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Influence of water saturation on propagation of elastic waves in transversely isotropic nearly saturated soil*

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Abstract: Biot's two-phase theory for fluid-saturated porous media was applied in a study carried out to investigate the influence of water saturation on propagation of elastic wave in transversely isotropic nearly saturated soil. The characteristic equations for wave propagation were derived and solved analytically. The results showed that there are four waves: the first and second quasi-longitudinal waves (QP1 and QP2), the quasi-transverse wave (QSV) and the anti-plane transverse wave (SH). Numerical results are given to illustrate the influence of saturation on the velocity, dispersion and attenuation of the four body waves. Some typical numerical results are discussed and plotted. The results can be meaningful for soil dynamics and earthquake engineering.

Key words: Saturation, Transversely isotropic, Nearly saturated, Dispersion, Attenuation
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INTRODUCTION

It has long been recognized that compressional waves propagation in saturated soils is strongly affected by the water filling the interstices of the soil grains. Hardin and Richart (1963) applied the two-phase theory of Biot (1956) and studied the influence of the confining pressure on compressional waves in saturated sands. Allen *et al*. (1980) determined the effects of void ratio and degree of saturation on the velocity of compressional waves within nearly saturated sands. White (1986) computed wave speeds and attenuation in rocks with partial gas saturation. His work showed that fluid-flow waves should be responsible for the dispersion and attenuation of body waves. The theories of Biot (1956) and Vardoulakis and Beskos (1986) were used by Bardet and Sayed (1993) to calculate the velocity and attenuation of compressional waves under one-dimension condition, respectively. They presented a new model for compressional waves in nearly saturated soils. Yang and Sato (1998) applied the two-phase theory of Biot (1956) to study the velocity and damping of elastic waves in nearly saturated soils. Dontsov and Nakoryakov (2001) used experimental methods to study the intensification of shock waves in a porous medium saturated with a liquid containing soluble-gas bubbles; and proposed a physical mechanism of shock wave intensification.

The writers mentioned above all investigated elastic waves in fluid-saturated isotropic porous solid. In fact, the stratum is remarkably anisotropic, and is usually represented as transversely isotropic poroelastic media (Gibson, 1974). Compressional waves and shear waves in fluid-filled transversely isotropic porous solid are all coupled waves. Influences of saturation on shear waves and compressional waves should be given attention to.

Following the introduction, the second section reviews the theory of nearly saturated soil given by Vardoulakis and Beskos (1986). The characteristic equations of the elastic wave propagation in transversely isotropic porous media are derived and the analytical expressions for wave speeds and attenuations are presented in this section. The last section provides numerical results to show the influence of water saturation on

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the velocity, dispersion and attenuation of body waves which propagate in transversely isotropic nearly saturated soil.

THEORETICAL FORMULATION

Theory of nearly saturated soil

One key difference between the fully water-saturated soil and incompletely saturated soil is the pore fluid. For the latter, pore fluid is a mixture of water and air whereas for the former pore fluid is water only. The relative proportions of constituent volumes for incompletely saturated soil can be characterized by porosity n and the degree of saturation S_r as

$$n = \frac{V_\nu}{V_t}, \quad S_r = \frac{V_w}{V_\nu} \tag{1}$$

in which V_ν, V_w are the volumes of pores and pore water, respectively, and V_t is total volume. For a typical case wherein the degree of saturation is sufficiently high (e.g. higher than 95%), the air embedded in the pore water is in the form of bubbles (Bardet and Sayed, 1993; Yang and Sato, 1998). For this special case the concept of homogeneous pore fluid may be applied to the theory of two-phase porous media (Biot, 1956) and the bulk modulus of the homogeneous fluid K_f can be approximately expressed in terms of the degree of saturation as (Verruijt, 1969)

$$K_f = \frac{1}{\dfrac{1}{K_w} + \dfrac{1 - S_r}{p_a}} \tag{2}$$

in which K_w is the bulk modulus of pore water and p_a the absolute fluid pressure. As shown in Fig.1, the air fluid modulus K_f is 2200 MPa for pure water, 100 MPa for $= 99.9\%$, 10 MPa for $S_r = 99\%$. The effect of S_r on K_f is reduced, but not eliminated, by increasing the absolute pressure p_a. As partial saturation significantly reduces the bulk modulus of pore fluid, the assumption of incompressible fluid generally adopted in soil mechanics is inappropriate here.

