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FINITE VOLUME METHOD BASED ON THE CROUZEIX-RAVIART ELEMENT FOR THE STOKES EQUATION*

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Abstract: The author provides a new discretization method-the finite volume method(FVM). For the Stokes equation the velocity space is approximated by the nonconforming linear element based on the dual partition and the pressure by the piecewise constant based on the primal triangulation . Under the suitable smoothness of the solution, the optimal convergence rate $O(h)$ is obtained, where h denotes the parameter of the space discretization.

Key words: Stokes equation, finite volume method, Crouzeix-Raviart element
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INTRODUCTION

We consider the Stokes equation:

$$-\Delta\bar{u} + \nabla p = \bar{f} \quad \text{in } \Omega \tag{1}$$

$$\text{div}\bar{u} = 0 \quad \text{in } \Omega \tag{2}$$

$$\bar{u} = 0 \quad \text{on } \partial\Omega$$

where $\bar{u} = (u_1, u_2)^T$ and p denote the velocity and the pressure of the fluid respectively. $\bar{f}$ denotes the external body forcing density. Ω is an open bounded and convex polygonal domain. $\partial\Omega$ denotes the boundary of Ω.

The discrete form for the conservative elliptic equation was recently obtained by FVM. The basic idea is that after the original continuous problem is integrated in a box, the integral equation is then discretized. Vanselow Reiner (1998) considered the equivalence between the FVM and the nonconforming Galerkin finite element method(FEM) for the simple Poisson equation. Generally speaking, FVM can be viewed as the Galerkin FEM or Petrov-Galerkin FEM. For example in (Li et al., 1993), FVM based on the box which is obtained by the inner vertex of the triangulation, is regarded as the Galerkin FEM combined with the numerical integration. In (Chatzipanteliols, 1993), the FVM based on the box obtained by the inner edge of the triangulation, is regarded as the nonconforming Galerkin FEM combined with suitable numerical integration. In (Mishev et al., 1999), FVM based on the inner vertex, is regarded as the Petrov-Galerkin FEM. For the Stokes equation, FVM can also be used for discretization, especially for the rectangular domain. In (Chou, 1998), the velocity is approximated by the nonconforming rotated bilinear element, and the pressure by the piecewise constant. In (Chou, 1997), the pressure is estimated by the nonconforming bilinear element.

In this paper based on the idea in (Chatzipanteliols, 1993), we first construct a box based on the inner edge, and the velocity is approximated by the nonconforming linear element and the pressure by the piecewise constant. Under the suitable smoothness of the solution, the optimal convergence $O(h)$ is obtained, where h is the parameter of space discretization(In this paper, c independent of h, denotes a general constant but may be different in different places).

DOMAIN PARTITION AND NOTATION

Denote $T_h = \{K\}$ the regular partition of triangulation and h_K is the diameter of K. $h = \max_{K\in T_h} h_K$. Denote E_h^{in} the set of inner edges of triangulars. $E_h(K)$ is the set of edges of K. $E_h = \bigcup_{K\in T_h} E_h(K)$. For any $K \in T_h$, we choose an

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inner point Z_K in K. For any $e \in E_h^{in}$, a box b_e is constructed as follows: suppose e is the common edge of two triangulars K_1, K_2 and denote Z_1, Z_2 the inner points of K_1, K_2, respectively and A, B denote the end points of e. We connect the four points A, Z_1, B, Z_2 and get a box b_e. $\{b_e\}_{e \in E_h^{in}}$ is called the dual partition of T_h. m_e is the midpoint of e. For the given domain $K \subset R^2$, $L^2(K)$ denotes the set of functions which is square integrable in K. $(\cdot,\cdot)_K$ denotes the scale inner product in $L^2(K)$. $|\cdot|_{s,K}$, $\|\cdot\|_{s,K}$ denote the seminorm and norm in Sobleve space $H^s(K)$, $s \in N$. We introduce norms $\|\cdot\|_{s,h}$ andseminorms $|\cdot|_{s,h}$ depending on the mesh.

$\|v\|_{s,h} = (\sum_{K \in T_h} \|v\|_{s,K}^2)^{1/2}$. The seminorm $|\cdot|_{s,h}$ is defined analogously. $|\cdot|$ denotes the domain area in R^2 or the length of segment. For the vector function $\bar{v} = (v_1, v_2)^{\mathrm{T}}$. we can also define $|\bar{v}|_{s,K}$, $\|v\|_{s,K}(\cdot,\cdot)_K$ analogously.

