &= [(c_{11}-c_{12})\Theta_{1j}(\lambda,\theta,i) + (h_{1i}-h_{2i})\Theta_{2j}(\lambda,\theta,i)]\sin\theta\cos\theta + h_{3i}\Theta_{3j}(\lambda,\theta,i)\cos2\theta,\\
\tau_{3j}^1(\theta) &= [(c_{11}-c_{12})s_2\Theta_{1j}(\lambda-1,\theta,2) + s_2(h_{12}-h_{22})\Theta_{2j}(\lambda-1,\theta,2)]\cdot\sin\theta\cos^2\theta + [s_2h_{32}\Theta_{3j}(\lambda-1,\theta,2)\cdot\\
&\quad \cos\theta + h_{33}\Theta_{4j}(\lambda-1,\theta,2)]\cdot\cos2\theta + (h_{13}-h_{23})\Theta_{5j}(\lambda-1,\theta,2)\sin\theta\cos\theta\\
D_{ij}^1(\theta) &= h_{4i}\Theta_{3j}(\lambda,\theta,i)\cos\theta - h_{5i}\Theta_{2j}\cdot(\lambda,\theta,i)\sin\theta,\\
D_{3j}^1(\theta) &= s_2[h_{42}\Theta_{3j}(\lambda,\theta,2)\cos\theta - h_{52}\Theta_{2j}(\lambda,\theta,2)\sin\theta]\cos\theta + h_{43}\Theta_{4j}(\lambda,\theta,2)\cos\theta - h_{53}\Theta_{5j}(\lambda,\theta,2)\sin\theta
\end{aligned}\tag{3}$$

2. The case of $s_1^2 = s_2^2 = s_3^2$

$$\left.\begin{aligned}
u_r &= r^{\lambda}[u_{k1}^2(\theta)C_k + u_{k2}^2(\theta)D_k],\\
v_\theta &= r^{\lambda}[v_{k1}^2(\theta)C_k + v_{k2}^2(\theta)D_k]\\
\phi &= r^{\lambda}[\phi_{k1}^2(\theta)C_k + \phi_{k2}^2(\theta)D_k],\\
\sigma_\theta &= r^{\lambda-1}[\sigma_{k1}^2(\theta)C_k + \sigma_{k2}^2(\theta)D_k]\\
\tau_{r\theta} &= r^{\lambda-1}[\tau_{k1}^2(\theta)C_k + \tau_{k2}^2(\theta)D_k],\\
D_\theta &= r^{\lambda-1}[D_{k1}^2(\theta)C_k + D_{k2}^2(\theta)D_k]
\end{aligned}\right\}\tag{4}$$

