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Thermoelastic stresses in a uniformly heated functionally graded isotropic hollow cylinder*

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Abstract: The axisymmetric thermoelastic problem of a uniformly heated, functionally graded isotropic hollow cylinder is considered. An analytical form of solution is proposed. For the case when the Young's modulus and thermal expansion coefficient have a power-law dependence on the radial coordinate, explicit exact solution is obtained. For the degenerated case, i.e. when the cylinder is homogeneous and isotropic, no stresses will occur provided it is subjected to a uniform temperature. Numerical results are finally given and some important inclusions are obtained.

Key words: thermoelastic stress, hollow cylinder, functionally graded material
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INTRODUCTION

The concept of functionally graded material (FGM) was first introduced by a group of Japanese scientists at the National Aerospace Laboratory in Japan. FGMs are spatial composites in which material properties vary continuously. This is achieved by gradually changing the volume fraction of the constituent materials usually in one direction to obtain a smooth variation of material properties.

The study of plates and shells of functionally graded materials (FGMs) has become an interesting subject due to their increasing application in industrial engineering (Chen, 2000; Chen et al., 1999; Horgan and Chan, 1999; Rooney and Ferrari, 2001). FGMs first appeared as heat-shielding materials and the associated thermal behaviors of FGM structures have attracted much interest. Noda and Tsuji (1991) discussed the steady-state thermal stresses and the thermal deformations in an FGM plate. Obata and Noda (1994) considered the one-dimensional steady-state problem of an FGM hollow cylinder and a hollow sphere. Tanigawa et al. (1997) investigated the one-dimensional transient thermal stress problem of an FGM hollow cylinder.

The axisymmetric thermoelastic problem of a functionally graded isotropic circular hollow cylinder was studied in this work. A solution having the form suggested by Spencer et al. (1992) for anisotropic laminates was adopted. An explicit solution was derived for the case when the material properties are power-law functions of the radial variable. Some useful discussions and numerical examples are presented.

BASIC EQUATIONS

In cylindrical coordinates (r, θ, z), the stress $\boldsymbol{\sigma}$, related to the infinitesimal strain $\boldsymbol{e}$ and the excess temperature T above the reference temperature in a stress-free state can be expressed by

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{\theta z}\\ \sigma_{rz}\\ \sigma_{r\theta}\end{bmatrix}=\begin{bmatrix}\lambda+2\mu & \lambda & \mu & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}e_{rr}-\alpha T\\ e_{\theta\theta}-\alpha T\\ e_{zz}-\alpha T\\ 2e_{\theta z}\\ 2e_{rz}\\ 2e_{r\theta}\end{bmatrix} \quad (1)$$

where, in terms of the displacement components u_r, u_θ, u_z in the r, θ, z directions,

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$$e_{rr}=\frac{\partial u_r}{\partial r} \qquad 2e_{\theta z}=\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}$$
$$e_{\theta\theta}=\frac{1}{r}\left[\frac{\partial u_\theta}{\partial \theta}+u_r\right] \qquad 2e_{rz}=\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}$$
$$e_{zz}=\frac{\partial u_z}{\partial z} \qquad 2e_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r} \tag{2}$$

In Eq. (1), λ and μ are two Lame constants, and α is the thermal expansion coefficient. All these constants are assumed to be functions of the radial coordinate r in this study.

The equations of equilibrium, in the absence of body forces, are

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0$$
$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0 \tag{3}$$
$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=0$$

MATHEMATICAL DEVELOPMENT OF SOLUTION

Spencer et al. (1992) suggested a special solution for the thermoelastic distortion of a homogeneous or laminated anisotropic tube. For a finite FGM cylinder of inner radius a, outer radius b, and length h, subjected to a uniform temperature, we seek similarly the solution to Eq. (1)-(3) in the following form

$$u_r=au(\xi) \quad u_\theta=0 \quad u_z=Ba\zeta \tag{4}$$

where $\xi=r/a$ and $\zeta=z/a$ are dimensionless coordinates, B is a constant to be determined. In Eq. (4), $u_r=au(\xi)$ represents a radial expansion or contraction, and $u_z=Ba\zeta$ is a uniform axial extension or contraction.