We introduce the finite element spaces V_h and M_h. For any $K \in T_h$, we define finite element (K, Σ_K, P_K) where Σ_K denotes the freedom degree of the function value at the midpoint in the three edges of K and P_K is the set of the linear polynomial in K. In fact the element is Crouzeix-Raviart element which is not continuous in Ω. We define the following spaces: S_h is the set of functions which are linear in every triangular and continuous at the midpoint of every inner edge. S_h^0 is the function which belongs to S_h and vanishes at the midpoint in every boundary side. $V_h = S_h^0 \times S_h^0$. M_h is the set of functions which are constant in every K and belong to M. where M, which is defined in the next section, is the pressure space.

DISCRETE FORM

Integrating (1) in b_e and using Green formula, we get

$$\int_{\partial b_e} -\frac{\partial \bar{u}}{\partial n} + p \cdot \bar{n}\,ds = \int_{b_e} \bar{f}\,ds \quad \forall e \in E_h^{in}, \tag{3}$$

Integrating (2) in K and using Green formula, we get

$$\int_{\partial K} \bar{u} \cdot \bar{n}\,ds = 0 \quad \forall K \in T_h. \tag{4}$$

The discrete form: find $(\bar{u}_h, p_h) \in V_h \times M_h$, such that

$$\int_{\partial b_e} -\frac{\partial \bar{u}_h}{\partial n} + p_h \cdot \bar{n}\,ds = \int_{b_e} \bar{f}\,ds \quad \forall e \in E_h^{in}, \tag{5}$$

$$\int_{\partial K} \bar{u}_h \cdot \bar{n}\,ds = 0 \quad \forall K \in T_h. \tag{6}$$

The above problem is equivalent to the following problem: find $(\bar{u}_h, p_h) \in V_h \times M_h$, such that

$$a_h(\bar{u}_h, \bar{v}) + b_h(\bar{v}, p_h) = F_h(\bar{v}) \quad \forall \bar{v} \in V_h \tag{7}$$

$$c_h(\bar{u}_h, q_h) = 0 \quad \forall q_h \in M_h, \tag{8}$$

where

$$a_h(\bar{u}_h, \bar{v}) = \sum_{e \in E_h^{in}} \bar{v}(m_e) \int_{\partial b_e} -\frac{\partial \bar{u}_h}{\partial n} ds \tag{9}$$

$$b_h(\bar{v}, p_h) = \sum_{e \in E_h^{in}} \bar{v}(m_e) \int_{\partial b_e} p_h \cdot \bar{n}\,ds \tag{10}$$

$$c_h(\bar{u}_h, q_h) = \sum_{K \in T_h} q_h(Z_K) \int_{\partial K} u_h \cdot \bar{n}\,ds \tag{11}$$

$$F_h(\bar{v}) = \sum_{e \in E_h^{in}} \bar{v}(m_e) \int_{b_e} \bar{f}\,dx \tag{12}$$

The standard variational form of (1), (2): find $(\bar{u}, p) \in V \times M$, such that

$$a(\bar{u}, \bar{v}) + b(\bar{v}, p) = F(\bar{v}) \quad \forall \bar{v} \in V \tag{13}$$

$$b(\bar{u}, q) = 0 \quad \forall q \in M \tag{14}$$

where

$$V = (H_0^1(\Omega))^2, M = \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}$$

$$a(\bar{u}, \bar{v}) = \sum_{K \in T_h} \int_K \nabla \bar{u} \cdot \nabla \bar{v}\,dx \tag{15}$$

$$b(\bar{v}, p) = -\sum_{K \in T_h} \int_K p \cdot (\nabla \cdot \bar{v})\,dx \tag{16}$$

$$F(\bar{v}) = \sum_{K \in T_h} \int_K \bar{f} \cdot \bar{v}\,dx. \tag{17}$$

Obviously, the bilinear forms a and b can also be defined in $(V + (S_h)^2 \times (V + (S_h)^2)$ and $(V + (S_h)^2) \times M$ respectively. Set $|\bar{v}|_{1,h}^2 = a(\bar{v}, \bar{v})$.

EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE DISCRETE FORM

In order to prove the existence and uniqueness of the discrete form, we need the following lemma.

Lemma 1. For any $(\bar{v}, q) \in V_h \times M_h$, $b_h(\bar{v}, q) = -c_h(\bar{v}, q)$.

Proof: For any $(\bar{v}, q) \in V_h \times M_h$,

$$b_h(\bar{v}, q) = \sum_{e \in E_h^{in}} \bar{v}(m_e) \int_{\partial b_e} q \cdot \bar{n} ds =$$

$$\sum_{K \in T_h} \sum_{e \in E_h(K)} \bar{v}(m_e) \int_{\partial b_e \cap K} q \cdot \bar{n} ds =$$

$$\sum_{K \in T_h} q(Z_K) \sum_{e \in E_h(K)} \int_{\partial b_e \cap K} \bar{v}(m_e) \cdot \bar{n} ds =$$

$$\sum_{K \in T_h} q(Z_K) \sum_{e \in E_h(K)} \int_{e} -\bar{v}(m_e) \cdot \bar{n} ds =$$

$$-\sum_{K \in T_h} q(Z_K) \int_{\partial K} \bar{v} \cdot \bar{n} ds = -c_h(\bar{v}, q). \text{Q.E.D.}$$

Lemma 2. $a_h(\bar{u}, \bar{v}) = a(\bar{u}, \bar{v})$, for any $(\bar{u}, \bar{v}) \in S_h^0 \times S_h^0$.

The proof is similar to that of Chatzipanteliols's(1993) and is omitted.

Lemma 3. $b_h(\bar{v}, q) = b(\bar{v}, q)$, for any $(\bar{v}, q) \in V_h \times M_h$.

Proof:

$$| b_h(\bar{v}, q) - b(\bar{v}, q) | =$$

$$\left| \sum_{e \in E_h^{in}} \bar{v}(m_e) \int_{\partial b_e} q \cdot \bar{n} ds + \sum_{K \in T_h} \int_K q \cdot (\nabla \cdot \bar{v}) dx \right| =$$

$$\left| \sum_{K \in T_h} \sum_{e \in E_h(K)} \bar{v}(m_e) \int_{\partial b_e \cap K} q \cdot \bar{n} ds + \sum_{K \in T_h} \int_K q \cdot (\nabla \cdot \bar{v}) dx \right| =$$

$$\left| \sum_{K \in T_h} q(Z_K) \left(\sum_{e \in E_h(K)} \int_{\partial b_e \cap K} \bar{v}(m_e) \cdot \bar{n} ds + \int_{\partial K} \bar{v} \cdot \bar{n} ds \right) \right|,$$

from the proof of lemma 1,

$$\sum_{e \in E_h(K)} \int_{\partial b_e \cap K} \bar{v}(m_e) \cdot \bar{n} ds + \int_{\partial K} \bar{v} \cdot \bar{n} ds = 0.$$

This completes the proof.

Theorem 1. The discrete form (7), (8) has one and only one solution.

Proof: From the coecivity of a and the fact that b satisfies the L-B-B condition and lemma 1.2.3. a_h satisfies coecivity in $S_h \times S_h$ and b_h satisfies the discrete L-B-B condition. So, the problem(7) - (8) has a solution. In the finite dimensional linear system, the existence is equivalent to the uniqueness. This proves the theorem.

CONVERGENCE

Denote $I_h^1 \bar{u}$ and $I_h^0 p$ the standard interpolants of $\bar{u}$ and p in V_h and M_h respectively.

We can prove the following lemma.

Lemma 4.

$| a(I_h^1 \bar{u} - \bar{u}, \bar{v}_h) | \leqslant ch | \bar{u} |_{2,\Omega} | \bar{v}_h |_{1,h}$

$\forall \bar{u} \in (H^2(\Omega))^2$, $\bar{v}_h \in V_h$

$| b(\bar{v}_h, I_h^0 p - p) | \leqslant ch | p |_{1,\Omega} | \bar{v}_h |_{1,h}$

$\forall p \in H^1(\Omega)$, $\bar{v}_h \in V_h$

$| b(I_h^1 \bar{u} - \bar{u}, q_h) | \leqslant ch | q_h |_{0,\Omega} | \bar{u} |_{2,\Omega}$