where $(j = 1, 2)$

$$\begin{aligned}
u_{1j}^2(\theta) &= [\alpha_{11}\Theta_{5j}(\lambda,\theta,1)\cos\theta + s_1\Theta_{4j}\cdot(\lambda,\theta,1)\sin\theta]/s_1,\\
u_{2j}^2(\theta) &= [\alpha_{41}P_{5j}(\lambda-1,\theta,1) + \alpha_{11}\Theta_{5j}\cdot(\lambda-1,\theta,1)\cos\theta + s_1\Theta_{4j}(\lambda-1,\theta,1)\sin\theta]\cos\theta,\\
u_{3j}^2(\theta) &= \cos^2\theta[\alpha_{11}\Theta_{2j}(\lambda-1,\theta,1)\cos\theta + s_1\Theta_{3j}(\lambda-1,\theta,1)\sin\theta] + 2\alpha_{41}\Theta_{5j}(\lambda-1,\theta,1)\cos^2\theta + \alpha_{51}P_{5j}(\lambda-1,\theta,1)\cos\theta\\
v_{1j}^2(\theta) &= [s_1\Theta_{4j}(\lambda,\theta,1)\cos\theta - \alpha_{11}\Theta_{5j}\cdot(\lambda,\theta,1)\sin\theta]/s_1,\\
v_{2j}^2(\theta) &= s_1\Theta_{4j}(\lambda-1,\theta,1)\cos^2\theta - \alpha_{41}P_{5j}\cdot(\lambda-1,\theta,1)\sin\theta - \alpha_{11}\Theta_{5j}(\lambda-1,\theta,1)]\sin\theta\cos\theta,\\
v_{3j}^2(\theta) &= \cos^2\theta[s_1\Theta_{3j}(\lambda-1,\theta,1)\cos\theta - \alpha_{11}\Theta_{2j}(\lambda-,\theta,1)\sin\theta] - \alpha_{41}\Theta_{5j}\cdot(\lambda-1,\theta,1)\sin2\theta - \alpha_{51}P_{5j}(\lambda-1,\theta,1)\sin\theta\\
\phi_{1j}^2(\theta) &= \alpha_{12}\Theta_{5j}(\lambda,\theta,1)/s_1,\\
\phi_{2j}^2(\theta) &= \alpha_{42}\Theta_{5j}(\lambda-1,\theta,1) + \alpha_{12}\Theta_{5j}\cdot(\lambda-1,\theta,1),\cos\theta\\
\phi_{3j}^2(\theta) &= \alpha_{12}\Theta_{2j}(\lambda-1,\theta,1)\cos^2\theta + 2\alpha_{42}\Theta_{5j}(\lambda-1,\theta,1)\cos\theta + \alpha_{52}P_{5j}(\lambda-1,\theta,1),\\
\sigma_{1j}^2(\theta) &= (c_{11}-c_{12})\Theta_{1j}(\lambda,\theta,1)\cos^2\theta + (n_1\cos^2\theta + n_4\sin^2\theta)\Theta_{2j}(\lambda,\theta,1) - n_7\sin2\theta\Theta_{3j}(\lambda,\theta,1)/s_1\\
\sigma_{2j}^2(\theta) &= (n_2\cos^2\theta + n_5\sin^2\theta)\Theta_{5j}(\lambda-1,\theta,1) - n_8\Theta_{4j}(\lambda-1,\theta,1)\sin2\theta - n_7\Theta_{3j}(\lambda-1,\theta,1)\cos\theta\sin2\theta +\\
&\quad (c_{11}-c_{12})s_1\Theta_{1j}(\lambda-1,\theta,1)\cdot
\end{aligned}$$

$\cos^3\theta + s_1(n_1\cos^2\theta + n_4\sin^2\theta)\Theta_{2j}\cdot(\lambda-1,\theta,1)\cos\theta$

$$\sigma_{3j}^2(\theta) = 2[(n_2\cos^2\theta + n_5\sin^2\theta)\Theta_{2j}(\lambda-1,\theta,1) - n_8\Theta_{3j}(\lambda-1,\theta,2)\cdot\sin2\theta]\cos\theta - n_9\Theta_{4j}(\lambda-1,\theta,1)\cdot\sin2\theta + (n_3\cos^2\theta + n_6\sin^2\theta)\cdot\Theta_{5j}(\lambda-1,\theta,1) + (c_{11}-c_{12})s_1\Theta_{6j}(\lambda-1,\theta,1)\cos^4\theta - n_7\Theta_{7j}(\lambda-1,\theta,1)\cdot\cos^2\theta\sin2\theta + (n_1\cos^2\theta + n_4\sin^2\theta)\cdot s_1\Theta_{8j}(\lambda-1,\theta,1)\cos^2\theta$$

$$\tau_{1j}^2(\theta) = \sin2\theta[(c_{11}-c_{12})\Theta_{1j}(\lambda,\theta,1) + (n_1-n_4)\Theta_{2j}(\lambda,\theta,1)]/2 + n_7\cos2\theta\Theta_{3j}(\lambda,\theta,1)/s_1,$$