Field equations and general solutions

The stress-strain relations of transversely isotropic porous media can be expressed as (Biot, 1962)

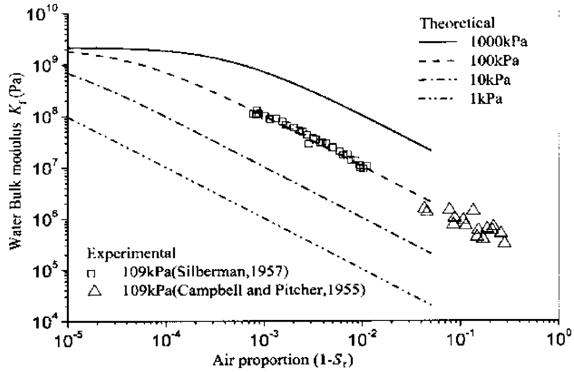

Fig.1 Bulk modulus of water versus degree of saturation, S_r, for various absolute hy drostatic pressure, p_a

$$\left.\begin{aligned}
\tau_{xx} &= (2B_1 + B_2)e_{xx} + B_2 e_{yy} + B_3 e_{zz} + B_6 \zeta \\
\tau_{yy} &= B_2 e_{xx} + (2B_1 + B_2)e_{yy} + B_3 e_{zz} + B_6 \zeta \\
\tau_{zz} &= B_3 e_{xx} + B_3 e_{yy} + B_4 e_{zz} + B_7 \zeta \\
\tau_{yz} &= 2B_5 e_{yz}, \quad \tau_{zx} = 2B_5 e_{zx}, \quad \tau_{xy} = 2B_1 e_{xy} \\
p_f &= B_6 e_{xx} + B_6 e_{yy} + B_7 e_{zz} + B_8 \zeta
\end{aligned}\right\} \tag{3}$$

in which $B_1 - B_8$ are the eight independent elastic constants for transversely isotropic liquid-saturated porous solids and P_f is the pore pressure; $e_{ij} = (u_{i,j} + u_{j,i})/2$ and $\zeta = w_{i,i}$ denote, respectively, the volumetric strain of the solid skeleton and the increment of fluid content; u_i and w_i are, respectively, the displacement of solid skeleton and pore fluid with respect to solid phase.

The equations of motion can be expressed as (Biot, 1956)

$$\tau_{ij,j} = \rho\ddot{u}_i + \rho_f \ddot{w}_i, \quad -p_{f,i} = \rho\ddot{u}_i + \frac{\rho_f}{n}\ddot{w}_i + \frac{\eta}{k_i}\dot{w}_i \tag{4}$$

in which $\rho = (1-n)\rho_s + n\rho_f$ is the total density; ρ_f and ρ_s are the density of pore fluid and grains, respectively; η is the fluid viscosity and $k_i (i = 1,2,3)$ the permeability; $k_1 = k_2$ and k_3 denote, respectively, the horizontal and vertical intrinsic permeabilities.

Substitution of Eq.(4) in Eq.(3) yields the following governing equations for waves propagation

$$(2B_1 + B_2)\frac{\partial^2 u_x}{\partial x^2} + B_1 \frac{\partial^2 u_x}{\partial y^2} + B_5 \frac{\partial^2 u_x}{\partial z^2} +$$
$$(B_1 + B_2)\frac{\partial^2 u_y}{\partial x \partial y} + (B_3 + B_5)\frac{\partial^2 u_z}{\partial x \partial z} -$$

$$B_6 \frac{\partial^2 w_x}{\partial x^2} - B_6 \frac{\partial^2 w_y}{\partial x \partial y} - B_6 \frac{\partial^2 w_z}{\partial x \partial z} = \rho \ddot{u}_x + \rho_f \ddot{w}_x \tag{5a}$$

$$(B_1 + B_2)\frac{\partial^2 u_x}{\partial x \partial y} + B_1 \frac{\partial^2 u_y}{\partial x^2} + (2B_1 + B_2)\frac{\partial^2 u_y}{\partial y^2} + B_5 \frac{\partial^2 u_y}{\partial z^2} + (B_3 + B_5)\frac{\partial^2 u_z}{\partial y \partial z} - B_6 \frac{\partial^2 w_x}{\partial x \partial y} - B_6 \frac{\partial^2 w_y}{\partial y^2} - B_6 \frac{\partial^2 w_z}{\partial y \partial z} = \rho \ddot{u}_y + \rho_f \ddot{w}_y \tag{5b}$$

