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Adaptive terminal sliding mode control for high-order nonlinear dynamic systems*

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Abstract: An adaptive terminal sliding mode control (SMC) technique is proposed to deal with the tracking problem for a class of high-order nonlinear dynamic systems. It is shown that a function augmented sliding hyperplane can be used to develop a new terminal sliding mode for high-order nonlinear systems. A terminal SMC controller based on Lyapunov theory is designed to force the state variables of the closed-loop system to reach and remain on the terminal sliding mode, so that the output tracking error then converges to zero in finite time which can be set arbitrarily. An adaptive mechanism is introduced to estimate the unknown parameters of the upper bounds of system uncertainties. The estimates are then used as controller parameters so that the effects of uncertain dynamics can be eliminated. It is also shown that the stability of the closed-loop system can be guaranteed with the proposed control strategy. The simulation of a numerical example is provided to show the effectiveness of the new method.

Key words: Terminal sliding mode control, Finite time convergence, Adaptive laws
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INTRODUCTION

Considerable attention has been focused on the control problem of uncertain dynamical nonlinear systems subject to external disturbances and parameter variations. One approach to deal with this problem is by means of SMC due to its excellent properties, such as insensitivity to parameter variations and certain external disturbances.

In general SMC, the sliding surfaces commonly are linear hyperplanes. Such linear hyperplanes can ensure the asymptotic convergence of tracking error on the sliding mode, so that the converging speed can be adjusted arbitrarily by choosing the parameters of the sliding surfaces. But in any case, the state tracking error on the sliding mode can't converge to zero in finite time.

A new technique called terminal SMC to get better performance was reported (Venkataraman et al., 1993; Man et al., 1994; Park et al., 1996; Yu. et al., (1997). Nonlinear functions were introduced into the sliding surfaces in such a way that the tracking error could converge to zero in finite time. Furthermore, the controller gain could be decreased compared with that obtained by using linear sliding surfaces. However, the introduction of nonlinear functions into the sliding surfaces of the controller is not easy to implement in practice. In addition, singularity problems may occur if the parameters are not chosen properly, especially in muti-input/muti-output systems. Park et al. (1999) proposed another terminal sliding manifold to overcome the above problems, but only second-order nonlinear uncertain systems were discussed. Moreover, the time derivative of the proposed sliding manifold is not continuous.

In this work, inspired by Park et al. (1999), a new terminal SMC was studied with the aim of obtaining a solution for the problems

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of terminal sliding mode. A terminal sliding surface adaptable to high-order systems was first designed; then a terminal SMC controller based on Lyapunov theory was derived for high-order nonlinear dynamical systems. However, the application of the SMC is limited as the prior knowledge of the upper bound of uncertainties (which include parameter variations and external disturbances) must be available in general SMC. Here we propose simple adaptation laws for the upper bounds on the norm of the uncertainties to overcome the above limitation. Finally, a numerical example was studied and the simulation results confirmed the validity of the proposed control method.

PROBLEM FORMULATION

Consider the following n th-order nonlinear system as following

$$\boldsymbol{y}^{(n)} = \boldsymbol{f}(\boldsymbol{y}^{(n-1)}, \cdots, \dot{\boldsymbol{y}}, \boldsymbol{y}, \boldsymbol{t}) + \Delta\boldsymbol{f}(\boldsymbol{y}^{(n-1)}, \cdots, \dot{\boldsymbol{y}}, \boldsymbol{y}, \boldsymbol{t}) + \boldsymbol{b}(\boldsymbol{y}^{(n-1)}, \cdots, \dot{\boldsymbol{y}}, \boldsymbol{y}, \boldsymbol{t})\boldsymbol{u} + \boldsymbol{d}(t) \quad (1)$$

where $\boldsymbol{y} \in \boldsymbol{R}^m$ is the output vector; $\boldsymbol{u} \in \boldsymbol{R}^m$ is the control input vector; $\boldsymbol{f} \in \boldsymbol{R}^m$ and $\boldsymbol{b} \in \boldsymbol{R}^{m\times m}$ are known nonlinear function matrices of the system states with rank $(\boldsymbol{b}) = m$; $\Delta\boldsymbol{f}$ represents the plant uncertainty and $\boldsymbol{d}(t)$ denotes the external disturbance.