From Eqs. (2) and (4), we obtain

$$e_{rr}=u' \quad e_{\theta\theta}=u/\xi \quad e_{zz}=B$$
$$2e_{\theta z}=2e_{rz}=2e_{r\theta}=0 \tag{5}$$

where a prime denotes differentiation with respect to ξ. Obviously all the strains and stresses depend only on the variable r. We have also $\sigma_{rz}=\sigma_{r\theta}=\sigma_{\theta z}=0$ from Eq. (1). Thus the equilibrium equations, Eq. (3), become

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \tag{6}$$

Substituting Eqs. (1) and (5) into Eq. (6) yields

$$(\lambda+2\mu)u''+\left[(\lambda+2\mu)'+(\lambda+2\mu)\frac{1}{\xi}\right]u'+\left[\frac{\lambda'}{\xi}-\frac{\lambda+2\mu}{\xi^2}\right]u=[(3\lambda+2\mu)\alpha]'T-\lambda'B \tag{7}$$

We consider the case when the Young's modulus E and the thermal expansion coefficient α have a power-law dependence on the radial coordinate, while the Poisson's ratio v is a constant (Horgan and Chan, 1999). Since

$$\lambda=\frac{Ev}{(1+v)(1-2v)} \qquad \mu=\frac{E}{2(1+v)} \tag{8}$$

we know that the two Lame constants λ and μ also have a power-law dependence on the radial coordinate. We thus can take

$$\lambda=\lambda^0\xi^n \quad \mu=\mu^0\xi^n \quad \alpha=\alpha^0\xi^n \tag{9}$$

where λ^0, μ^0 and α^0 are constants, and n is the inhomogeneity parameter. In view of Eq. (9), we obtain from Eq. (7)

$$(\lambda^0+2\mu^0)\xi^2u''+(n+1)(\lambda^0+2\mu^0)\xi u'+[(n-1)\lambda^0-2\mu^0]u=2n(3\lambda^0+2\mu^0)\alpha^0T\xi^{n+1}-n\lambda^0B\xi \tag{10}$$

It is obvious that the homogeneous solution to Eq. (10) can be obtained by assuming

$$u=C\xi^\beta \tag{11}$$

where C is an arbitrary constant. Substituting Eq. (11) into Eq. (10) and omitting the right-hand side, we obtain

$$\beta^2+n\beta+\left(\frac{n\lambda^0}{\lambda^0+2\mu^0}-1\right)=0 \tag{12}$$

Thus

$$\beta_1=\frac{-n+\sqrt{n^2-4[n\lambda^0/(\lambda^0+2\mu^0)-1]}}{2}$$
$$\beta_2=\frac{-n-\sqrt{n^2-4[n\lambda^0/(\lambda^0+2\mu^0)-1]}}{2} \tag{13}$$

By virtue of Eq. (8), we can verify that the discriminant of Eq. (12)

$$\Delta=n^2-4[n\lambda^0/(\lambda^0+2\mu^0)-1]=n^2-4nv/(1-v)+4$$

is always greater than zero for a given Poisson's ratio $0 \leqslant \upsilon < 1/2$. That is to say, the two roots in Eq. (13) are real and distinct.

The particular solution to Eq. (10) is easy to obtain, and the complete solution to it is as follows:

$$u = C_1 \xi^{\beta_1} + C_2 \xi^{\beta_2} + C_3 \xi^{n+1} + C_4 \xi \quad (14)$$

where C_1 and C_2 are two arbitrary constants, and

$$C_3 = \frac{2(1+v)}{(2n+3)(1-v)+v}\alpha^0 T \qquad C_4 = -vB \quad (15)$$

Note that if $n = 0$ the right-hand side of Eq. (10) vanishes which results in $C_3 = C_4 = 0$.

