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A unified convergence theory of a numerical method, and applications to the replenishment policies*

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Abstract: In determining the replenishment policy for an inventory system, some researchers advocated that the iterative method of Newton could be applied to the derivative of the total cost function in order to get the optimal solution. But this approach requires calculation of the second derivative of the function. Avoiding this complex computation we use another iterative method presented by the second author. One of the goals of this paper is to present a unified convergence theory of this method. Then we give a numerical example to show the application of our theory.

Key words: Inventory, Shortages, Deterioration, Zero of derivative, Iteration, Optimization theory
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INTRODUCTION

In this paper we will discuss some problems on inventory replenishment policies for deteriorating items with shortages in a declining market. Replenishment policies of deteriorating commodities have been an interesting research topic in inventory managements in the past, and it will remain likely the same in the future. Since Ghare and Schrader (1963) published the first paper considering deterioration of inventory, many papers presented various models with different assumptions on patterns of deterioration with or without shortages. All of them tried to find the optimal solution to make the total cost minimum. So it comes to the basic problem of optimization theory, one dimensional search, which can be reduced to the problem of finding zeros of the derivative of a differentiable function f defined on an open domain D. Naturally, we can use Newton's method or the cubic interpolation method with derivatives to find the zeros. However, Newton's method requires computation of the second derivative while the cubic interpolation method requires computation of the square roots (Yuan and Sun, 1997). In order to avoid these complex computations, Wang (1979) introduced the following convergent iteration method of order two:

$$x_{n+1} = P(f; x_n, x_{n-1}) = x_n - \frac{f'(x_n)}{\delta(f; x_n, x_{n-1})},\quad n = 0, 1, 2, \cdots, \tag{1}$$

where

$$\delta(f; x, y) = \frac{1}{x-y}\left\{4f'(x) - 6\frac{f(x)-f(y)}{x-y} + 2f'(y)\right\}. \tag{2}$$

Under two global conditions:

$$\left|f'''(x)\right| \leqslant M, \qquad \left|f^{(IV)}(x)\right| \leqslant N,$$

the convergence of the iterations (1) was shown in Wang (1979). Later Wang and Li (2001) improved this convergence theorem with only one global condition:

$$\left|f'''(x_0)/f''(x_0)\right| = \gamma,$$
$$\left|f^{(IV)}(x)/f''(x_0)\right| \leqslant L.$$

In this paper we will give a unified convergence theory on the basis of Wang (1999) and Wang

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and Li (2003). Then we give some special cases as consequences, and apply our results to an optimization problem in determining the replenishment policy for an inventory system (Chu and Chen, 2002) in order to get the optimal solution.

Let $\mathscr{K} = \mathscr{R}$ or $\mathscr{C}$, be real or complex, and let $D \subset \mathscr{K}$ be an open convex set. Assume that $f: D \to \mathscr{K}$ has derivatives up to the fourth order. For convenience, we write $F(x) = f'(x)$. Then the iteration method (1) can be rewritten as

$$x_{n+1} = x_n - \frac{F(x_n)}{D(F; x_n, x_{n-1})},\quad n = 0, 1, 2, \cdots, \tag{3}$$

where

$$D(F; x, y) = \frac{1}{x-y}\left\{4F(x) - 6\int_0^1 F(y + \theta(x-y))\mathrm{d}\theta + 2F(y)\right\}, \tag{4}$$

while $F: D \to \mathscr{K}$ has derivatives up to the third order.

In the sequel, for $r > 0$, $x_0 \in \mathscr{K}$, we always set $B(x_0, r) = \{x \in \mathscr{K}: |x - x_0| < r\}$, and $\overline{B(x_0, r)} = \{x \in \mathscr{K}: |x - x_0| \leqslant r\}$. In order to study the convergence we require some lemmas, some of which are directly taken from Wang (1999) and Chu and Chen (2002).