In order to design a proper control law for the system (1), we define $\boldsymbol{x}_1 = \boldsymbol{y}, \cdots, \boldsymbol{x}_n = \boldsymbol{y}^{(n-1)}$ and rewrite Eq. (1) in the following form

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2 \\ &\vdots \\ \dot{\boldsymbol{x}}_n &= \boldsymbol{f}(\boldsymbol{x}, t) + \Delta\boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{b}(\boldsymbol{x}, t)\boldsymbol{u} + \boldsymbol{d}(t) \end{aligned} \quad (2)$$

where $\boldsymbol{x} = [\boldsymbol{x}_1^{\mathrm{T}}, \cdots, \boldsymbol{x}_n^{\mathrm{T}}]^{\mathrm{T}}$.

DESIGN OF TERMINAL SLIDING MODE

The control problem is to design a control law such that the states of the closed-loop system $\boldsymbol{x} = [\boldsymbol{x}_1^{\mathrm{T}}, \cdots, \boldsymbol{x}_n^{\mathrm{T}}]^{\mathrm{T}}$ could track the desired states $\boldsymbol{x}_d = [\boldsymbol{x}_{1\mathrm{d}}^{\mathrm{T}}, \cdots, \boldsymbol{x}_{n\mathrm{d}}^{\mathrm{T}}]^T$ belonging to the class of $\boldsymbol{C}^1$ functions on $[t_0, \infty)$. We also define the error vector as

$$\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}_{\mathrm{d}} = [e^{\mathrm{T}}, \dot{\boldsymbol{e}}^{\mathrm{T}}, \cdots, \boldsymbol{e}^{(n-1)^{\mathrm{T}}}]^{\mathrm{T}} \quad (3)$$

where $\boldsymbol{e} = [e_1, e_2, \cdots, e_m]^{\mathrm{T}}$. Thus, we design the sliding surface as

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{A}\boldsymbol{e}(t) - \boldsymbol{w}(t) \quad (4)$$

where $\boldsymbol{A} = [\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n]$ is a matrix, $\boldsymbol{A}_i = \mathrm{diag}(A_{i1}, A_{i2}, \cdots, A_{im})$ and A_{ij} ($i = 1, \cdots n$; $j = 1, \cdots m$) are positive constants, $\boldsymbol{w}(t) = \boldsymbol{A}\boldsymbol{v}(t)$, $\boldsymbol{v}(t) = [\boldsymbol{v}(t)^{\mathrm{T}}, \dot{\boldsymbol{v}}(t)^{\mathrm{T}}, \cdots, \boldsymbol{v}^{(n-1)}(t)^{\mathrm{T}}]^{\mathrm{T}}$. *Let* $\boldsymbol{v} = [v_1, v_2, \cdots, v_m]^{\mathrm{T}}$ and $v_i(t)$ is designed such that the following assumption holds.

Assumption 1: For $i = 1, 2, \cdots m$, $v_i(t)$: $\boldsymbol{R}_+ \to \boldsymbol{R}$, $v_i(t) \in \boldsymbol{C}^n(0, \infty)$, $\dot{v}_i, \cdots v_i^{(n)} \in \boldsymbol{L}^\infty$, the support of $v_i(t)$ is a bounded interval $[0, T]$ for some $T > 0$, $v_i(0) = e_i(0)$, $\dot{v}_i(0) = \dot{e}_i(0)$, $\cdots$, $v_i^{(n)}(0) = e_i^{(n)}(0)$, where $\boldsymbol{C}^n[0, \infty)$ represents the set of all nth-order differentiable continuous functions defined on $[0, \infty)$.