$\forall \bar{u} \in (H^2(\Omega))^2$, $q_h \in M_h$

$| F(\bar{v}_h) - F_h(\bar{v}_h) | \leqslant ch | \bar{v}_h |_{1,h} \| \bar{f} \|_{0,\Omega}$

$\forall \bar{f} \in (L^2(\Omega))^2$, $\bar{v}_h \in V_h$

Proof:

$$| a(I_h^1 \bar{u} - \bar{u}, \bar{v}_h) | \leqslant$$

$$\sum_K \left| \int_K \nabla (I_h^1 \bar{u} - \bar{u}) \nabla \bar{v}_h dx \right| \leqslant$$

$$\sum_K | I_h^1 \bar{u} - \bar{u} |_{1,K} | \bar{v}_h |_{1,K} \leqslant$$

$$ch \sum_K | \bar{u} |_{2,K} | \bar{v}_h |_{1,K} \leqslant$$

$$ch | \bar{u} |_{2,\Omega} | \bar{v}_h |_{1,h},$$

$$b(\bar{v}_h, I_h^0 p - p) | =$$

$$\left| -\sum_K \int_K (I_h^0 p - p) \nabla \cdot \bar{v}_h dx \right| \leqslant$$

$$\sum_K \| I_h^0 p - p \|_{0,K} | \bar{v}_h |_{1,K} \leqslant$$

$$ch \sum_K | p |_{1,K} | \bar{v}_h |_{1,K} \leqslant$$

$$ch | p |_{1,\Omega} | \bar{v}_h |_{1,h},$$

$$| b(I_h^1 \bar{u} - \bar{u}, q_h) | =$$

$$\left| -\sum_K \int_K q_h (\nabla \cdot (I_h^1 \bar{u} - \bar{u})) dx \right| \leqslant$$

$$\sum_K \| q_h \|_{0,K} | I_h^1 \bar{u} - \bar{u} |_{1,K} \leqslant$$

$$ch \sum_k \| q_h \|_{0,K} | \bar{u} |_{2,K} \leqslant$$

$$ch \| q_h \|_{0,\Omega} | \bar{u} |_{2,\Omega}$$

$$| F(\bar{v}_h) - F_h(\bar{v}_h) | =$$

$$\left| \sum_e \bar{v}_h(m_e) \int_{b_e} \bar{f} dx - \sum_K \int_K \bar{f} \cdot \bar{v}_h dx \right| =$$

$$\left| \sum_K \left(\sum_{e \in E_h^{in}} \bar{v}_h(m_e) \int_{b_e \cap K} \bar{f} dx - \int_K \bar{f} \cdot \bar{v}_h dx \right) \right| =$$

$$\left|\sum_K \sum_{e\in E_h(K)} \int_{b_e\cap K} (\bar v_h(m_e) - \bar v_h)\cdot \bar f \mathrm{d}x\right| \leqslant$$
$$\left|\sum_K \sum_{e\in E_h(K)} \| \bar v_h(m_e) - \bar v_h \|_{0,b_e\cap K} \| \bar f \|_{0,b_e\cap K}\right|,$$

by the standard interpolant result, above mentioned formula

$$\leqslant \sum_K \sum_{e\in E_h(K)} ch_K \, | \bar v_h |_{1,b_e\cap K} \| \bar f \|_{0,b_e\cap K} \leqslant$$
$$ch \| \bar v_h \|_{1,h} \| \bar f \|_{0,\Omega} \cdot \text{Q.E.D.}$$

Using the above lemma, we can prove the following main theorem.

Theorem 2. If the solution of (13), (14) $(\bar u, p)\in (H^2(\Omega))^2\times H^1(\Omega)$ and $(\bar u_h, p_h)$ is the solution of (7) − (8), the following estimation holds:

$$| \bar u - \bar u_h |_{1,h} + \| p - p_h \|_{0,\Omega} \leqslant$$
$$ch(\| \bar u \|_{2,\Omega} + \| p \|_{1,\Omega} + \| \bar f \|_{0,\Omega}),$$

where c is not dependent on h.