$$(B_3 + B_5)\frac{\partial^2 u_x}{\partial x \partial z} + (B_3 + B_5)\frac{\partial^2 u_y}{\partial y \partial z} + B_5 \frac{\partial^2 u_z}{\partial x^2} + B_5 \frac{\partial^2 u_z}{\partial y^2} + B_4 \frac{\partial^2 u_z}{\partial z^2} - B_7 \frac{\partial^2 w_x}{\partial x \partial z} - B_7 \frac{\partial^2 w_y}{\partial y \partial z} - B_7 \frac{\partial^2 w_z}{\partial z^2} = \rho \ddot{u}_z + \rho_f \ddot{w}_z \tag{5c}$$

$$B_6 \frac{\partial^2 u_x}{\partial x^2} + B_6 \frac{\partial^2 u_y}{\partial x \partial y} + B_7 \frac{\partial^2 u_z}{\partial x \partial z} - B_8 \frac{\partial^2 w_x}{\partial x^2} - B_8 \frac{\partial^2 w_y}{\partial x \partial y} - B_8 \frac{\partial^2 w_z}{\partial x \partial z} = -\rho_f \ddot{u}_x - \frac{\rho_f}{n}\ddot{w}_x - \frac{\eta}{k_1}\dot{w}_x \tag{5d}$$

$$B_6 \frac{\partial^2 u_x}{\partial x \partial y} + B_6 \frac{\partial^2 u_y}{\partial y^2} + B_7 \frac{\partial^2 u_z}{\partial x \partial z} - B_8 \frac{\partial^2 w_x}{\partial x \partial y} - B_8 \frac{\partial^2 w_y}{\partial y^2} - B_8 \frac{\partial^2 w_z}{\partial y \partial z} = -\rho_f \ddot{u}_y - \frac{\rho_f}{n}\ddot{w}_y - \frac{\eta}{k_1}\dot{w}_y \tag{5e}$$

$$B_6 \frac{\partial^2 u_x}{\partial x \partial z} + B_6 \frac{\partial^2 u_y}{\partial y \partial z} + B_7 \frac{\partial^2 u_z}{\partial z^2} - B_8 \frac{\partial^2 w_x}{\partial x \partial z} - B_8 \frac{\partial^2 w_y}{\partial y \partial z} - B_8 \frac{\partial^2 w_z}{\partial z^2} = -\rho_f \ddot{u}_z - \frac{\rho_f}{n}\ddot{w}_z - \frac{\eta}{k_3}\dot{w}_z \tag{5d}$$

$B_1 - B_8$ can be given as (Kazi-Aoual *et al.*, 1988)

$$\left.\begin{aligned}
&B_1 = c_{66},\ B_2 = c_{12} + \frac{B_6^2}{B_8},\ B_3 = c_{33} + \frac{B_6 B_7}{B_8} \\
&B_4 = c_{33} + \frac{B_7^2}{B_8},\ B_5 = c_{44}, \\
&B_6 = -\left[1 - \frac{c_{11} + c_{12} + c_{13}}{3K_s}\right] B_8 \\
&B_7 = -\left[1 - \frac{c_{13} + c_{33}}{3K_s}\right] B_8 \\
&B_8 = \left[\frac{1-n}{K_s} + \frac{n}{K_f} - \frac{1}{9K_s^2}(2c_{11} + 2c_{12} + 4c_{13} + c_{33})\right]^{-1}
\end{aligned}\right\} \tag{6}$$

in which c_{ij} is the elastic constants of the solid skeleton; K_s is the bulk modulus of solid grains.

We seek Eq. (5)'s plane harmonic wave solution of the form

$$\{\boldsymbol{u}, \boldsymbol{w}\}^T = \{u_i, w_i\}^T = \{\boldsymbol{a}_i, \boldsymbol{b}_i\}^T \exp[ik(l_1 x + l_2 y + l_3 z - ct)] \tag{7}$$

where $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are polarization vectors; $k = \omega / c$ is wavenumber; ω and c are, respectively, the angular frequency and the phase velocity of the wave; $l_i (i = 1, 2, 3)$ is the direction cosine of the waves. We rotate the elastic tensor by the usual tensor transformation so that the plane $y = 0$ contains the propagation vector making an angle θ with the vertical axis (axis). Without loss of generality, we take $l_1 = \sin\theta$, $l_2 = 0$, $l_3 = \cos\theta$.

Substituting the expressions for displacements from Eq. (7) into Eq. (5), we obtain

$$\begin{bmatrix} \omega^2 \rho - (B_1 k_1^2 + B_5 k_3^2) k^2 & \omega^2 \rho_f \\ \omega^2 \rho_f & \omega^2 (\rho_f / n + i\eta / (k_1 \omega)) \end{bmatrix} \begin{Bmatrix} a_y \\ b_y \end{Bmatrix} = 0 \tag{8a}$$

$$[d_{ij}] \{a_x, a_z, b_x, b_z\}^T = 0 \tag{8b}$$

where $[d_{ij}]$ is a 4×4 matrix, and the elements will be shown in APPENDIX Ⅰ.