Here $v_i(t)$ is designed as the following

$$v_i(t) = \begin{cases} \displaystyle\sum_{k=0}^{n} \frac{1}{k!} e_i(0)^{(k)} t^k + \sum_{j=0}^{n}\left(\sum_{l=0}^{n} \frac{a_{jl}}{(T)^{j-l+n+1}} e_i(0)^{(l)}\right) t^{j+n+1} & \text{if } 0 \leqslant t \leqslant T \\ 0 & \text{if } t > T \end{cases} \quad (5)$$

where the parameters a_{jl} can be obtained from the conditions of Assumption 1. Without loss of generality, we consider a second-order system as an example to show the procedure of designing $v_i(t)$. Thus $n = 2$ and $v_i(t)$ can be rewritten as

$$v_i(t) = \begin{cases} e_i(0) + \dot{e}_i(0)t + \dfrac{1}{2}\ddot{e}_i(0)t^2 + \left[\dfrac{a_{00}}{T^3}e_i(0) + \dfrac{a_{01}}{T^2}\dot{e}_i(0) + \dfrac{a_{02}}{T}\ddot{e}_i(0)\right]t^3 + \\ \left[\dfrac{a_{10}}{T^4}e_i(0) + \dfrac{a_{11}}{T^3}\dot{e}_i(0) + \dfrac{a_{12}}{T^2}\ddot{e}_i(0)\right]t^4 + \left[\dfrac{a_{20}}{T^5}e_i(0) + \dfrac{a_{21}}{T^4}\dot{e}_i(0) + \dfrac{a_{22}}{T^3}\ddot{e}_i(0)\right]t^5 & \text{if } 0 \leqslant t \leqslant T \\ 0 & \text{if } t > T \end{cases} \quad (6)$$

From Assumption 1, $v_i(t)$ should be second-differentiable continuous function at $t = T$; then we can obtain three groups of simple equations

$$\begin{cases}1 + a_{00} + a_{10} + a_{20} = 0\\ 3a_{00} + 4a_{10} + 5a_{20} = 0\\ 6a_{00} + 12a_{10} + 20a_{20} = 0\end{cases}$$

$$\begin{cases}1 + a_{01} + a_{11} + a_{21} = 0\\ 1 + 3a_{01} + 4a_{11} + 5a_{21} = 0\\ 6a_{01} + 12a_{11} + 20a_{21} = 0\end{cases}$$

$$\begin{cases}\frac{1}{2} + a_{02} + a_{12} + a_{22} = 0\\ 1 + 3a_{02} + 4a_{12} + 5a_{22} = 0\\ 1 + 6a_{02} + 12a_{12} + 20a_{22} = 0\end{cases}$$

Based on the above three groups of equations, parameters a_{jl} ($j = 0, 1, 2$; $l = 0, 1, 2$) can be calculated easily as

$$\begin{cases}a_{00} = -10\\ a_{10} = 15\\ a_{20} = -6\end{cases} \quad \begin{cases}a_{01} = -6\\ a_{11} = 8\\ a_{21} = -3\end{cases} \quad \begin{cases}a_{02} = -\frac{3}{2}\\ a_{12} = \frac{3}{2}\\ a_{22} = -\frac{1}{2}\end{cases}$$

Similarly, we can calculate the corresponding parameters a_{jl} for nth-order systems and obtain the terminal sliding surface. For nth-order systems, we can obtain $n + 1$ groups of the equations and $(n + 1)^2$ parameters should be solved. But all these work can be done off-line and the procedure for solving the equations is very simple.

After selecting the terminal surface, the next step is to choose the SMC controller such that the sliding surface will attract the system trajectories and that upon the intersection with the sliding surface, the states variables will remain there for all subsequent time.

