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Extreme value distributions of mixing two sequences with the same MDA

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Abstract: Suppose $\{X_i, i\geq 1\}$ and $\{Y_i, i\geq 1\}$ are two independent sequences with distribution functions $F_X(x)$ and $F_Y(x)$, respectively. $Z_{i,n}$ is the combination of X_i and Y_i with a probability p_n for each i with $1\leq i\leq n$. The extreme value distribution $G_Z(x)$ of this particular triangular array of the i.i.d. random variables $Z_{1,n}, Z_{2,n}, \cdots, Z_{n,n}$ is discussed. We found a new form of the extreme value distributions i) $\Phi_{\alpha_1}^A(x)\Phi_{\alpha_2}(x)$ and ii) $\Psi_{\alpha_1}^A(x)\Psi_{\alpha_2}(x)$ ($\alpha_1<\alpha_2$), which are not max-stable. It occurs if F_X and F_Y belong to the same $MDA(\Phi)$ or $MDA(\Psi)$.

Key words: Extreme value distribution, Maximum domain of attraction (MDA), Mixed distribution functions
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INTRODUCTION

Suppose $\{X_n, n\geq 1\}$ is a sequence of independent and identically distributed random variables with common continuous distribution function $F_X(x)$. Define $M_n(X) = \max(X_1, X_2, \ldots, X_n)$. We consider the nondegenerated limit distribution of $\Pr\{M_n(X)\leq \alpha_n x+\beta_n\}$ where α_n and β_n are some normalizing constants

$$G_X(x) = \lim_{n\to\infty}\Pr\left\{M_n(X)\leq \alpha_n x+\beta_n\right\}.$$

For i.i.d. random variables X, Fisher and Tippett found in 1928 that limit distributions exist and that there are only three types of distributions, the so-called Extreme Value Distributions and that $G_X(x)$ is either of the following three types:

$$G_X(x) = \begin{cases} \Phi_\alpha(x) = \exp\left\{-x^{-\alpha}\right\} I(x\geq 0), \\ \Lambda(x) = \exp\left\{-\mathrm{e}^{-x}\right\}, x\in\mathbb{R}, \\ \Psi_\alpha(x) = I(x\geq 0) + \exp\left\{-(-x)^\alpha\right\} I(x<0) \end{cases}$$

where α is a positive parameter and Φ_α is called Fréchet Distribution, Λ is called Gumbel Distribution, while Ψ_α is called Weibull Distribution. These three types distributions identify with the so-called max-stable distributions.

One then says that $F(x)$ belongs to the maximum domain of attraction (MDA) of the limit distribution G_X. We can use the one-parameter family of the extreme value distributions

$$H_\rho(x) = \begin{cases} \exp\left\{-(1+\rho x)^{-1/\rho}\right\} & \text{if } \rho\neq 0, \\ \exp\left\{-\exp\left\{-x\right\}\right\} & \text{if } \rho = 0, \end{cases} \tag{1}$$

where $1+\rho x>0$. This family is called standard generalized extreme value distribution (GEV) or Jenkinson-von Mises representation of the extreme value distributions.

We consider in the following work the case of a particular triangular array of the random variables $Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n}$. Let $\{X_i, i\geq 1\}$ and $\{Y_i, i\geq 1\}$ be two independent sequences of independent and identically distributed random variables with distribution functions $F_X(x)\in MDA(G_X)$ and $F_Y(x)\in MDA(G_Y)$, respectively; i.e. there exists normalizing sequences $\alpha_{1,n}$, $\beta_{1,n}$ and $\alpha_{2,n}$, $\beta_{2,n}$ such that

$$\lim_{n\to+\infty} F_X^n(\alpha_{1,n}x+\beta_{1,n}) = G_X(x),$$
$$\lim_{n\to+\infty} F_Y^n(\alpha_{2,n}x+\beta_{2,n}) = G_Y(x).$$

We deal with the case that $\{Z_{1,n}, 1\le i\le n\}$ is a mixture of two independent sequences $\{X_i, i\ge 1\}$ and $\{Y_i, i\ge 1\}$, for $p_n\in[0,1)$ which is defined by:

$$Z_{i,n} = \begin{cases} X_i & \text{with probability } p_n \\ Y_i & \text{with probability } 1-p_n. \end{cases}$$