Nondependence of a_y and b_y on a_x, a_z, b_x and b_z in the above equations shows the decoupling of SH type of wave. Hence, from Eq. (8a), we have

$$k = \left[\frac{\rho[\rho_f / n + i\eta / (k_1 \omega)] - \rho_f^2}{\rho_f / n + i\eta / (k_1 \omega)} \cdot \frac{\omega^2}{B_1 l_1^2 + B_5 l_3^2}\right]^{\frac{1}{2}} \tag{9}$$

Hence from Eq. (9) we get the phase velocity c and attenuation coefficient α of SH type of wave as

$$c = \omega / \mathrm{Re}(k), \quad \alpha = \mathrm{Im}(k) \tag{10}$$

Now from Eq. (8b) coupling of longitudinal and transverse motion is clear. For a nontrivial solution of these equations, we must have

$$|d_{ij}| = 0 \tag{11}$$

which is equivalent to

$$ak^6 + bk^4 + ck^2 + d = 0 \tag{12}$$

the coefficients of this equation can be found in APPENDIX II. From Eq.(12) we can get k_i ($i = 1, 2, 3$) which are the respective wavenumbers of QP1, QP2, QSV type wave; and the phase velocity and attenuation coefficient of each type of wave can be obtain

$$c_i = \omega/\mathrm{Re}(k_i),\quad \alpha_i = \mathrm{Im}(k_i) \qquad (13)$$

The method for solving a complex coefficients equation can be found in mathematical reference books.

NUMERICAL RESULTS AND CONCLUSIONS

Table 1 gives the physical parameters of the poroelastic formations used in this study. They are given in Schmitt (1989), Bardet and Sayed (1993), Yang and Sato (1998).

Table 1 Transversely isotropic poroelastic formations parameters

c_{11} (GPa)	c_{12} (GPa)	c_{13} (GPa)	c_{33} (GPa)	c_{44} (GPa)	K_s (GPa)	K_w (GPa)	k_1 (m^2)	k_3 (m^2)	ρ_s (kg/m^3)	ρ_f (kg/m^3)	η (Pa•s)	n
0.0957	0.0119	0.0233	0.0832	0.030	35	2.2	10^{-10}	10^{-11}	2600	1000	10^{-3}	0.2

Effects of saturation on the velocities and attenuations of four waves

The computed phase velocities and attenuations of the four types of waves are shown in Fig.2 as functions of air proportion $(1 - S_r)$ at a specific frequency of 10 Hz, for $\theta = \pi/4$. The dimensionless velocity factor ζ_c and attenuation factor ζ_α are, respectively, the ratios of the phase velocity and attenuation of the waves in nearly saturated soil with specific air proportion and the phase velocity and attenuation of the waves in completely saturated soil.