ADAPTIVE TERMINAL SLIDING MODE CONTROLLER

In sliding mode control, the control input should force all trajectories to intersect the surface $\boldsymbol{\sigma}(\boldsymbol{x}, t) = 0$ and remain there for all subsequent time. The sliding mode behavior reflects the stability of the state trajectories on the sliding surface. In other words, the control law should force the tracking error to be zero for arbitrary initial states.

From Eqs (2), (3), and (4), we have

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{x}, t) = \boldsymbol{A}_n \{\boldsymbol{f}(\boldsymbol{x}, t) + \Delta\boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{b}(\boldsymbol{x}, t)\boldsymbol{u} + \boldsymbol{d}(t) - \boldsymbol{x}_{1d}^{(n)} - \boldsymbol{v}(t)^{(n)}\} + \sum_{k=1}^{n-1} \boldsymbol{A}_k \{\boldsymbol{e}^{(k)} - \boldsymbol{v}(t)^{(k)}\} \tag{7}$$

Assumption 2 The uncertainty $\Delta\boldsymbol{f}(\boldsymbol{x}, t)$ and the external disturbance $\boldsymbol{d}(t)$ satisfy the following inequalities

$$\| \Delta\boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{d}(t) \| \leqslant c_0 + c_1 \| \boldsymbol{x} \| \tag{8}$$

where c_0, c_1 are positive constants.

Now, consider the simple adaptive laws for the uncertainty $\Delta\boldsymbol{f}(\boldsymbol{x}, t)$ and the external disturbance $\boldsymbol{d}(t)$

$$\dot{\tilde{c}}_0(t, \boldsymbol{x}) = q_0^{-1} \| \boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma} \|, \quad \dot{\tilde{c}}_1(t, \boldsymbol{x}) = q_1^{-1} \| \boldsymbol{A}_n^{T}\boldsymbol{\sigma} \| \cdot \| \boldsymbol{x} \|, \tag{9}$$

where $\tilde{c}_0(t, \boldsymbol{x}) = \bar{c}_0(t, \boldsymbol{x}) - c_0$ and $\tilde{c}_1(t, \boldsymbol{x}) = \bar{c}_1(t, \boldsymbol{x}) - c_1$ are parameter adaptation errors, q_0 and q_1 are adaptation gains with positive values. $\bar{c}_0(t, \boldsymbol{x})$ and $\bar{c}_1(t, \boldsymbol{x})$ are the adaptive parameters about c_0 and c_1, respectively. Since c_0 and c_1 are constants, the adaptive laws can also be written as

$$\dot{\bar{c}}_0(t, \boldsymbol{x}) = q_0^{-1} \| \boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma} \|,$$

$$\dot{\bar{c}}_1(t, \boldsymbol{x}) = q_0^{-1} \| \boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma} \| \cdot \| \boldsymbol{x} \|,$$

Consider the Lyapunov function candidate as follows

$$2V(\boldsymbol{\sigma}, \tilde{c}_0, \tilde{c}_1) = \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma} + q_0\tilde{c}_0^2 + q_1\tilde{c}_1^2$$

Differentiating $V(\boldsymbol{\sigma}, \tilde{c}_0, \tilde{c}_1)$ with respect to time t yields