Hence $\{Z_{i,n}, 1\le i\le n\}$, $n\ge 1$, is an array of independent random variables for fixed n, with distribution

$$F_{Z,n}(x) = p_n F_X(x) + (1-p_n)F_Y(x). \tag{2}$$

We consider the limit of $M_n(Z)=\max(Z_{1,n}, Z_{2,n}, \dots, Z_{n,n})$ and assume

$$G_Z(x) = \lim_{n\to\infty} \Pr\{M_n(Z)\le \alpha_n x+\beta_n\}, \tag{3}$$

for some sequence p_n with $p_n\to p\in[0,1)$ and normalizing sequences α_n, β_n. We investigate the following questions. Does the extreme value distribution $G_z(x)$ of $\{Z_{i,n}, 1\le i\le n\}$ still belong to one of the above three type extreme value distributions? If not, what distributions $G_z(x)$ are possible? What is the influence of $p_n\to p\in[0,1)$ on the form of $G_z(x)$?

In case $p_n\to 0$, we might believe that $G_z(x)$ is equal or close to $G_Y(x)$, since p_n is small and moreover $p_n\to 0$, as $n\to\infty$. But the question is under which conditions can the influence from $\{X_i, i\ge 1\}$ be ignored.

This model is motivated for instance by the idea that the extreme values can be contaminated by some other random variables. In other situations, extreme values are based on observations which stem from several sources with rather different distributions.

PRELIMINARY RESULTS

Define the right endpoint x_F of the distribution $F(x)$ to be $x_F=\sup\{x;\ F(x)<1\}$. It is well-known that if $F(x)\in MDA(\Phi_\alpha)$, then $x_F=\infty$; if $F(x)\in MDA(\Psi_\alpha)$, then $x_F<\infty$ is finite; if $F(x)\in MDA(\Lambda)$, then x_F can be either finite or infinite. If the extreme value distribution $G_Z(x)$ exists, hence Eq.(3) holds, then

$$\begin{aligned} n\overline{F}_{Z,n}(\alpha_n x+\beta_n) &= n(1-F_{Z,n}(\alpha_n x+\beta_n)) \\ &= np_n\overline{F}_X(\alpha_n x+\beta_n) \\ &\quad +n(1-p_n)\overline{F}_Y(\alpha_n x+\beta_n) \\ &\to -\log G_Z(x). \end{aligned}$$

Proposition 1 Suppose $F_X\in MDA(G_X)$ and $F_Y\in MDA(G_Y)$ and satisfy

$$\lim_{x\to x'_F} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = d\in[0,\infty),$$

where $x'_F=\max(x_{F_X}, x_{F_Y})$. If $p_n\to 0$, then with normalizing sequences $\alpha_{2,n}$ and $\beta_{2,n}$,

$$G_Z(x) = G_Y(x).$$

Proof Since $\lim_{n\to\infty}(\alpha_{2,n}x+\beta_{2,n}) = x'_F$, we have

$$n\overline{F}_Y(\alpha_{2,n}x+\beta_{2,n}) \to -\log G_Y(x).$$

Thus

$$\begin{aligned} np_n\overline{F}_X(\alpha_{2,n}x+\beta_{2,n}) &\sim np_n d\overline{F}_Y(\alpha_{2,n}x+\beta_{2,n}) \\ &\sim -p_n d\log G_Y(x)\to 0, \end{aligned}$$

this completes the proof.

From Proposition 1 we note that the mixture influences G_Z such that $G_Z\ne G_Y$, only if F_X and F_Y satisfy: $\lim_{x\to x'_F} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = \infty$. An interesting case occurs if $np_n \overline{F}_X(\alpha_{2,n}x+\beta_{2,n}) \to \text{Const}(x) \neq 0$.

If $np_n \to \infty$, let $\alpha'_{1,n} = \alpha_{1,[np_n]}$, $\beta'_{1,n} = \beta_{1,[np_n]}$, then normalizing sequences $\alpha'_{1,n}$ and $\beta'_{1,n}$ satisfy

$$np_n \overline{F}_X(\alpha'_{1,n}x+\beta'_{1,n}) \to -\log G_X(x).$$

Now we derive the limit results when $F_X(x)$ and $F_Y(x)$ have different right endpoints.