$$\tau_{2j}^2(\theta) = (n_2-n_5)\Theta_{5j}(\lambda-1,\theta,1)\sin\theta\cdot\cos\theta + n_8\Theta_{4j}(\lambda-1,\theta,1)\cos2\theta + n_7\Theta_{3j}(\lambda-1,\theta,1)\cos\theta\cos2\theta + (c_{11}-c_{12})s_1\Theta_{1j}(\lambda-1,\theta,1)\cdot\sin\theta\cos^2\theta + s_1(n_1-n_4)\Theta_{2j}\cdot(\lambda-1,\theta,1)\sin\theta\cos^2\theta,$$

$$\tau_{3j}^2(\theta) = [(n_2-n_5)\Theta_{2j}(\lambda-1,\theta,1)\sin2\theta + 2n_8\Theta_{3j}(\lambda-1,\theta,1)\cos2\theta]\cos\theta + n_9\Theta_{4j}(\lambda-1,\theta,1)\cos2\theta + (n_3-n_6)\Theta_{5j}(\lambda-1,\theta,1)\sin\theta\cos\theta + (c_{11}-c_{12})s_1\Theta_{6j}(\lambda-1,\theta,1)\cdot\sin\theta\cos^3\theta + n_7\Theta_{7j}(\lambda-1,\theta,1)\cdot\cos^2\theta\cos2\theta' + (n_1-n_4)s_1\Theta_{8j}\cdot(\lambda-1,\theta,1)\sin\theta\cos^3\theta$$

$$D_{1j}^2(\theta) = [n_{10}\Theta_{3j}(\lambda,\theta,1)\cos\theta - s_1 n_{13}\Theta_{2j}(\lambda,\theta,1)\sin\theta]/s_1,$$

$$D_{2j}^2(\theta) = n_{10}\Theta_{3j}(\lambda-1,\theta,1)\cos^2\theta + n_{11}\Theta_{4j}(\lambda-1,\theta,1)\cos\theta - n_{14}\Theta_{5j}\cdot(\lambda-1,\theta,1)\sin\theta - n_{13}s_1\Theta_{2j}\cdot(\lambda-1,\theta,1)\cos\theta\sin\theta$$

$$D_{3j}^2(\theta) = n_{10}\Theta_{7j}(\lambda-1,\theta,1)\cos^3\theta + 2n_{11}\cdot(\lambda-1,\theta,1)\cos^2\theta + n_{12}\Theta_{4j}(\lambda-1,\theta,1)\cos\theta - n_{13}s_1\Theta_{8j}(\lambda-1,\theta,1)\cdot\cos^2\theta\sin\theta - n_{14}\Theta_{2j}(\lambda-1,\theta,1)\cdot\sin2\theta - n_{15}\Theta_{5j}(\lambda-1,\theta,1)\sin\theta \tag{5}$$

The repeated subscripts $k(k=1,2,3)$ in Eqs. (2) and (4) mean taking the summation.

SINGULARITY AT THE INTERFACE EDGE

In the coordinate system (r,ϑ,θ) the continuity condition on the interface and the free boundary condition can be represented as

$$\begin{aligned} u_{r1}(r,-\theta_0) &= u_{r2}(r,-\theta_0),\\ v_{\theta1}(r,-\theta_0) &= v_{\theta2}(r,-\theta_0),\\ \sigma_{\theta1}(r,-\theta_0) &= \sigma_{\theta2}(r,-\theta_0),\\ \tau_{r\theta1}(r,-\theta_0) &= \tau_{r\theta2}(r,-\theta_0),\\ D_{\theta1}(r,-\theta_0) &= D_{\theta2}(r,-\theta_0),\\ \phi_1(r,-\theta_0) &= \phi_2(r,-\theta_0). \end{aligned} \tag{6}$$

$$\begin{aligned} \sigma_{\theta1}(r,-(\theta_0+\theta_1)) &= 0,\\ \tau_{r\theta1}(r,-(\theta_0+\theta_1)) &= 0,\\ D_{\theta1}(r,-(\theta_0+\theta_1)) &= 0,\\ \sigma_{\theta2}(r,-(\theta_2-\theta_0)) &= 0,\\ \tau_{r\theta2}(r,-(\theta_2-\theta_0)) &= 0,\\ D_{\theta2}(r,-(\theta_2-\theta_0)) &= 0. \end{aligned} \tag{7}$$