$$\begin{aligned}\dot{V}(\boldsymbol{\sigma}, \tilde{c}_0, \tilde{c}_1) = {} & \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} + q_0\tilde{c}_0\dot{\tilde{c}}_0 + q_1\tilde{c}_1\dot{\tilde{c}}_1 = \\ & \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{A}_n \{\boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{x}_{1d}^{(n)} - v(t)^n + \\ & \boldsymbol{A}_n^{-1}\sum_{k=1}^{n-1} \boldsymbol{A}_k(\boldsymbol{e}^{(k)} - \boldsymbol{v}(t)^{(k)})\} + \\ & \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{A}_n\boldsymbol{b}(\boldsymbol{x}, t)\boldsymbol{u} + \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{A}_n \{\Delta\boldsymbol{f}(\boldsymbol{x}, t) + \\ & \boldsymbol{d}(t)\} + q_0\tilde{c}_0\dot{\tilde{c}}_0 + q_1\tilde{c}_1\dot{\tilde{c}}_1 \leqslant \\ & \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{A}_n \{\boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{x}_{1d}^{(n)} - v(t)^{(n)} + \\ & \boldsymbol{A}_n^{-1}\sum_{k=1}^{n-1} \boldsymbol{A}_k(\boldsymbol{e}^{(k)} - \boldsymbol{v}_{(t)}^{(k)})\} + \\ & \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{A}_n\boldsymbol{b}(\boldsymbol{x}, t)\boldsymbol{u} + \| \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{A}_n \| \cdot (c_0 + \\ & c_1 \| \boldsymbol{x} \|) + q_0\tilde{c}_0\dot{\tilde{c}}_0 + q_1\tilde{c}_1\dot{\tilde{c}}_1\end{aligned} \tag{10}$$

Here, the control input $\boldsymbol{u}(t)$ is chosen as follows

$$\boldsymbol{u}(t) = -\boldsymbol{b}(\boldsymbol{x},t)^{-1}\{\boldsymbol{f}(\boldsymbol{x},t) - \boldsymbol{x}_{1d}^{(n)} - v(t)^{(n)} + \boldsymbol{A}_n^{-1}\sum_{k=1}^{n-1}\boldsymbol{A}_k(\boldsymbol{e}^{(k)} - \boldsymbol{v}(t)^{(k)})\} - \boldsymbol{b}(\boldsymbol{x},t)^{-1}\frac{\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}}{\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|}\cdot\{(\bar{c}_0 + \bar{c}_1\|\boldsymbol{x}\|) + K\} \quad (11)$$

where K is a positive constant, then

$$\dot{V}(\boldsymbol{\sigma},\tilde{c}_0,\tilde{c}_1) \leqslant -\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot(\bar{c}_0 + \bar{c}_1\|\boldsymbol{x}\| + K) + \|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot(c_0 + c_1\|\boldsymbol{x}\|) + q_0\tilde{c}_0\dot{\tilde{c}}_0 + q_1\tilde{c}_1\dot{\tilde{c}}_1$$
$$= -K\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\| + \tilde{c}_0(q_0\dot{\tilde{c}}_0 - \|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|) + \tilde{c}_1(q_0\dot{\tilde{c}}_1 - \|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot\|\boldsymbol{x}\|) \quad (12)$$

Taking Eq.(9) as the adaptation laws, we can derive

$$\dot{V}(\boldsymbol{\sigma},\tilde{c}_0,\tilde{c}_1) \leqslant -K\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\| < 0 \quad (13)$$

Since K is positive, the system origin $\boldsymbol{\sigma} = 0$ is globally uniformly asymptotically stable (Slotine et al., 1991).

Next, let us discuss the convergence rate of the sliding variable vector $\boldsymbol{\sigma}$. Using Eq.(10) and Eq.(13), we have

$$\dot{V}(\boldsymbol{\sigma},\tilde{c}_0,\tilde{c}_1) = \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} + q_0\tilde{c}_0\dot{\tilde{c}}_0 + q_1\tilde{c}_1\dot{\tilde{c}}_1 \leqslant -K\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|$$

or

$$\boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} + q_0\tilde{c}_0 q_0^{-1}\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\| + q_1\tilde{c}_1 q_1^{-1}\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot\|\boldsymbol{x}\| \leqslant -K\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\| \text{ or } -\boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} - \tilde{c}_0\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\| - \tilde{c}_1\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot\|\boldsymbol{x}\| \geqslant K\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|$$

or

$$\|\boldsymbol{\sigma}\|\cdot\|\dot{\boldsymbol{\sigma}}\| - \tilde{c}_0\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\| - \tilde{c}_1\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot\|\boldsymbol{x}\| \geqslant K\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|$$

then

$$\|\dot{\boldsymbol{\sigma}}\| \geqslant \frac{\|\boldsymbol{A}_n^{T}\boldsymbol{\sigma}\|(\tilde{c}_0 + \tilde{c}_1\cdot\|\boldsymbol{x}\| + K}{\|\boldsymbol{\sigma}\|},$$
$$\text{for } \|\boldsymbol{\sigma}\| \neq 0 \quad (14)$$