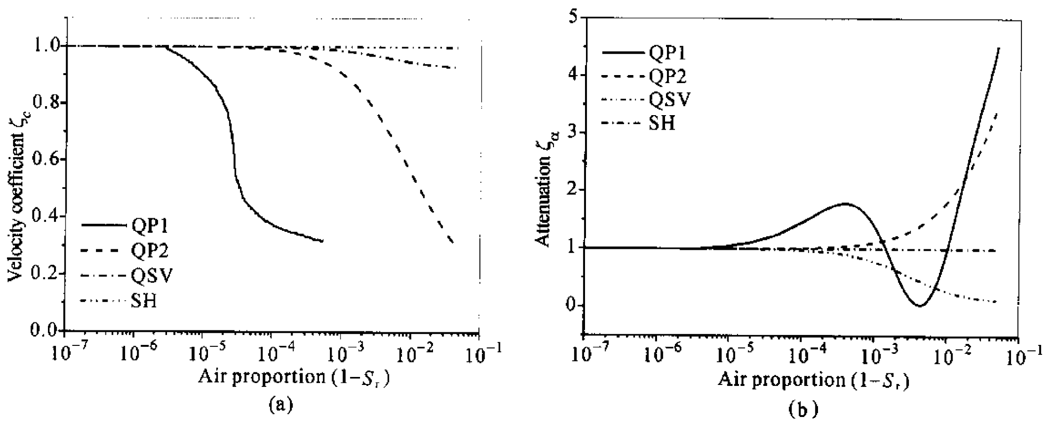

Fig.2 Effects of saturation on the (a) velocitiy and (b) attenuation

It can be seen that the velocity of QP1 wave decreases substantially with even a slight decrease below complete saturation. The velocity reaches maximum in the case of full saturation, and when the degree of saturation is 95%, ζ_c reduces from 1.0 to 0.12. The influence of air inclusion on the QP2 wave is also noticeable. When the degree of saturation is 95%, ζ_c reduces from 1.0 to 0.28. For SH wave, as expected, the influence of saturation is negligible. QSV wave is no longer classical transverse wave, but quasi-transverse wave. The velocity of QSV also slightly decreases with the increase of air proportion in soil, and when the degree of saturation is 95%, ζ_c reduces from 1.0 to 0.92.

The attenuation of QP1 wave rapidly fluctuates while air proportion increases in soil; when the degree of saturation is 99.96%, ζ_α reaches maximum of 1.79, and when the degree of saturation is 99.6%, ζ_α reaches minimum of 0.03. The attenuation of QP2 wave increases rapidly with increasing air proportion, and when the degree of saturation is 95%, ζ_α increases from 1 to 3.46. Opposite to

QP2 wave, the attenuation of QSV wave decreases with increasing air proportion, and when the degree of saturation is 95%, ζ_{α} decreases from 1 to 0.12. As expected, saturation has no influence on the attenuation of SH wave.

Three quasi-body waves in deferent propagation direction

Fig. 3 shows the velocities and attenuations of quasi-body waves as a function of the angle of propagation direction. The frequency is taken as 10 Hz for computation.