Theorem 1 Suppose the continuous distribution functions $F_X \in MDA(G_X)$ and $F_Y \in MDA(G_Y)$.

i) Assume $x_{F_X} > x_{F_Y}$ and let $np_n \to A \in [0, \infty]$, then

$$G_Z(x) = \begin{cases} G_Y(x) & \text{with } \alpha_n = \alpha_{2,n}, \beta_n = \beta_{2,n}, \text{ if } A = 0 \\ T_X(x) & \text{with } \alpha_n = 1, \beta_n = 0, \quad \text{if } A > 0 \\ G_X(x) & \text{with } \alpha_n = \alpha'_{1,n}, \beta_n = \beta'_{1,n}, \text{ if } A = \infty \end{cases}$$

Where

$$T_X(x) = \begin{cases} 0, & \text{if } x < x_{F_Y} \\ \exp\{-A(1-F_X(x))\}. & \text{if } x_{F_Y} < x < x_{F_X} \end{cases}$$

ii) Assume $x_{F_X} < x_{F_Y}$ then with the normalizing sequence $\alpha_n = \alpha_{2,n}$ and $\beta_n = \beta_{2,n}$

$$G_Z(x) = G_Y(x).$$

Proof i) Setting $\alpha_n = \alpha_{2,n}$ and $\beta_n = \beta_{2,n}$, we have

$$n\overline{F}_{Z,n}(\alpha_{2,n}x+\beta_{2,n}) \sim np_n \overline{F}_X(\alpha_{2,n}x+\beta_{2,n}) - \log G_Y(x). \tag{4}$$

If A=0, the result follows from Eq.(4). If A>0 and

a) if $x < x_{F_Y}$, then $F_X(x)$<1 and $F_Y(x)$<1.

For $n \to \infty$

$$\Pr\{M_n(Z) \le x\} = \{p_n F_X(x) + (1-p_n)F_Y(x)\}^n \le \{\max(F_X(x), F_Y(x))\}^n \to 0.$$

b) if $x_{F_Y} \le x \le x_{F_X}$, then $F_X(x)$<1=$F_Y(x)$=1.

For $n \to \infty$

$$\begin{aligned}\Pr\{M_n(Z) \le x\} &= \{p_n F_X(x) + (1-p_n)F_Y(x)\}^n \\ &= \{1 - p_n(1-F_X(x))\}^n \\ &\sim \exp\{-np_n(1-F_X(x))\} \\ &\to \exp\{-A(1-F_X(x))\},\end{aligned}$$

which implies the claim.

If $A = \infty$, we set $\alpha_n = \alpha'_{1,n}$ and $\beta_n = \beta'_{1,n}$. Since $\alpha'_{1,n}x + \beta'_{1,n} \to x_{F_X}$, for large n, $F_Y(\alpha'_{1,n}x+\beta'_{1,n}) = 0$.

Hence

$$n\overline{F}_{Z,n}(\alpha'_{1,n}x+\beta'_{1,n}) \to -\log G_X(x),$$

the statement follows.

ii) Setting $\alpha_n = \alpha_{2,n}$ and $\beta_n = \beta_{2,n}$. Since $\alpha_{2,n}x+\beta_{2,n} \to x_{F_Y}$, it follows $\overline{F}_X(\alpha_{2,n}x+\beta_{2,n}) = 0$ for large n. Hence, by Eq.(4) the claim follows.

In the following work we derive $G_Z(x)$ if $p_n \to p$, $0 < p < 1$.

Theorem 2 Suppose $F_X \in MDA(G_X)$ and $F_Y \in MDA(G_Y)$. Let $V = \max(X, Y)$ and its extreme value distribution be $G_V(x)$. Then the extreme value distributions $G_Z(x)$ and $G_V(x)$ are of the same type. Let $x'_F = \max(x_{F_X}, x_{F_Y})$. In particular,

i) if $\lim_{x \to x'_F} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = \infty$, then

(a) with normalizing sequences $\alpha_n = \alpha_{1,n}$ and $\beta_n = \beta_{1,n}$

$$G_V(x) = G_X(x).$$

(b) with normalizing sequences $\alpha_n = \alpha'_{1,n}$ and $\beta_n = \beta'_{1,n}$

$$G_Z(x) = G_X(x).$$

ii) if $\lim_{x \to x'_F} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = d \in (0, \infty)$, then the extreme value distributions $G_X(x)$, $G_Y(x)$, $G_Z(x)$ and $G_V(x)$ are of the same type.

iii) If $\lim_{x \to x'_F} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = 0$, then $G_Z(x)$, $G_V(x)$, and

$G_Y(x)$ are of the same type.