It is noted from Eq.(9) that

$$\tilde{c}_0 = \bar{c}_0(0,\boldsymbol{x}) + \int_0^t q_0^{-1}\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\,\mathrm{d}t - c_0 \quad (15a)$$
$$\tilde{c}_1 = \bar{c}_1(0,\boldsymbol{x}) + \int_0^t q_1^{-1}\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|\cdot\|\boldsymbol{x}\|\,\mathrm{d}t - c_1 \quad (15b)$$

Because c_0 are c_1 unknown constants, we can suitably choose $\bar{c}_0(0)$, $\tilde{c}_1(0)$, q_0 and q_1 such that

$$\tilde{c}_0 > 0 \quad \text{and} \quad \tilde{c}_1 > 0$$

Without loss of generality, we can let $\boldsymbol{A}_n = \boldsymbol{I}$, then we have

$$\|\dot{\boldsymbol{\sigma}}\| = \frac{\|\boldsymbol{\sigma}\|\cdot(\tilde{c}_0 + \tilde{c}_1\cdot\|\boldsymbol{x}\| + K)}{\|\boldsymbol{\sigma}\|} = \tilde{c}_0 + \tilde{c}_1\cdot\|\boldsymbol{x}\| + K > K$$
$$\text{for } \|\boldsymbol{\sigma}\| \neq 0. \quad (16)$$

Because K is a positive constant, Eq.(16) means that the sliding surfaces $\boldsymbol{\sigma}$ can converge to zero in finite time.

Remark From Assumption 1 and the definition of the proposed sliding hyperplane Eq.(4), it is clear that the system states are on the sliding hyperplane at initial instant $\boldsymbol{\sigma}(\boldsymbol{x},0) = 0$. Moreover, based on the above proof, it implies that the sliding mode exists $\forall\, t \geqslant 0$. From the definite of $\boldsymbol{\sigma}(\boldsymbol{x},t)$, it is easy to know that

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{A}\boldsymbol{e}(t) - \boldsymbol{w}(t) = \boldsymbol{A}\boldsymbol{e}(t) - \boldsymbol{A}\boldsymbol{v}(t) = \boldsymbol{A}\boldsymbol{\zeta}(t)$$

where $\boldsymbol{\zeta}(t) = \boldsymbol{e}(t) - \boldsymbol{v}(t)$. From Assumption 1, it is obvious that $\boldsymbol{\zeta}(0) = 0$. Since the sliding mode exists $\forall\, t \geqslant 0$($\boldsymbol{\sigma} = 0$, $\forall\, t \geqslant 0$), it can be known that $\boldsymbol{\zeta}(0) = 0$, $\forall\, t \geqslant 0$, which is equivalent to $\boldsymbol{e}(t) \equiv \boldsymbol{v}(t)$. The function $\boldsymbol{v}(t)$ can be designed arbitrarily if Assumption 1 holds, and Assumption 1 shows that $\boldsymbol{v}(t)$ is designed so that $\boldsymbol{v}(t) = 0$, $\forall\, t \geqslant T$. Thus, the tracking error $\boldsymbol{e}(t)$ converges to zero in finite time T.

Theorem 1 Consider the nonlinear system (1) with Assumptions 1-2, if the sliding surface $\boldsymbol{\sigma}$ is chosen as Eq.(4) and the sliding mode control $\boldsymbol{u}$ is given by Eq.(11), the output tracking error of the closed-loop system can be guaranteed to converge to zero in finite time T which can be set arbitrarily.