FURTHER INVESTIGATION

1. The homogeneous case

We first consider the homogeneous case for which $n = 0$ and all material constants are independent of the radial coordinate. In this case, we have $\beta_1 = 1$, $\beta_2 = -1$, $C_3 = C_4 = 0$. Thus the solution takes the form $u = C_1 \xi + C_2 \xi^{-1}$. The boundary conditions at the cylindrical surfaces $r = a, b$ are

$$\sigma_{rr} = 0 \quad \sigma_{r\theta} = 0 \quad \sigma_{rz} = 0 \quad (r = a, b) \quad (16)$$

where the last two conditions have been automatically satisfied at both surfaces as shown earlier in the paper. The first condition in Eq. (16) gives

$$2(\lambda + \mu)C_1 - 2\mu C_2 + \lambda B = (3\lambda + 2\mu)\alpha T \quad (17)$$

$$2(\lambda + \mu)C_1 - 2\mu C_2/\eta^2 + \lambda B = (3\lambda + 2\mu)\alpha T \quad (18)$$

where $\eta = b/a$ is the outer radius-to-inner radius ratio. Apart from the boundary conditions at $r = a, b$, we should also consider the conditions at the two ends $z = 0$ and $z = h$ as follows:

$$\sigma_{zz} = 0 \quad \sigma_{\theta z} = 0 \quad \sigma_{rz} = 0 \quad (z = 0, h) \quad (19)$$

The last two conditions in Eq. (19) are automatically satisfied and the first condition gives

$$2\lambda C_1 + (\lambda + 2\mu)B = (3\lambda + 2\mu)\alpha T \quad (20)$$

From Eqs. (17), (18) and (20), we obtain $C_1 = B = \alpha T$ and $C_2 = 0$. Now it is immediately shown that for a homogeneous hollow cylinder, a well-known conclusion can be achieved, i.e. all stress components vanish everywhere when it is uniformly heated.

2. The inhomogeneous case

By virtue of Eq. (14), the expressions of the radial, circumferential and axial stresses are derived as follows:

$$\begin{aligned}\sigma_{rr} = {} & [(\lambda^0 + 2\mu^0)\beta_1 + \lambda^0]C_1 \xi^{n+\beta_1-1} + \\ & [(\lambda^0 + 2\mu^0)\beta_2 + \lambda^0]C_2 \xi^{n+\beta_2-1} + \\ & [(\lambda^0 + 2\mu^0)(n+1) + \lambda^0]C_3 \xi^{2n} + \\ & 2(\lambda^0 + \mu^0)C_4 \xi^n - (3\lambda^0 + 2\mu^0) \cdot \\ & \alpha^0 T\xi^{2n} + \lambda^0 B\xi^n \\ \sigma_{\theta\theta} = {} & [\lambda^0\beta_1 + (\lambda^0 + 2\mu^0)]C_1 \xi^{n+\beta_1-1} \\ & + [\lambda^0\beta_2 + (\lambda^0 + 2\mu^0)]C_2 \xi^{n+\beta_2-1} \\ & + [\lambda^0(n+1) + (\lambda^0 + 2\mu^0)]C_3 \xi^{2n} \\ & + 2(\lambda^0 + \mu^0)C_4 \xi^n - (3\lambda^0 + 2\mu^0) \cdot \\ & \alpha^0 T\xi^{2n} + \lambda^0 B\xi^n \\ \sigma_{zz} = {} & \lambda^0(\beta_1 + 1)C_1 \xi^{n+\beta_1-1} + \lambda^0(\beta_2 + 1) \cdot \\ & C_2 \xi^{n+\beta_2-1} + \lambda^0(n+2)C_3 \xi^{2n} + \\ & 2\lambda^0 C_4 \xi^n - (3\lambda^0 + 2\mu^0)\alpha^0 T\xi^{2n} + \\ & (\lambda^0 + 2\mu^0)B\xi^n \end{aligned} \quad (21)$$

Then from the boundary conditions Eq. (16), we have

$$[\beta_1(\lambda^0 + 2\mu^0) + \lambda^0]C_1 + [\beta_2(\lambda^0 + 2\mu^0) + \lambda^0]C_2 + [(n+1)(\lambda^0 + 2\mu^0) + \lambda^0]C_3 + 2(\lambda^0 + \mu^0)C_4 - (3\lambda^0 + 2\mu^0)\alpha^0 T + \lambda^0 B = 0 \quad (22)$$

$$[\beta_1(\lambda^0 + 2\mu^0) + \lambda^0]C_1 \eta^{\beta_1-1} + [\beta_2(\lambda^0 + 2\mu^0) + \lambda^0]C_2 \eta^{\beta_2-1} + [(n+1)(\lambda^0 + 2\mu^0) + \lambda^0]C_3 \eta^n + 2(\lambda^0 + \mu^0)C_4 - (3\lambda^0 + 2\mu^0)\alpha^0 \cdot T\eta^n + \lambda^0 B = 0 \quad (23)$$