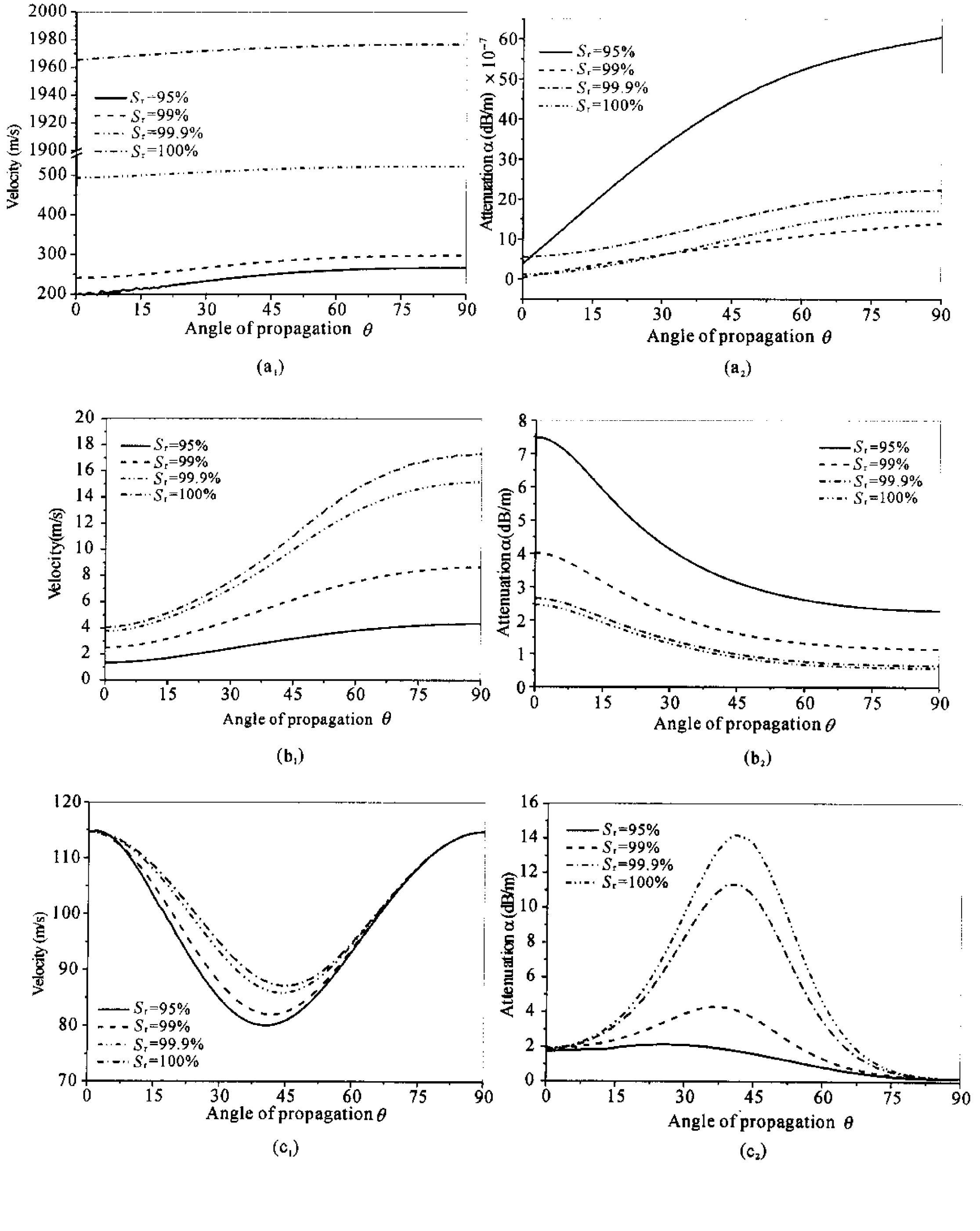

Fig.3 Velocities and attenuations of waves in deferent propagation directions

(a_1) Velocity of QP1 wave; (a_2) Attenumtion of QP1 wave; (b_1) Velocity of QP2 wave; (b_2) Attenumtion of QP2 wave; (c_1) Velocity of QSV wave; (c_2) Attenumtion of QSV wave

It can be seen that the velocity of QP1 wave increases slightly as θ increases, while the velocity of QP2 wave increases significantly. And in any propagation direction, the velocities of QP1 wave and QP2 wave decrease substantially with decreasing saturation. For QP1 wave, as the degree of saturation decreases, the influences of propagation direction on the velocity and attention increase accordingly. The velocity of QP1 wave increases less than 1% in the case of full saturation, and when the degree of saturation is 95%, it increases 37%. The other way round, as the degree of saturation decreases, the influences of propagation direction on the velocity and attention of QP2 wave decrease accordingly. The velocity of QP2 wave increases from 4.05 m/s to 17.3 m/s in the case of full saturation, and when the degree of saturation is 95%, it increases from 1.33 m/s to 4.32 m/s.

The velocity of QSV wave first decreases with increasing θ, and when $\theta = 42°$ the velocity reaches minimum; the velocity of QSV wave increases with increasing θ, and when $\theta = 90°$ the velocity reaches maximum. The effects of propagation direction on velocity and attenuation of the three quasi-body waves increase as the degree of saturation decreases, accordingly, and the effect is more apparent for QSV wave.

Velocity-frequency curve and attenuation-frequency curve of three quasi-body waves

The calculated dispersion and attenuation of the three quasi-body waves are shown in Fig.4 as functions of frequency. Two concepts are introduced here: critical frequency f_{ci} (Biot, 1956) and attenuation quality factor ζ_α (Schmitt, 1989)

$$f_{c1} = \frac{n\eta}{2\pi\rho_f k_1} = 318.3 \text{ Hz} \qquad (14a)$$

$$f_{c3} = \frac{n\eta}{2\pi\rho_f k_3} = 3183.1 \text{ Hz} \qquad (14b)$$

$$\zeta_\alpha = \frac{2\mathrm{Im}(k)}{\mathrm{Re}(k)}$$

where k is the complex wavenumber.

a) Low frequency zone ($f < f_{c1}$): The velocities of QP1 and QSV hardly change with frequencies, but as the frequencies increase, the attenuations increase accordingly. In the low-frequency zone, viscosity is the most important factor causing attenuation, which increases with increasing frequency. For QP2 wave, the dispersion is remarkable and the velocity gradually increases from zero; the attenuation is greater than that of the two waves mentioned above. The degree of saturation has obvious impact on the attenuations of the QP1 wave and QSV wave because of the coupling between fluid-flow waves and body waves.

b) Critical frequency zone ($f_{c1} \leqslant f < f_{c3}$): Here the velocities of the three quasi-body waves fluctuate, and the attenuations obviously peak value or begin to decrease. Effects of the degree of saturation on the dispersions of the three quasi-body waves are significant too. The dispersions of QP1 and QP2 become weaker as the degrees of saturation decrease. The case for QSV wave is more complex: when the degree of saturation exceeds 99.9%, the velocity decreases, otherwise the velocity increases with increasing frequency. The influence of the degree of saturation on the attenuations of the three quasi-body waves is also obvious. The attenuation caused by viscosity and fluid-flow waves is gradually counteracted by inertia.

c) High frequency zone ($f_{c3} \leqslant f$): The velocities of the three quasi-body waves tend to a steady-state value, and the influence of the degree of saturation on the velocities reaches a maximum. As the frequencies increase, the attenuations of the three quasi-body waves decrease gradually, as the effect of inertia gets stronger and stronger while the effects of viscosity and fluid-flow waves become weaker and weaker.