CHATTERING

For the obtained control law Eq.(11), chattering phenomenon maybe appear for the closed-loop system due to the function vector $\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}/\|\boldsymbol{A}_n^{\mathrm{T}}\boldsymbol{\sigma}\|$. Chattering is not undesirable because it involves high control activity and may excite high-frequency dynamics. To reduce the effect of chattering, the discontinuous function can be replaced by a proper continuous function (Chern et al., 1991)

$$S_\delta(A_n^T\sigma) = \frac{A_n^T\boldsymbol{\sigma}}{\| A_n^T\boldsymbol{\sigma} \| + \delta} \quad (17)$$

where $\boldsymbol{\delta}$ is chosen as a function of the error $\boldsymbol{e}$, i.e.

$$\delta = \delta_0 + \delta_1 \| \boldsymbol{e} \| \quad (18)$$

where δ_0, δ_1 are positive constants. With a proper S_δ, the chattering phenomenon can be eliminated.

NUMERICAL EXAMPLE

In this section, a second-order system of the form Eq. (1) is studied. The corresponding parameters are given as follows

$$\boldsymbol{f}(\dot{\boldsymbol{y}}, \boldsymbol{y}, t) = \begin{bmatrix} -1.5\dot{y}_1^2\cos(3y) \\ 3\dot{y}_2\sin(y_1) \end{bmatrix};$$

$$\boldsymbol{b}(\dot{y}, y, t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Delta\boldsymbol{f}(\dot{\boldsymbol{y}}, \boldsymbol{y}, t) = \begin{bmatrix} (\sin t)\dot{y}_2^2\cos(3y_2) \\ e^{-t}\dot{y}_1\sin(y_1) \end{bmatrix};$$

$$\boldsymbol{d}(t) = 0$$

where $\boldsymbol{y} = [y_1, y_2]^T \in \boldsymbol{R}^2$ and $\boldsymbol{u} = [u_1, u_2]^T \in \boldsymbol{R}^2$.

Then the system description of the form Eq. (2) can be obtained as follows

$$\dot{x}_{11} = x_{12}$$
$$\dot{x}_{12} = -1.5x_{12}^2\cos(3x_{21}) + (\sin t)x_{22}^2\cos(3x_{21}) + u_1$$
$$\dot{x}_{21} = x_{22}$$
$$\dot{x}_{22} = 3x_{22}\sin(x_{11}) + e^{-t}x_{12}\sin(x_{11}) + u_2$$
$$y = [x_{11}, x_{21}]^T$$

From Eq.(4) and Eq.(6), Let $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}$, then the sliding surface can be derived as

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \dot{e}_1 + 4e_1 - \dot{v}_1 - 4v_1 \\ \dot{e}_2 + 4e_2 - \dot{v}_2 - 4v_2 \end{bmatrix}$$

$$v_i(t) = \begin{cases} e_i(0) + \dot{e}_i(0)t + \frac{1}{2}\ddot{e}(0)t^2 - \\ \left[\frac{10}{T^3}e_i(0) + \frac{6}{T^2}\dot{e}(0) + \frac{3}{2T}\ddot{e}_i(0)\right]t^3 + \\ \left[\frac{15}{T^4}e_i(0) + \frac{8}{T^3}\dot{e}_i(0) + \frac{3}{2T^2}\ddot{e}_i(0)\right]t^4 - \\ \left[\frac{6}{T^5}e_i(0) + \frac{3}{T^4}\dot{e}_i(0) + \frac{1}{2T^3}\ddot{e}_i(0)\right]t^5 & \text{if } 0 \leqslant t \leqslant T \\ 0 & \text{if } t > T \end{cases}$$