Proof Since $F_V(x)=F_X(x)F_Y(x)$, we get

$$\overline{F}_V(x)=\overline{F}_X(x)+\overline{F}_Y(x)-\overline{F}_X(x)\overline{F}_Y(x).$$

i) Since $\lim\limits_{x\to x'_F}\dfrac{\overline{F}_X(x)}{\overline{F}_Y(x)}=\infty$, we have $x_{F_X}\geq x_{F_Y}$ and $\lim\limits_{n\to\infty}(\alpha_{1,n}x+\beta_{1,n})=x'_F$. Thus

$$n\overline{F}_V(\alpha_{1,n}x+\beta_{1,n})\sim n\overline{F}_X(\alpha_{1,n}x+\beta_{1,n})\to-\log G_X(x),$$

implies (a) follows. On the other hand,

$$n\overline{F}_{Z,n}(\alpha'_{1,n}x+\beta'_{1,n})\sim np\overline{F}_X(\alpha'_{1,n}x+\beta'_{1,n})\to-\log G_X(x)$$

implies (b) follows.

ii) Since $\lim\limits_{x\to x'_F}\dfrac{\overline{F}_X(x)}{\overline{F}_Y(x)}=d\in(0,\infty)$, $G_X(x)$ and $G_Y(x)$ are of the same type. Moreover

$$n\overline{F}_V(\alpha_{1,n}x+\beta_{1,n})\to-(1+d^{-1})\log G_X(x)$$

changing the normalizing sequences, we get $G_V(x)=G_X(x)$. On the other hand,

$$n\overline{F}_{Z,n}(\alpha_{1,n}x+\beta_{1,n})\to-(p+(1-p)d^{-1})\log G_X(x),$$

changing the normalizing sequences, we also get $G_Z(x)=G_X(x)$. This completes the proof.

iii) If $\lim\limits_{x\to x'_F}\dfrac{\overline{F}_X(x)}{\overline{F}_Y(x)}=0$. By using the same approach or interchanging $F_X(x)$ and $F_Y(x)$, $G_Z(x)$, $G_Y(x)$ and $G_V(x)$ are of the same type.

By Theorem 1 and Theorem 2 the situation where $x_{F_X}=x_{F_Y}=x_F\leq\infty$ and $p_n\to 0$ is not yet dealt with. According to the different types of *MDA* of distribution functions F_X and F_Y, the following cases can be considered.

1. F_X and F_Y belong to the same *MDA*.

i) F_X and $F_Y\in MDA(\Phi_\alpha)$;

ii) F_X and $F_Y\in MDA(\Psi_\alpha)$;

iii) F_X and $F_Y\in MDA(\Lambda)$.

2. F_X and F_Y belong to different *MDA*'s.

i) F_X and $F_Y\in MDA(\Phi_\alpha)$ or $MDA(\Lambda)$;

ii) F_X and $F_Y\in MDA(\Psi_\alpha)$ or $MDA(\Lambda)$.

In this paper we only discuss the cases in which i) F_X and $F_Y\in MDA(\Phi_\alpha)$ and ii) F_X and $F_Y\in MDA(\Psi_\alpha)$

RESULTS FOR THE SAME RIGHT ENDPOINTS

According to the known theory of extreme value distribution, it is wellknown:

1) $F(x)\in MDA(\Phi_\alpha)$ iff there exists a slowly varying function $L(x)$ with $x_F=\infty$ such that $\overline{F}(x)=x^{-\alpha}L(x)$, where α is a positive parameter.

By Karamata's representation, there exists a measurable function α: $R_+\to R_+$ and a function c: $R_+\to R_+$ and a constant $c>0$ such that

$$\lim_{x\to\infty}c(x)=c>0,\ \lim_{x\to\infty}\alpha(x)=\alpha>0,$$

and for $x>0$

$$\overline{F}(x)=c(x)\exp\left\{-\int_1^x t^{-1}\alpha(t)\mathrm{d}t\right\}.$$

2) $F(x)\in MDA(\Lambda)$ iff there exists a Von Mises function $F^*(x)$ and measurable functions $c(x)$ and $f(x)$ such that $\lim\limits_{x\to x_F}c(x)=c>0$, and

$$\overline{F}(x)=c(x)(1-F^*(x))=c(x)\exp\left\{-\int_{x_0}^x\frac{1}{f(t)}\mathrm{d}t\right\}.$$

for $x_0<x<x_F$ and where f is an auxiliary function with $f>0$ on (x_0, x_F) and f is absolutely continuous with $f'(x)\to 0$, as $x\to x_F$.

3) $F(x)\in MDA(\Psi_\alpha)$ and $x_F<\infty$ iff there exists a slowly varying function $L(x)$ such that

$$\overline{F}(x_F - x^{-1}) = x^{-\alpha} L(x). \tag{5}$$

where α is a positive parameter.

In this section we discuss the situation when $x_{F_X} = x_{F_Y}$. According Proposition 1, it is necessary to consider $\lim_{x\to\infty}\frac{\overline{F}_Y(x)}{\overline{F}_X(x)}$.