CONCLUSIONS

The two-phase theory of Biot (1956) was used to derive expressions for the velocity and attenuation of elastic waves within transversely isotropic nearly saturated soil. Influence of the degree of saturation on the four body waves is considered in detail. The above obtained results are useful for soil dynamics and earthquake engineering, and are especially relevant to the dynamic analysis of ground motion amplification in geotechnical engineering.

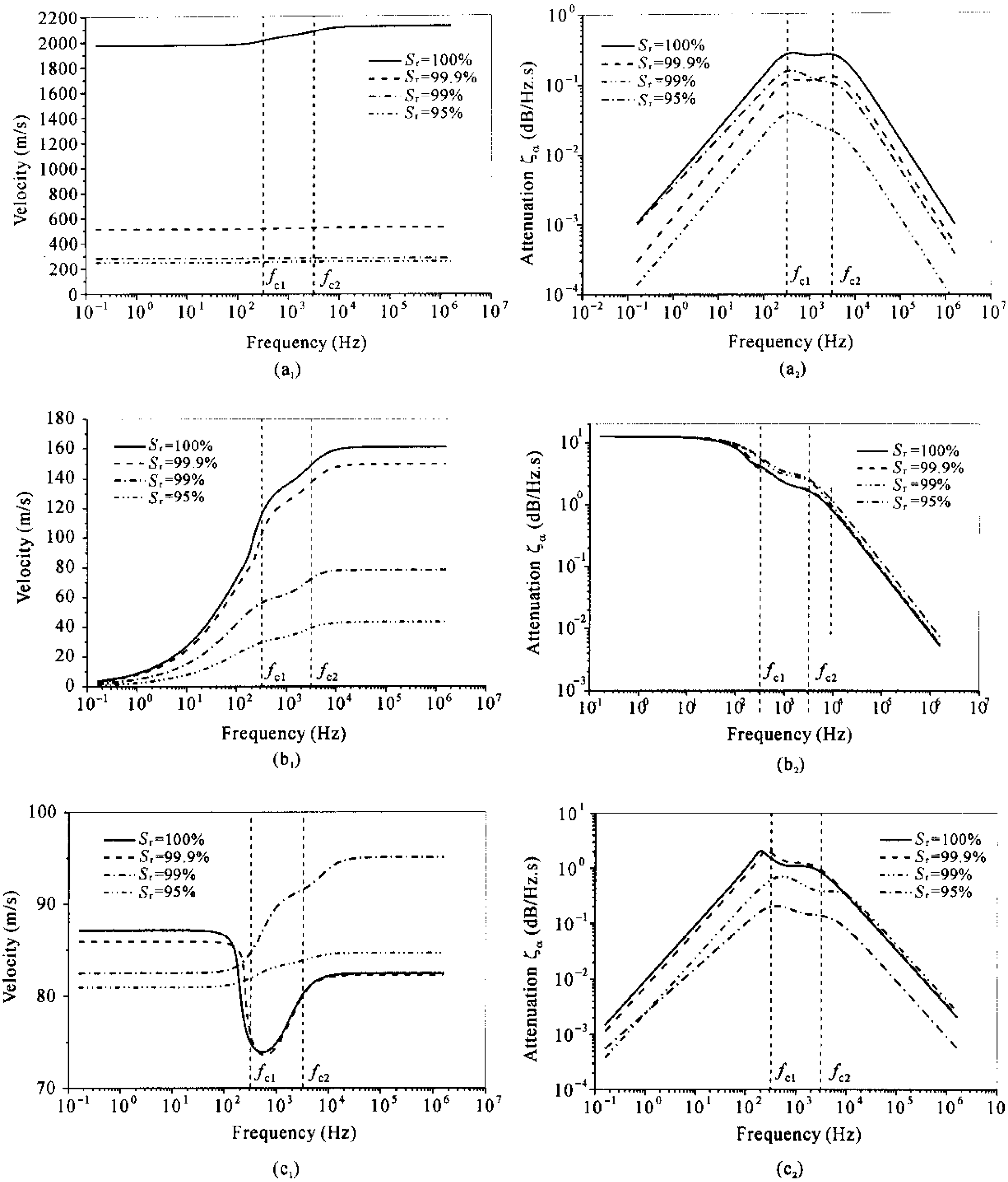

Fig.4 Velocities-frequency curve and attenuations-frequency curve

(a_1) Velocity of QP1 wave; (a_2) Attenumtion of QP1 wave; (b_1) Velocity of QP2 wave; (b_2) Attenumtion of QP2 wave; (c_1) Velocity of QSV wave; (c_2) Attenumtion of QSV wave