PRELIMINARIES AND LEMMAS

Through out the paper, we always assume that $F'(x_0)^{-1}$ exists, and that $L(u)$ is a positive integrable function on the internal $[0, R]$ for some sufficienly large number $R > 0$, with $\int_0^R (R-u)L(u)\mathrm{d}u = R$. Moreover, we assume the first derivative of $L(u)$, $L'(u)$ is positive integrable and not decreasing monotonically in the internal $[-\eta, R]$, where $\eta = |x_{-1} - x_0|$.

Take $r_0 > 0$ such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1.$$

Set

$$b = \int_0^{r_0} uL(u)\mathrm{d}u.$$

$\bar\beta \in (0, b]$, define

$$h(t) = \bar\beta - t + \int_0^t (t-u)L(u)\mathrm{d}u,\quad \forall\, t \in [0, R].$$

Lemma 1 (Wang 1999) The function h is decreasing monotonically in $[0, r_0]$, but it is increasing monotonically in $[r_0, R]$. Moreover, if $\bar\beta \leqslant b$,

$$h(\bar\beta) > 0,\quad h(r_0) = \bar\beta - b \leqslant 0,\quad h(R) = \bar\beta > 0.$$

Thus h has a unique zero in each interval which are denoted by r_1 and r_2 respectively. They satisfy:

$$\bar\beta < r_1 < \frac{r_0}{b}\bar\beta < r_0 < r_2 < R.$$

$\bar\beta < b$ and $r_1 = r_2$ when $\bar\beta = b$.

Lemma 2 For any $x, y \in D$,

$$D(F; x, y) = F'(x) - (x-y)^2\int_0^1 F'''(y + \theta(x-y))\theta^2(1-\theta)\mathrm{d}\theta. \tag{5}$$

Proof Define $f(x) = \int_0^x F(u)\mathrm{d}u$. Then

$$\frac{f(x) - f(y)}{x-y} = \frac{1}{x-y}\int_y^x F(u)\mathrm{d}u = \int_0^1 F(y + \theta(x-y))\mathrm{d}\theta.$$

Therefore,

$$D(F; x, y) = \frac{1}{x-y}\left\{4f'(x) - 6\frac{f(x)-f(y)}{x-y} + 2f'(y)\right\} = 4f[x, x, y] - 2f[x, y, y] = 2f[x, x, x] - 2f[x, x, x, y](x-y) + 2f[x, x, y, y](x-y) = 2f[x, x, x] - 2f[x, x, x, y, y](x-y)^2 = f''(x) - (x-y)^2\left(\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}\right)f[x, y] = F'(x) - (x-y)^2\int_0^1 F'''(y + \theta(x-y))\theta^2(1-\theta)\mathrm{d}\theta.$$

The proof is completed.

Lemma 3 Let $\{t_n\}$ be given by applying method (3) to h, then t_n increases monotonically and tends to r_1.

Proof From Lemma 2, we have that

$$t_{n+1} \leqslant t_n - \frac{h(t_n)}{h'(t_n)}.$$

It follows that $t_{n+1} \leqslant r_1$ provided that $t_n \leqslant r_1$. Observe that $t_{-1} \leqslant t_0 < r_1$. By induction, we get

$$t_n \leqslant r_1, \quad n = -1, 0, \cdots.$$

Moreover, since

$$h(t) > 0, \; D(h; t, \bar{t}) \leqslant h'(t) < 0, \quad \forall t \in [-\eta, r_1], \; \bar{t} < t,$$

here $\eta = |x_0 - x_{-1}|$. So t_n increases monotonically and tends to r_1. The proof is completed.