Lemma 1 Suppose $F_i(x)$ satisfies

$$\overline{F}_i(x) = c_i(x)\exp\left\{-\int_{x_{0i}}^{x}\frac{1}{g_i(t)}\mathrm{d}t\right\},$$

for $x_{0i}<x<\infty$ and where g_i is an absolutely continuous function with $g_i>0$ on (x_{0i}, ∞), $g'_i(x)$ exists and measurable functions $c_i(x)$ satisfies

$$\lim_{x\to\infty} c_i(x) = c_i > 0, \text{ for } i=1, 2.$$

i) If $\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)} > 1$ and $\lim_{t\to\infty}\frac{t}{g_2(t)} = a \in (0,\infty]$, then

$$\lim_{x\to\infty}\frac{\overline{F}_2(x)}{\overline{F}_1(x)} = \infty.$$

ii) If $\lim_{x\to\infty}\frac{\overline{F}_2(x)}{\overline{F}_1(x)} = d \in (0,\infty)$, then

$$\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)} = 1.$$

Proof i) If $\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)} = \rho > 1$; then

$$\lim_{t\to\infty}\frac{\frac{1}{g_1(t)} - \frac{1}{g_2(t)}}{\frac{1}{g_2(t)}} = \rho - 1 > 0,$$

which implies, for any ε with $\rho - 1 > \varepsilon > 0$, there exists $x_0^*(\varepsilon)$, such that for $\forall t \ge x_0^*(\varepsilon)$,

$$\frac{\frac{1}{g_1(t)} - \frac{1}{g_2(t)}}{\frac{1}{g_2(t)}} \ge \rho - 1 - \varepsilon > 0,$$

and hence

$$\frac{1}{g_1(t)} - \frac{1}{g_2(t)} \ge (\rho - 1 - \varepsilon)\frac{1}{g_2(t)}.$$

Since $\lim_{t\to\infty}\frac{t}{g_2(t)} = a \in (0,\infty]$, we discuss the following cases:

(a) If $a=\infty$, there exists x_1^*, such that for $t\ge x_1^*$, $t\ge g_2(t)$. Thus $\forall t \ge x^*(\varepsilon) = \max(x_0^*(\varepsilon), x_1^*)$,

$$\frac{1}{g_1(t)} - \frac{1}{g_2(t)} \ge (\rho - 1 - \varepsilon)\frac{1}{t} = h\frac{1}{t}. \tag{6}$$

Let $x_0 = \max(x_{01}, x_{02})$

and $b = \exp\left\{-\int_{x_{02}}^{x_0}\frac{1}{g_2(t)}\mathrm{d}t\right\}$ (if $x_{01} \ge x_{02}$)

and $b = \exp\left\{\int_{x_{01}}^{x_0}\frac{1}{g_1(t)}\mathrm{d}t\right\}$ (if $x_{01} < x_{02}$).

For fixed $x^*(\varepsilon)$ and by Eq.(6),

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\overline{F}_2(x)}{\overline{F}_1(x)} \\
&= \lim_{x\to\infty}\frac{c_2(x)}{c_1(x)} b \lim_{x\to\infty}\exp\left\{-\int_{x_0}^{x}\left(\frac{1}{g_2(t)} - \frac{1}{g_1(t)}\right)\mathrm{d}t\right\} \\
&\ge \frac{bc_2}{c_1}\exp\left\{\int_{x_0}^{x^*(\varepsilon)}\left(\frac{1}{g_1(t)} - \frac{1}{g_2(t)}\right)\mathrm{d}t\right\} \cdot \\
&\quad \lim_{x\to\infty}\exp\left\{h\int_{x^*(\varepsilon)}^{x}\frac{1}{t}\mathrm{d}t\right\} \to \infty
\end{aligned}$$

(b) If $a\in(0, \infty)$, the proof is similar as that in (a).

ii) By using L'Hospital's rule for the second factor, we have

$$\lim_{x\to\infty}\frac{\overline{F}_2(x)}{\overline{F}_1(x)}=\frac{c_2}{c_1}\lim_{x\to\infty}\frac{\exp\left\{-\int_{x_{02}}^{x}\frac{1}{g_2(t)}\mathrm{d}t\right\}}{\exp\left\{-\int_{x_{01}}^{x}\frac{1}{g_1(t)}\mathrm{d}t\right\}}\lim_{x\to\infty}\frac{g_1(x)}{g_2(x)}$$

$$=\lim_{x\to\infty}\frac{g_1(x)}{g_2(x)}\lim_{x\to\infty}\frac{\overline{F}_2(x)}{\overline{F}_1(x)},$$

hence

$$\left(1-\lim_{x\to\infty}\frac{g_1(x)}{g_2(x)}\right)\lim_{x\to\infty}\frac{\overline{F}_2(x)}{\overline{F}_1(x)}=0, \tag{7}$$

the result follows.