As pointed out by Spencer et al. (1992) the form of solution considered does not permit the point-by-point specification of traction at the two ends $z = 0, h$. Only certain resultant forces and moments can be specified on the basis of Saint-Venant's principle. For the problem considered, we have

$$2\pi\int_a^b \sigma_{zz} r\,\mathrm{d}r = 0 \quad (24)$$

which gives rise to

$$\frac{\beta_1+1}{n+\beta_1+1}\lambda^0 C_1(\eta^{n+\beta_1+1}-1)+\frac{\beta_2+1}{n+\beta_2+1}\cdot \lambda^0 C_2(\eta^{n+\beta_2+1}-1)+\frac{n+2}{2n+2}\lambda^0 C_3(\eta^{2n+2}-1)+\frac{2\lambda^0}{n+2}C_4(\eta^{n+2}-1)-\frac{3\lambda^0+2\mu^0}{2n+2}\alpha^0 T\cdot(\eta^{2n+2}-1)+\frac{\lambda^0+2\mu^0}{n+2}B(\eta^{n+2}-1)=0 \quad (25)$$

It is noted that Eq. (25) is not valid for $n=-\beta_j-1(j=1,2)$, -2 and -1. If, for example $n=-2$, Eq. (25) should be modified to

$$\frac{\beta_1+1}{\beta_1-1}\lambda^0 C_1(\eta^{\beta_1-1}-1)+\frac{\beta_2+1}{\beta_2-1}\lambda^0 C_2(\eta^{\beta_2-1}-1)+2\lambda^0 C_4\ln\eta+\frac{3\lambda^0+2\mu^0}{2}\alpha^0 T(\eta^{-2}-1)+(\lambda^0+2\mu^0)B\ln\eta=0 \quad (26)$$

For $n=-\beta_j-1$ $(j=1,2)$ and $n=-1$, similar equations can be derived and are omitted here for brevity.

With Eq. (15), the three unknowns C_1, C_2 and B can be determined from Eqs. (22), (23), and (25) or Eq. (26).

NUMERICAL EXAMPLE

The distributions of nondimensional stresses $R=\sigma_{rr}/(E^0\alpha^0 T)$, $\Theta=\sigma_{\theta\theta}/(E^0\alpha^0 T)$ and $Z=\sigma_{zz}/(E^0\alpha^0 T)$ are shown in Figs. 1-3 for a thin hollow cylinder with $\eta=b/a=1.05$, while those for a thick hollow cylinder with $\eta=b/a=1.5$ are given in Figs. 4-6 for comparison. Several combinations of Poisson's ratio and inhomogeneity parameter n are considered.

It is seen that the Poisson's ratio, the inhomogeneous parameter and the outer radius-to-inner radius ratio all have significant influence on the distributions of the thermoelastic stresses. In particular, a negative n will lead to compressive radial stress and vice versa. Also, for a negative n, both the circumferential and axial stress components change from negative at the inner surface to positive at the outer surface and vice versa. It is also seen that the distributions of $\sigma_{\theta\theta}$ and σ_{zz} are very similar and the difference of magnitude between them is very small. But the magnitude