Proof: Denote $\bar e_h = I_h^1 \bar u - \bar u_h$, $\lambda_h = I_h^0 p - p_h$ then $(\bar e_h, \lambda_h)\in V_h\times M_h$ satisfies

$$a_h(\bar e_h, \bar v_h) + b_h(\bar v_h, \lambda_h) =$$
$$a_h(I_h^1\bar u, \bar v_h) + b_h(\bar v_h, I_h^0 p) - F_h(\bar v_h) =$$
$$a_h(I_h^1\bar u, \bar v_h) + b_h(\bar v, I_h^0 p) - F_h(\bar v_h) +$$
$$F(\bar v_h) - F(\bar v_h), \forall \bar v_h \in V_h. \tag{18}$$

Multiply both sides of (1) by $\bar v_h$ and integrate the formula in K and sum with K, to obtain

$$F(\bar v_h) = a(\bar u, \bar v_h) + b_h(\bar v_h, p) +$$
$$\sum_K \Big(\int_{\partial K} p(\bar n\cdot \bar v_h)\mathrm{d}s - \int_{\partial K} \frac{\partial \bar u}{\partial n}\cdot \bar v_h \mathrm{d}s\Big).$$

Substitute this formula into (18) and use lemma 1 and lemma 2, to get

$$a_h(\bar e_h, \bar v_h) + b_h(\bar v_h, \lambda_h) =$$
$$a_h(I_h^1\bar u - \bar u, \bar v_h) + b(\bar v_h, I_h^0 p - p) + F(\bar v_h) -$$
$$F_h(\bar v_h) + \sum_K\Big(\int_K \frac{\partial \bar u}{\partial n}\cdot \bar v_h\mathrm{d}s - \int_{\partial K} p(\bar n\cdot \bar v_h)\mathrm{d}s\Big) \equiv$$
$$R(\bar v_h), \forall \bar v_h \in V_h. \tag{19}$$
$$b_h(\bar e_h, q_h) = b_h(I_h^1\bar u, q_h) - b_h(\bar u_h, q_h) =$$
$$b(I_h^1\bar u - \bar u, q_h), \forall q_h \in M_h. \tag{20}$$

We estimate two terms $(1) \equiv \sum_K \int_{\partial K} p(\bar n\cdot \bar v_h)\mathrm{d}s$ and $(2) \equiv \sum_K \int_{\partial K} -\frac{\partial \bar u}{\partial n}\cdot \bar v_h\mathrm{d}s$. Denote $I_{KP}^0 p$ the piecewise constant interpolant of p in ∂K, i.e,

$$I_{KP}^0 p \,|_e = p(m_e), \ \forall e\in E_h(K). \tag{21}$$
$$|(1)| \leqslant \left|\sum_K \int_{\partial K} (p - I_{KP}^0 p)(\bar n\cdot \bar v_h)\mathrm{d}s\right| +$$
$$\left|\sum_K \int_{\partial K} I_{KP}^0 p(\bar n\cdot \bar v_h)\mathrm{d}s\right| =$$
$$\left|\sum_K \int_{\partial K} (p - I_{KP}^0 p)(\bar n\cdot \bar v_h)\mathrm{d}s\right| \leqslant$$
$$\sum_K \| p - I_{KP}^0 p \|_{-1/2,\partial K} \| \bar v_h \|_{1/2,\partial K} \leqslant$$
$$\sum_K ch \, | p |_{1/2,\partial K} \| \bar v_h \|_{1/2,\partial K} \leqslant$$
$$\sum_K ch \, | p |_{1,K} \| \bar v_h \|_{1,K} \leqslant$$
$$ch \| p \|_{1,\Omega} \| \bar v_h \|_{1,\Omega} \leqslant ch \| p \|_{1,\Omega} \| \bar v_h \|_{1,h}, \tag{22}$$

where we use $\left|\sum_K \int_{\partial K} I_{KP}^0 p(\bar n\cdot \bar v_h)\mathrm{d}s\right| = 0$ and trace theorem.