Lemma 2 $F_1(x)$ is defined in Lemma 1. Assume that $\lim_{n\to\infty}\frac{\overline{F}_1(v_{2,n})}{\overline{F}_1(v_{1,n})}=d\in[0,\infty]$ for sequences $v_{i,n}\to\infty$, as $n\to\infty$, i=1, 2. For any ε with $0<\varepsilon<1$ and large n,

i) if $d\in(0,\infty)$, then $\frac{\overline{F}_2(v_{2,n})}{\overline{F}_2(v_{1,n})}\le d^{1-\varepsilon}$;

if d=0, then $\lim_{n\to\infty}\frac{\overline{F}_2(v_{2,n})}{\overline{F}_2(v_{1,n})}=0$ for any $F_2(x)$ defined in Lemma 1 with $\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)}\le 1$.

ii) if $d\in(0,\infty)$, then $\frac{\overline{F}_2(v_{2,n})}{\overline{F}_2(v_{1,n})}\ge d^{1+\varepsilon}$;

if $d=\infty$, then $\lim_{n\to\infty}\frac{\overline{F}_2(v_{2,n})}{\overline{F}_2(v_{1,n})}=\infty$ for any $F_2(x)$ defined in Lemma 1 with $\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)}\ge 1$.

Proof i) Since $\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)}\le 1$, for any ε with $1>\varepsilon>0$, there exists $t_0(\varepsilon)$, such that

$$\forall t\ge t_0(\varepsilon),\ \frac{1}{g_2(t)}\ge(1-\varepsilon)\frac{1}{g_1(t)}.$$

Hence, for any ε with $1>\varepsilon>0$ and large n, we have for $d>0$

$$\frac{\overline{F}_2(v_{2,n})}{\overline{F}_2(v_{1,n})}\sim\exp\left\{-\int_{v_{1,n}}^{v_{2,n}}\frac{1}{g_2(t)}\mathrm{d}t\right\}$$

$$\le\exp\left\{-\int_{v_{1,n}}^{v_{2,n}}\frac{1}{g_1(t)}(1-\varepsilon)\mathrm{d}t\right\}$$

$$\sim\left(\frac{\overline{F}_1(v_{2,n})}{\overline{F}_1(v_{1,n})}\right)^{1-\varepsilon}\to d^{1-\varepsilon}\in[0,\infty),$$

the statements follow.

ii) Since $\lim_{t\to\infty}\frac{g_2(t)}{g_1(t)}\ge 1$, for any ε with $1>\varepsilon>0$, there exists $t_0(\varepsilon)$, such that

$$\forall t\ge t_0(\varepsilon),\ \frac{1}{g_2(t)}\le(1+\varepsilon)\frac{1}{g_1(t)}$$

Hence,

$$\frac{\overline{F}_2(v_{2,n})}{\overline{F}_2(v_{1,n})}\ge\exp\left\{-\int_{v_{1,n}}^{v_{2,n}}\frac{1}{g_1(t)}(1+\varepsilon)\mathrm{d}t\right\}$$

$$\sim\left(\frac{\overline{F}_1(v_{2,n})}{\overline{F}_1(v_{1,n})}\right)^{1+\varepsilon}\to d^{1+\varepsilon}\in[0,\infty],$$

the statements follow.

F_X and $F_Y\in MDA(\Phi_\alpha)$

In this subsection the case of F_X and $F_Y\in MDA(\Phi_\alpha)$ is dealt with.

Theorem 3 Suppose $F_X\in MDA(\Phi_{a_1})$ and $F_Y\in MDA(\Phi_{a_2})$.

i) If $\frac{a_1}{a_2}>1$, then with the normalizing sequences $\alpha_n=\alpha_{2,n}$ and $\beta_n=0$

$$G_Z(x)=\Phi_{a_2}(x).$$

ii) If $\frac{a_1}{a_2}\le 1$ and $np_n\overline{F}_X(\alpha_{2,n})\to A\in[0,\infty]$ as $n\to\infty$, then

$$G_Z(x)=\begin{cases}\Phi_{a_1}^{A}(x)\Phi_{a_2}(x) & \text{with } \alpha_n=\alpha_{2,n},\ \beta_n=0, \text{ if } A\ge 0;\\ \Phi_{a_1}(x) & \text{with } \alpha_n=\alpha'_{1,n},\ \beta_n=0, \text{ if } A=\infty.\end{cases}$$