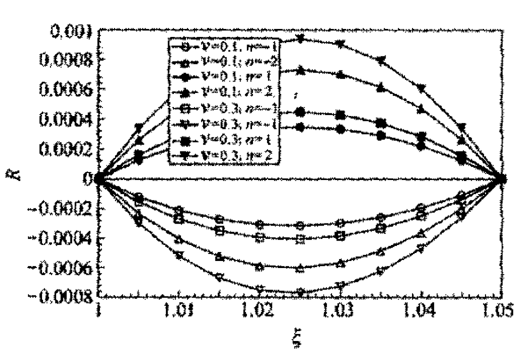

Fig.1 Curves of *R* versus for *ξ* a thin cylinder

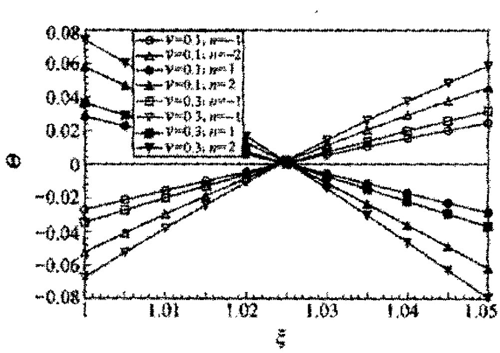

Fig.2 Curves of *Θ* versus *ξ* for a thin cylinder

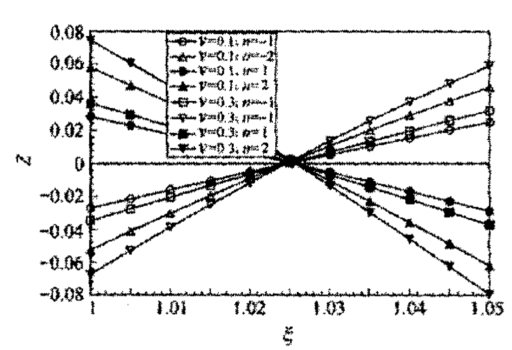

Fig.3 Curves of *Z* versus *ξ* for a thin cylinder

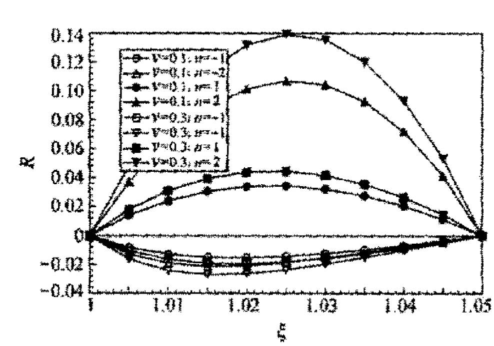

Fig.4 Curves of *R* versus *ξ* for a thick cylinder

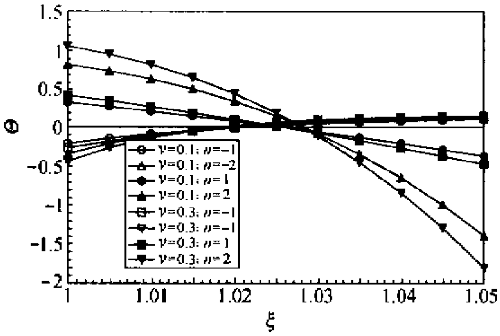

Fig.5 Curves of Θ versus ξ for a thick cylinder

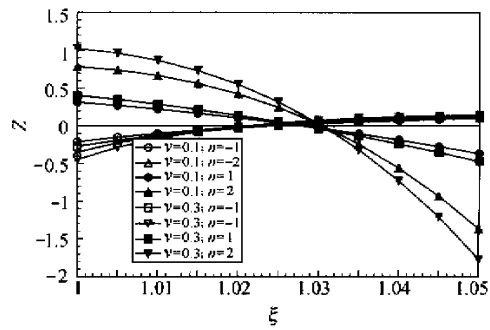

Fig.6 Curves of Z versus ξ for a thick cylinder

of σ_{rr} is much smaller than that of $\sigma_{\theta\theta}$ and σ_{zz}, even in the thick cylinder. The distributions of $\sigma_{\theta\theta}$ and σ_{zz} in the thin cylinder are almost linear along the radial direction, while in the thick cylinder they are no longer linear.

CONCLUSIONS

The two-dimensional axisymmetric problem of a functionally graded isotropic circular hollow cylinder subjected to a uniform temperature field was investigated in the paper. An analytical solution was obtained when the material properties are power-law functions of the radial coordinate. It was shown that for a homogeneous hollow cylinder, no stress occurs when it is uniformly heated.

Numerical results showed that the inhomogeneity parameter n has great effect on the distributions of thermoelastic fields. For example, a negative n will yield compressive circumferential stress at the inner surface and tensile circumferential stress at the outer surface, while a positive n gives a contrary result. Thus by selecting a proper value of n, it is possible for engineers to design a cylinder that can meet some special requirements.

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