$$|(2)| \leqslant \left|\sum_K \int_{\partial K} \frac{\partial \bar u}{\partial n}\cdot \bar v_h\mathrm{d}s\right| \leqslant$$
$$\left|\sum_K \int_{\partial K} \Big(\frac{\partial \bar u}{\partial n} - I_K^0\Big(\frac{\partial \bar u}{\partial n}\Big)\Big)\cdot \bar v_h\mathrm{d}s\right| +$$
$$\left|\sum_K \int_{\partial K} I_K^0\Big(\frac{\partial \bar u}{\partial n}\Big)\cdot \bar v_h\mathrm{d}s\right| =$$
$$\left|\sum_K \int_{\partial K} \Big(\frac{\partial \bar u}{\partial n} - I_K^0\Big(\frac{\partial \bar u}{\partial n}\Big)\Big)\cdot \bar v_h\mathrm{d}s\right| \leqslant$$
$$\sum_K \left\| \frac{\partial \bar u}{\partial n} - I_K^0 \frac{\partial \bar u}{\partial n} \right\|_{-1/2,\partial K} \| \bar v_h \|_{1/2,\partial K} \leqslant$$
$$\sum_K ch \left| \frac{\partial \bar u}{\partial n} \right|_{1/2,\partial K} \| \bar v_h \|_{1,K} \leqslant$$
$$\sum_K ch \left| \frac{\partial \bar u}{\partial n} \right|_{1,K} \| \bar v_h \|_{1,K} \leqslant$$
$$\sum_K ch \, | \bar u |_{2,K} \| \bar v_h \|_{1,K} \leqslant$$
$$ch \| \bar u \|_{2,\Omega} \| \bar v_h \|_{1,h}, \tag{23}$$

by $\left|\sum_K \int_{\partial K} I_K^0\Big(\frac{\partial \bar u}{\partial n}\Big)\cdot \bar v_h\mathrm{d}s\right| = 0$ and trace theorem.

From (22), (23) and lemma 4, set $\bar v_h = \bar e_h$ in (19) to get $| R(\bar e_h) | \leqslant cT \| \bar e_h \|_{1,h}$, where $T = h(\| \bar u \|_{2,\Omega} + \| p \|_{1,\Omega} + \| \bar f \|_{0,\Omega})$, and

$$a_h(\bar e_h, \bar e_h) + b_h(\bar e_h, \lambda_h) = R(\bar e_h) \tag{24}$$

From (20) and lemma 4 we get

$$| b_h(\bar e_h, \lambda_h) | = | b(I_h^1\bar u - \bar u, \lambda_h) | \leqslant$$
$$ch \| \lambda_h \|_{0,\Omega} \| \bar u \|_{2,\Omega}. \tag{25}$$

From (24) and $a_h = a$ in V_h, we get

$$| \bar{e}_h |_{1,h}^2 - ch \| \lambda_h \|_{0,\Omega} \| \bar{u} \|_{2,\Omega} \leqslant cT \| \bar{e}_h \|_{1,h}, \tag{26}$$

From the fact that b_h satisfies the discrete LBB condition and $a_h = a$ in V_h, we get

$$c \| \lambda_h \|_{0,\Omega} \leqslant \sup_{\bar{v}_h \in V_h} \frac{b_h(\bar{v}_h, \lambda_h)}{| \bar{v}_h |_{1,h}} = \sup_{\bar{v}_h \in V_h} \frac{R(\bar{v}_h) - a_h(\bar{e}_h, \bar{v}_h)}{| \bar{v}_h |_{1,h}} \leqslant \sup_{\bar{v}_h \in V_h} (cT + c | \bar{e}_h |_{1,h}) \leqslant c(T + | \bar{e}_h |_{1,h}), \tag{27}$$

By substituting (27) into (26), we get

$$| \bar{e}_h |_{1,h}^2 \leqslant cT | \bar{e}_h |_{1,h} + ch \| \bar{u} \|_{2,\Omega}(T + | \bar{e}_h |_{1,h}) \leqslant c | \bar{e}_h |_{1,h}(T + h \| \bar{u} \|_{2,\Omega}) + ch \| \bar{u} \|_{2,\Omega} T \leqslant c | \bar{e}_h |_{1,h} T + cT^2 \leqslant \frac{1}{2} | \bar{e}_h |_{1,h}^2 + cT^2 + cT^2,$$

so

$$| \bar{e}_h |_{1,h} \leqslant cT.$$

Substituting this formula into (26), we get $\| \lambda_h \|_{0,\Omega} \leqslant cT$.

From the standard interpolant result,

$$| \bar{u} - I_h^1 \bar{u} |_{1,h} + \| p - I_h^0 p \|_{0,\Omega} \leqslant ch(| \bar{u} |_{2,\Omega} + | p |_{1,\Omega}).$$

We complete the proof.

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