Lemma 4 Let $\{x_n\}$ be given by the iteration (3); then

$$F(x_n) = \frac{1}{2}F''(x_0)(x_n - x_{n-1})^2 + \frac{1}{2}\int_0^1 F'''(x_0 + \theta(x_{n-1} - x_0))\mathrm{d}\theta(x_{n-1} - x_0)(x_n - x_{n-1})^2 + \frac{1}{2}\int_0^1 F'''(x_{n-1} + \theta(x_n - x_{n-1}))(1-\theta)^2\mathrm{d}\theta(x_n - x_{n-1})^3 + \int_0^1 F'''(x_{n-2} + \theta(x_{n-1} - x_{n-2}))\theta^2(1 - \theta)\mathrm{d}\theta(x_{n-1} - x_{n-2})^2(x_n - x_{n-1}).$$

Proof It follows from Lemma 2 that

$$F(x_n) = F(x_n) - F(x_{n-1}) - D(F; x_{n-1}, x_{n-2})(x_n - x_{n-1}) = F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1}) - \frac{1}{2}F''(x_{n-1})(x_n - x_{n-1})^2 + \frac{1}{2}F''(x_{n-1})(x_n - x_{n-1})^2 - \frac{1}{2}F''(x_0)(x_n - x_{n-1})^2 + \frac{1}{2}F''(x_0)(x_n - x_{n-1})^2 + \int_0^1 F'''(x_{n-2} + \theta(x_{n-1} - x_{n-2}))\theta^2(1 - \theta)\mathrm{d}\theta(x_{n-1} - x_{n-2})^2(x_n - x_{n-1}).$$

Using Taylor formula, we have the desired result. The proof is completed.

By applying Taylor formula to $F'(x)$, it is not difficult to get

Lemma 5 For any $x_0, x, y \in D$,

$$D(F; x, y) = F'(x_0) + F''(x_0)(x - x_0) + (y - x_0)^2\int_0^1 F'''(x_0 + \theta(y - x_0))(1 - \theta)\mathrm{d}\theta + (x - y)(y - x_0)\int_0^1 F'''(x_0 + \theta(y - x_0))\mathrm{d}\theta + (x - y)^2\int_0^1 F'''(y + \theta(x - y))(1 - \theta)^2(1 + \theta)\mathrm{d}\theta. \tag{6}$$

THE UNIFIED CONVERGENCE THEOREM

In this section we give the unified convergence theory on the basis of Wang (1999) and Wang and Li (2003).

Theorem 1 Suppose that $x_{-1}, x_0 \in D$, $F'(x_0) \neq 0$, and F satisfy

$$|F''(x_0)/F'(x_0)| = L(0),$$

$$|F'''(x)/F'(x_0)| \leqslant L'(|x - x_0|), \quad \forall x \in D \tag{7}$$

with $B(x_0, r_0) \subset D$, $\eta \leqslant r_0$, $\bar{\beta} \leqslant b$, where

$$\eta = |x_0 - x_{-1}|, \quad \beta = |x_1 - x_0|,$$

$$\bar{\beta} = \beta(1 + \eta^2\int_0^1 L'(-\theta\eta)(-1 + 2\theta + \theta^2 - \theta^3)\mathrm{d}\theta),$$

then $\{x_n\} \subset B(x_0, r_0)$ converge to a zero x^* of F.

Proof We shall prove inductively that

$$|x_{n+1} - x_n| \leqslant t_{n+1} - t_n, \tag{8}$$

holds for any $n = 0, 1, \cdots$. In fact the inequality holds trivially for $n = 0, 1$. We assume Eq.(8) holds for all indices from 0 to n; obviously x_{-1}, x_0, x_1, ..., $x_n \in D$, then from Lemma 4 and 5 and the assumption (7) we have that

$$|F(x_n)/F'(x_0)| \leqslant \frac{1}{2}L(0)(t_n - t_{n-1})^2 + \frac{1}{2}\int_0^1 L'(\theta(t_{n-1})\mathrm{d}\theta t_{n-1}(t_n - t_{n-1})^2 + \frac{1}{2}\int_0^1 L'(t_{n-1} + \theta(t_n - t_{n-1}))(1 - \theta)^3\mathrm{d}\theta(t_n - t_{n-2})^2 + \int_0^1 L'(t_{n-2} + \theta(t_{n-1} - t_{n-2}))\theta^2(1 - \theta)\mathrm{d}\theta(t_{n-1} - t_{n-2})^2(t_n - t_{n-1}) = h(t_n). \tag{9}$$