where the subscripts 1 and 2 denote the material 1 and 2, respectively. Substituting (2) and (4) into (6) and (7), a system of twelve homogeneous linear algebraic equations can be obtained for the twelve unknown coefficients A_j, B_j, C_j, $D_j (j=1,2,3)$. The nontrivial condition leads to the following characteristic equation for λ

$$\det\begin{bmatrix} X_1 & Y_1 & 0 \\ X_2 & 0 & Y_2 \end{bmatrix}^{\mathrm{T}} = 0 \tag{8}$$

where $(k=1,2)$

$$X_k = (-1)^{k+1}\begin{bmatrix} u_{11}^k(-\theta_0) & u_{12}^k(-\theta_0) & u_{21}^k(-\theta_0) & u_{22}^k(-\theta_0) & u_{31}^k(-\theta_0) & u_{32}^k(-\theta_0)\\ v_{11}^k(-\theta_0) & v_{12}^k(-\theta_0) & v_{21}^k(-\theta_0) & v_{22}^k(-\theta_0) & v_{31}^k(-\theta_0) & v_{32}^k(-\theta_0)\\ \phi_{11}^k(-\theta_0) & \phi_{12}^k(-\theta_0) & \phi_{21}^k(-\theta_0) & \phi_{22}^k(-\theta_0) & \phi_{31}^k(-\theta_0) & \phi_{32}^k(-\theta_0)\\ \sigma_{11}^k(-\theta_0) & \sigma_{12}^k(-\theta_0) & \sigma_{21}^k(-\theta_0) & \sigma_{22}^k(-\theta_0) & \sigma_{31}^k(-\theta_0) & \sigma_{32}^k(-\theta_0)\\ \tau_{11}^k(-\theta_0) & \tau_{12}^k(-\theta_0) & \tau_{21}^k(-\theta_0) & \tau_{22}^k(-\theta_0) & \tau_{31}^k(-\theta_0) & \tau_{32}^k(-\theta_0)\\ D_{11}^k(-\theta_0) & D_{12}^k(-\theta_0) & D_{21}^k(-\theta_0) & D_{22}^k(-\theta_0) & D_{31}^k(-\theta_0) & D_{32}^k(-\theta_0) \end{bmatrix}$$

$$Y_k = (-1)^{k+1}\begin{bmatrix} \sigma_{11}^k(\varphi_k) & \sigma_{12}^k(\varphi_k) & \sigma_{21}^k(\varphi_k) & \sigma_{22}^k(\varphi_k) & \sigma_{31}^k(\varphi_k) & \sigma_{32}^k(\varphi_k) \\ \tau_{11}^k(\varphi_k) & \tau_{12}^k(\varphi_k) & \tau_{21}^k(\varphi_k) & \tau_{22}^k(\varphi_k) & \tau_{31}^k(\varphi_k) & \tau_{32}^k(\varphi_k) \\ D_{11}^k(\varphi_k) & D_{12}^k(\varphi_k) & D_{21}^k(\varphi_k) & D_{22}^k(\varphi_k) & D_{31}^k(\varphi_k) & D_{32}^k(\varphi_k) \end{bmatrix} \quad (9)$$

in which, $\varphi_1 \doteq -\theta_1 - \theta_0$, $\varphi_2 = -\theta_2 + \theta_0$. The determination of admissible values of λ from the characteristic equation (8) is usually done using a numerical method.

CONCLUSIONS

The singularity characteristic equation as well as singular stress and electric displacement fields near the axisymmetric interface edge of the bonded dissimilar transversely isotropic piezoelectric materials with arbitrary bonding angles and the interface angles are deduced. The results showed that the electric displacement field and the stress field have the same singularity, and the singular order is not only dependent on the interface edge shape and the mechanical properties, but also on the piezoelectric property of the dissimilar piezoelectric materials.

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