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APPENDIX Ⅰ

$$\left.\begin{aligned}
d_{11} &= \omega^2\rho - [(2B_1 + B_2)l_1^2 + B_5 l_3^2]k^2 \\
d_{12} &= d_{21} = -(B_3 + B_5)l_1 l_3 k^2 \\
d_{13} &= d_{31} = \omega^2\rho_f + B_6 l_1^2 k^2 \\
d_{14} &= d_{41} = B_6 l_1 l_3 k^2 \\
d_{22} &= \omega^2\rho - (B_5 l_1^2 + B_4 l_3^2)k^2 \\
d_{23} &= d_{32} = B_7 l_1 l_3 k^2 \\
d_{24} &= d_{42} = \omega^2\rho_{\mathrm{f}} + B_7 l_3^2 k^2 \\
d_{33} &= \omega^2\left(\frac{\rho_{\mathrm{f}}}{n} + \frac{i\eta}{\omega k_1}\right) - B_8 l_1^2 k^2 \\
d_{34} &= d_{43} = -B_8 l_1 l_3 k^2 \\
d_{44} &= \omega^2\left(\frac{\rho_{\mathrm{f}}}{n} + \frac{i\eta}{\omega k_3}\right) - B_8 l_3^2 k^2
\end{aligned}\right\} \quad (1)$$

APPENDIX Ⅱ

$$a = \omega^2(B_8 D_2^2 - 2B_6 B_7 D_2 l_1 l_3 - B_8 D_1 D_3 + B_6^2 D_3 l_1^2 + B_7^2 D_1 l_3^2)(\widetilde{m}_1 l_3^2 + \widetilde{m}_3 l_1^2) \quad (1)$$

$$b = \omega^4[D_1 D_3 - D_2^2)\widetilde{m}_1\widetilde{m}_3 + 2\rho_{\mathrm{f}} l_1(B_6 D_3 \widetilde{m}_3 l_1 - B_6 D_2 \widetilde{m}_1 l_3 - B_7 D_2 \widetilde{m}_3 l_3) + \rho\widetilde{m}_3 l_1^2(B_8 D_3 + B_8 D_1 - B_6 l_1^2) + \rho\widetilde{m}_1 l_3^2(B_8 D_1 + B_8 D_3 + 2B_7 D_1 - B_7^2 l_3^2) - \rho l_1^2 l_3^2(B_6^2\widetilde{m}_1 - B_7^2\widetilde{m}_3) + \rho_{\mathrm{f}}^2(B_6^2 l_1^4 - 2B_8 D_2 l_1 l_3 - B_8 D_1 l_1^2 - B_8 D_3 l_3^2 + 2B_6 B_7 l_1^2 l_3^2 + B_7^2 l_3^4)] \quad (2)$$

$$c = -\omega^6[-2\rho_{\mathrm{f}}^3(B_6 l_1^2 - B_7 l_3^2) + \rho\widetilde{m}_1\widetilde{m}_3(D_1 + D_3) + 2\rho_{\mathrm{f}}(B_7\widetilde{m}_1 l_3^2 + \rho B_6\widetilde{m}_3 l_1^2) + \rho^2 B_8(\widetilde{m}_1 l_3^2 + \widetilde{m}_2 l_1^2) + \rho_{\mathrm{f}}^2(D_3\widetilde{m}_3 - \rho B_8 l_3^2 - \rho B_8 l_1^2 - D_1\widetilde{m}_1)] \quad (3)$$

$$d = \omega^8(\rho\widetilde{m}_3 - \rho_{\mathrm{f}}^2)(\widetilde{m}_1\rho - \rho_{\mathrm{f}}^2) \quad (4)$$

$$\widetilde{m} = \frac{\rho_{\mathrm{f}}}{n} + \frac{i\eta}{\omega k_1}, \quad \widetilde{m}_3 = \frac{\rho_{\mathrm{f}}}{n} + \frac{i\eta}{\omega k_3} \quad (5)$$

$$\begin{aligned}
D_1 &= (2B_1 + B_2)l_1^2 + B_5 l_3^2 \\
D_2 &= (B_3 + B_5)l_1 l_3 \\
D_3 &= B_5 l_1^2 + B_4 l_3^2
\end{aligned} \quad (6)$$