$$|D(F; x_n, x_{n-1})/F'(x_0)| \geqslant 1 - h''(t_0)t_n - t_{n-1}^2\int_0^1 L'(\theta t_{n-1})(1 - \theta)\mathrm{d}\theta - t_{n-1}(t_n - t_{n-1})\int_0^1 L'(\theta t_{n-1})\mathrm{d}\theta - (t_n - t_{n-1})^2\int_0^1 L'(t_{n-1} + \theta(t_n - t_{n-1}))(1 - \theta)^2(1 + \theta)\mathrm{d}\theta =$$

$$- D(h; t_n, t_{n-1}). \qquad (10)$$

Eqs. (9) and (10) imply that Eq. (8) holds for $n+1$ and Eq. (8) is established for all n. This implies that $\{x_n\} \subset B(x_0, r_0)$ converge to a zero x^* of F. Then the proof is completed.

CONSEQUENCES OF THE CONVERGENCE THEOREM

In this section we will take L to be some particular function and then obtain a series of concrete results.

Kantorovich type theorem

Given fixed positive constants γ and L, take

$$L(u) = \gamma + Lu. \qquad (11)$$

Then r_0 is the solution of the equation

$$\int_0^{r_0} (\gamma + Lu)\mathrm{d}u = \gamma r_0 + \frac{1}{2} L r_0^2 = 1,$$

i.e.

$$r_0 = \frac{2}{\gamma + \sqrt{\gamma^2 + 2L}}. \qquad (12)$$

Therefore

$$b = \int_0^{r_0} u(\gamma + Lu)\mathrm{d}u = \frac{2(\gamma + 2\sqrt{\gamma^2 + 2L})}{3(\gamma + \sqrt{\gamma^2 + 2L})^2}. \qquad (13)$$

In this case the majorizing function is

$$h(t) = \bar{\beta} - t + \frac{1}{2}\gamma t^2 + \frac{1}{6} L t^3. \qquad (14)$$

and $r_1 \leqslant r_2$ are its two positive solutions when $\bar{\beta} \leqslant b$. Thus from Theorem 1 we immediately obtain the Kantorovich type theorem (see additional references here).

Theorem 2 (Wang and Li, 2001) Suppose that $x_{-1}, x_0 \in D$, $F'(x_0) \neq 0$, F satisfy

$$\left| F''(x_0)/F'(x_0) \right| = \gamma, \quad \left| F'''(x)/F'(x_0) \right| \leqslant L, \quad \forall x \in D$$

and

$$B(x_0, r_0) \subset D, \quad \eta \leqslant r_0, \quad \bar{\beta} \leqslant b,$$

where

$$\eta = \left| x_0 - x_{-1} \right|, \quad \beta = \left| x_1 - x_0 \right|,$$

$$r_0 = \frac{2}{\sqrt{\gamma^2 + 2L} + \gamma}, \quad \bar{\beta} = \beta\left(1 + \frac{L}{12}\eta^2\right),$$

$$b = \frac{2(\gamma + 2\sqrt{\gamma^2 + 2L})}{3(\gamma + \sqrt{\gamma^2 + 2L})^2}.$$

Then the iteration (3) is well defined and $\{x_n\}$ converge to a solution x^* of the equation $f'(x) = 0$ [$F(x) = 0$] satisfying

$$x^* \in \overline{B(x_0 - F(x_0)/F'(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)}.$$

Moreover, for each r satisfying $r_1 < r < r_2$ if $\bar{\beta} < b$ and $r = r_1$ if $\bar{\beta} = b$, the equation $F(x) = 0$ has a unique solution in the closed ball $\overline{B(x_0, r)}$.