Proof i) Since $\frac{a_1}{a_2} > 1$, by Lemma 1 we have $\lim_{x\to\infty}\frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = 0$. By Proposition 1, the result follows.

ii) Since $\frac{a_1}{a_2} \le 1$, by Eq.(2) we have

$$\begin{aligned} n\overline{F}_{Z,n}(\alpha_{2,n}x) &\sim np_n\overline{F}_X(\alpha_{2,n}x) + x^{-a_2} \\ &= (np_n\overline{F}_X(\alpha_{2,n}))\frac{\overline{F}_X(\alpha_{2,n}x)}{\overline{F}_X(\alpha_{2,n})} + x^{-a_2} \\ &\sim (np_n\overline{F}_X(\alpha_{2,n}))x^{-a_1} + x^{-a_2}. \end{aligned}$$

a) If $np_n\overline{F}_X(\alpha_{2,n}) \to A$, as $n\to\infty$, $A \ge 0$, then the results follow.

b) If $np_n\overline{F}_X(\alpha_{2,n}) \to \infty$, then $np_n \to \infty$. Setting $\alpha_n = \alpha'_{1,n}$, $np_n\overline{F}_X(\alpha'_{1,n}) \to 1$ and

$$\lim_{n\to\infty}\frac{\overline{F}_X(\alpha'_{1,n})}{\overline{F}_X(\alpha_{2,n})} = \lim_{n\to\infty}\frac{np_n\overline{F}_X(\alpha'_{1,n})}{np_n\overline{F}_X(\alpha_{2,n})} \to 0.$$

Now by Lemma 2

$$0 \le n\overline{F}_Y(\alpha'_{1,n}x) \sim x^{-\alpha_2}\frac{\overline{F}_Y(\alpha'_{1,n})}{\overline{F}_Y(\alpha_{2,n})} \to 0.$$

Hence, $n\overline{F}_Y(\alpha'_{1,n}x) \to 0$ and

$$\begin{aligned} n\overline{F}_{Z,n}(\alpha'_{1,n}x) &= np_n\overline{F}_X(\alpha'_{1,n}x) + n(1-p_n)\overline{F}_Y(\alpha'_{1,n}x) \\ &\sim x^{-\alpha_1} + n\overline{F}_Y(\alpha'_{1,n}x) \to x^{-\alpha_1}, \end{aligned}$$

the statement follows.

Remark 1 Only if $\alpha_1<\alpha_2$ and $A>0$, $G_Z(x)$ can exist as the mixture form $\Phi_{a_1}^A(x)\Phi_{a_2}(x)$, because if $\alpha_1 = \alpha_2$ and $np_n\overline{F}_X(\alpha_{2,n}) \to A > 0$, by Theorem 3

$$n\overline{F}_{Z,n}(\alpha_{2,n}x) \to Ax^{-a_2} + x^{-a_2} = (1+A)x^{-a_2}.$$

Setting $\alpha_n = (1+A)^{1/a_2}\alpha_{2,n}$, we get $n\overline{F}_{Z,n}(\alpha_n x) \to x^{-a_2}$, which implies with this new normalizing sequences we get $G_Z(x) = \Phi_{a_2}(x)$.

F_X* and *$F_Y \in MDA(\Psi_\alpha)$

In this subsection the case of F_X and $F_Y \in MDA(\Psi_\alpha)$ is dealt with.

Assume $x_{F_X} = x_{F_Y} = x_F < \infty$ and $\overline{F}_Y(\gamma_{2,n}) \sim \frac{1}{n}$.

If $np_n \to \infty$, let $\overline{F}_X(\gamma'_{1,n}) \sim \frac{1}{np_n}$.