The control objective is to maintain $x_{11} = x_{11\mathrm{d}}$ and $x_{21} = x_{21\mathrm{d}}$. Here, we adopt the following curves as the desired trajectory $x_{11\mathrm{d}}$, $x_{21\mathrm{d}}$

$$x_{11\mathrm{d}} = \sin(\pi t/2) \quad x_{21\mathrm{d}} = \cos(\pi t/2)$$

then $\boldsymbol{e} = [e_1 e_2]^T = [x_{11} - x_{11d} \quad x_{21} - x_{21}\mathrm{d}]^T$ and the required control law is

$$\boldsymbol{u}(t) = -\begin{bmatrix} -1.5x_{12}^2\cos(3x_{21}) + 4(x_{12} - \frac{\pi}{2}\cos(\pi t/2) - \dot{v}_1(t)) - \ddot{v}_1(t) + \frac{\pi^2}{4}\sin(\pi t/2) \\ 3x_{22}\sin(x_{11}) + 4(x_{22} + \frac{\pi}{2}\sin(\pi t/2) - \dot{v}_2(t)) - \ddot{v}_2(t) + \frac{\pi^2}{4}\cos(\pi t/2) \end{bmatrix} - \frac{\boldsymbol{\sigma}}{\| \boldsymbol{\sigma} \| + \delta_0 + \delta_1 \| \boldsymbol{e} \|}\{\bar{c}_0 + \bar{c}_1 \| \boldsymbol{x} \| + K\}$$

where $\dot{\bar{c}}_0 = q_0^{-1} \| \boldsymbol{\sigma} \|$, $\dot{\bar{c}} = q_1^{-1} \| \boldsymbol{\sigma} \| \cdot \| \boldsymbol{x} \|$

The initial state is taken as $\boldsymbol{y}_0 = [0.5, 0]^T$ and the system is simulated with a time interval of 0.002s by using Runge-Kutta method. Let $\delta_0 = 0.03$, $\delta_1 = 5$, $q_0 = 1$, $q_1 = 1$, $K = 10$, and $T^2 = 1.5$s, then the tracking performances of the states x_{11} and x_{21} are shown in Fig.1 and Fig.2, respectively(in the next page). From the simulation curves, we can easily know that the tracking errors can converge to zero within $T = 1.5$s.

CONCLUSIONS

The design approach for a new adaptive terminal SMC is presented in this paper for a class of high-order nonlinear dynamic systems. The proposed adaptive terminal SMC controller can force the trajectories of the closed-loop system to

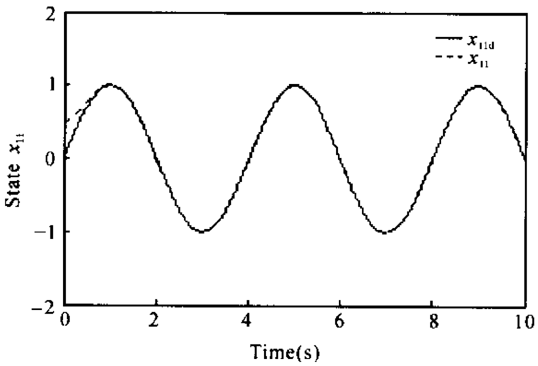

Fig.1 Tracking performance of state x_{11}

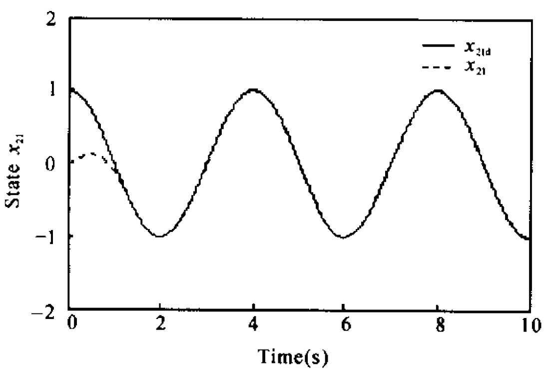

Fig.2 Tracking performance of state x_{21}

track the desired trajectories. In addition, by using the terminal sliding technique, the error state can converge to zero in finite time T. Simulation results showed that the closed-loop system has better robustness and zero steady state error than conventional closed-loop systems.

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