Smale type theorem

For fixed $\gamma > 0$, let

$$L(u) = \frac{2\gamma}{(1 - \gamma u)^3}. \qquad (15)$$

Then by Wang (1999) we have

$$r_0 = \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}, \quad b = (3 - 2\sqrt{2})\frac{1}{\gamma}, \qquad (16)$$

and the majorizing function is

$$h(t) = \beta - t + \frac{\gamma t^2}{(1 - \gamma t)}. \qquad (17)$$

The two positive roots of h are

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1 + \alpha \mp \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}, \qquad (18)$$

where $\alpha = \beta\gamma$. From Theorem 1 we deduce the following Smale type theorem:

Theorem 3 Suppose that $r > r_1$, $\left| F''(x_0)/F'(x_0) \right| = 2\gamma$ and

$$\left| F'''(x)/F'(x_0) \right| \leqslant \frac{6\gamma^2}{(1 - \gamma(\left| x - x_0 \right|))^4}, \quad \forall x \in \overline{B(x_0, r)}. \qquad (19)$$

Let $\beta = \left| F(x_0)/F'(x_0) \right|$ and $\alpha = \beta\gamma \leqslant 3 - 2\sqrt{2}$. Then the iteration (3) is well defined and $\{x_n\}$ converge to a solution x^* of the equation $f'(x) = 0$ [$F(x) = 0$] satisfying

$$x^* \in \overline{B(x_0 - F(x_0)/F'(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)}.$$

Moreover, for each r satisfying $r_1 \leqslant r < r_2$ if $\alpha < 3 - 2\sqrt{2}$ and $r = r_1$ if $\alpha = 3 - 2\sqrt{2}$, the equation $F(x) = 0$ has a unique solution in the closed ball $\overline{B(x_0, r)}$.

APPLICATION

In this section we apply our method (3) to the Example 1 in Chu and Chen(2002).

Assumptions

(a) A single item with a constant rate of deterioration is considered.
(b) Deterioration of the stored products is considered only after the products have been received into inventory.
(c) The system operates only for a prescribed period of H years.
(d) There is no replacement or repair of deteriorated units during the H years.
(e) Demand is known and decreased exponentially.
(f) The rate of replenishment is infinite; replenishment interval is constant.
(g) Carrying cost applies to good units only.
(h) Shortages are allowed except for the final period.
(i) The order quantity, inventory level and demand are treated as continuous variables, and the number of replenishment is treated as a discrete variable.

The simplified total cost function for partial backordering is:

$$TC(n, r) = (\theta C + C_2) e^{(\alpha/n)(1-r)H} \frac{A}{\alpha - \theta} \cdot \left(\frac{e^{(\alpha/n)rH} - 1}{\alpha} - \frac{e^{(\theta/n)rH} - 1}{\theta} \right) X(n) + \left(C + \frac{C_2}{\theta} \right) \cdot e^{-\alpha/n} A \left(\frac{e^{(\alpha/n)H} - e^{(\theta/n)H}}{\alpha - \theta} - \frac{e^{(\alpha/n)H} - 1}{\alpha} \right) + C_4 \cdot \frac{A(1-B)}{\alpha} (e^{(\alpha/n)(1-r)H} - 1) X(n)) + C_3 \frac{AB}{\alpha^2} \left(1 + \left(\frac{\alpha}{n}(1-r)H - 1 \right) e^{(\alpha/n)(1-r)H} \right) X(n) + nC_1, \tag{20}$$

where A notes Initial demand rate; θ Parameter of the deterioration rate function; α A constant governing the decreasing rate of the demand rate; H Length of the whole planning horizon; n Number of replenishments during H; T Length of the replenishment interval, that is, $nT = H$; C_1 Replenishment cost, \$ per order; C_2 Inventory carrying cost, \$ per unit per year; C_3 Backlogged shortage cost, \$ per unit per year; C_4 Lost sale shortage cost, \$ per unit; C Cost of a deteriorated unit; B Fraction of shortage backordered; r Fraction of each cycle in which there is no shortage (100% service level); $TC(n, r)$ Total cost function when there are n replenishmetns and $100r\%$ service;

$$X(n) = \frac{1 - e^{-\alpha H}}{1 - e^{(-\alpha/n)H}} - 1.$$

For a given positive integer n, the optimal value of r that minimizes $TC(n, r)$ must satisfy:

$$\frac{\partial}{\partial r} TC(n, r) = 0, \tag{21}$$

yielding

$$\frac{AH}{n} X(n) e^{(\alpha/n)(1-r)H} f(n, x) = 0, \tag{22}$$

with

$$f(n, r) = \left(C + \frac{C_2}{\theta} \right) (e^{(\theta/n)rH} - 1) - C_3 B(1 - r)\frac{H}{n} - C_4(1 - B). \tag{23}$$

We consider the Example 1 in Chu and Chen (2002). They assumed that $A = 200$, $\alpha = 0.98$, $C = 60$, $C_1 = 150$, $C_2 = 60$, $C_3 = 20$, $C_4 = 90$, $\theta = 0.1$, $B = 0.9$, $H = 10$.

First, we consider the case when $n = 2$. As r is the fraction of each cycle in which there are no shortages (100r% service level). Therefore, we have $0 \leqslant r \leqslant 1$; so we choose $x_{-1} = 0.9$, $x_0 = 1$, and apply our method to find the solution of Eq.(23) by choosing $F(x) = f(2, x)$. It can be easily seen that

$$e^{\frac{x}{2}} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{2^k k!} \leqslant 1 + \frac{x}{2} + \sum_{k=2}^{\infty} \frac{x^k}{2^{k+1}} = 1 + \frac{x}{2} + \frac{\frac{x^2}{8}}{1 - \frac{x}{2}} \leqslant \frac{17}{16} + \frac{x}{2} \quad (x < 1).$$

If we choose

$$L(u) = \frac{165}{4F'(x_0)} u^2 + \frac{165}{2F'(x_0)} \frac{25}{16} u + \frac{165}{F'(x_0)} e^{\frac{1}{2}},$$

$$L'(u)=\frac{165}{4F'(x_0)}u+\frac{165}{2F'(x_0)}\cdot\frac{25}{16},$$

then

$$\left|\frac{F''(x_0)}{F'(x_0)}\right|=L(0)$$
$$\left|\frac{F'''(x)}{F'(x_0)}\right|\leqslant L'(|x-x_0|).$$

Compared with $x=0.225353$ in Chu and Chen (2002), our solution

$$x^*=0.225353366173998326$$

is accurate enough.

Finally (Table 1), we show the solutions when $n=2,3,\cdots,10$ with the initial point x_{-1} calculated by the method in Chu and Chen (2002) and x_0 in Newton's method.

Table 1 Solution of method in Chu and Chen (2002), Newton's method, and our method

n	initial point of x_{-1} and x_0		Chu and Chen's	Newton's	ours
2	0.225162	0.225707	0.225162	0.225353	0.225353
3	0.238661	0.238951	0.238661	0.238661	0.238761
4	0.250767	0.250957	0.250767	0.250832	0.250832
5	0.262305	0.262445	0.262305	0.262353	0.262353
6	0.273557	0.273668	0.273557	0.273595	0.273595
7	0.284644	0.284736	0.284644	0.284675	0.284675
8	0.295627	0.295706	0.295627	0.295654	0.295654
9	0.306541	0.306611	0.306541	0.306565	0.306565
10	0.317407	0.317469	0.317407	0.317428	0.317428

From Table 1 above, our solutions have the same accuracy as that given by Newton's methods. But we do not need to calculate the second derivatives of functions.

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