Theorem 4 Suppose $F_X \in MDA(\Psi_{a_1})$ and $F_Y \in MDA(\Psi_{a_2})$.

i) If $\frac{a_1}{a_2} > 1$, then with the normalizing sequences $\alpha_n = x_F - \gamma_{2,n}$ and $\beta_n = x_F$, for $x < 0$

$$G_Z(x) = \exp\{-(-x)^{a_2}\} = \Psi_{a_2}(x).$$

ii) If $\frac{a_1}{a_2} \le 1$ and $np_n\overline{F}_X(\gamma_{2,n}) \to A \in [0,\infty]$, as $n\to\infty$, then

$$G_Z(x) = \begin{cases} \Psi_{a_1}^A(x)\Psi_{a_2}(x) \text{ with } \alpha_n = x_F - \gamma_{2,n}, \beta_n = x_F, \text{ if } A \ge 0; \\ \Psi_{a_1}(x) \quad \text{ with } \alpha_n = x_F - \gamma'_{1,n}, \beta_n = x_F, \text{ if } A = \infty. \end{cases}$$

Proof Since $x_{F_X} = x_{F_Y} = x_F < \infty$, by Eq.(5)

$$\overline{F^*}_X(x) = \overline{F}_X(x_F - x^{-1}) = x^{-a_1}L_1(x)$$

and

$$\overline{F^*}_Y(x) = \overline{F}_Y(x_F - x^{-1}) = x^{-a_2}L_2(x)$$

implying $F^*_X(\mathrm{x}) \in MDA(\Phi_{a_1})$ and $F^*_Y(\mathrm{x}) \in MDA(\Phi_{a_2})$ with $x_{F^*_X} = x_{F^*_Y} = \infty$, for all $x>0$. By using this relationship and Theorem 3, it is easy to prove this theorem.

CONCLUSION

Now we show that the extreme value distributions $\Phi^A_{\alpha_1}(x)\Phi_{\alpha_2}(x)$ and $\Psi^A_{\alpha_1}(x)\Psi_{\alpha_2}(x)$ ($\alpha_1<\alpha_2$) are not max-stable distributions, and hence they do not belong to the three types extreme value distributions.

Theorem 5 If $A\in(0, \infty)$, then the extreme value distributions $\Phi^A_{\alpha_1}(x)\Phi_{\alpha_2}(x)$ and $\Psi^A_{\alpha_1}(x)\Psi_{\alpha_2}(x)$ ($\alpha_1<\alpha_2$) are not max-stable distribution functions.

Proof i) Suppose $G_Z(x)=\Phi^A_{\alpha_1}(x)\Phi_{\alpha_2}(x)$ ($\alpha_1<\alpha_2$).

Suppose that $\Phi^A_{\alpha_1}(x)\Phi_{\alpha_2}(x)$ is a max-stable distribution, which means that there exist constants $a_k>0$ and b_k such that

$$\Phi^{kA}_{\alpha_1}(a_kx+b_k)\Phi^k_{\alpha_2}(a_kx+b_k)=\Phi^A_{\alpha_1}(x)\Phi_{\alpha_2}(x).$$

By taking logarithms and expanding it, this is equivalent to

$$Ak(a_kx+b_k)^{-\alpha_1}+k(a_kx+b_k)^{-\alpha_2}=Ax^{-\alpha_1}+x^{-\alpha_2}.$$

For fixed k, let $x\to 0$, then the right hand side of the above equation tends to ∞, if $b_k\neq 0$, then the left hand side is bounded, which is a contradiction. Hence $b_k=0$ and

$$Ak(a_kx)^{-\alpha_1}+k(a_kx)^{-\alpha_2}=Ax^{-\alpha_1}+x^{-\alpha_2} \quad (8)$$

Let $x\to 0$, by Eq.(8) we have $k(a_kx)^{-\alpha_2}\sim x^{-\alpha_2}$, implying $a_k=k^{1/\alpha_2}$. Putting it into Eq.(8)

$$Ak^{1-\frac{\alpha_1}{\alpha_2}}x^{-\alpha_1}+x^{-\alpha_2}=Ax^{-\alpha_1}+x^{-\alpha_2},$$

which for all k leads to $k^{1-\frac{\alpha_1}{\alpha_2}}=1$. Hence, $\alpha_1=\alpha_2$, which is a contradiction.

ii) Suppose $G_Z(x)=\Psi^A_{\alpha_1}(x)\Psi_{\alpha_2}(x)$ ($\alpha_1<\alpha_2$), we can use the same method as in proof of i).

References

Bingham, N.H., Goldie, C.M., and Teugels, J.L., 1983. Regular Variation. Springen-Verlag, Berlin.

Embrechts, P., Klueppelberg, C., Mikosch, T., 1997. Modelling Extremal Events. Springer-Verlag, Berlin.

Jiang, Y.X., 2002. Mixed Extreme Value Distributions. Ph.D. Thesis, Berne University, Switzerland.

Leedbetter, M.R., Lingen, G., Rootzan, H., 1983. Extremes and Related Properties of Random Sequences and Processes. Springen-Verlag, Berlin.

Resnick, S.I., 1987. Extreme Values, Regular Variation, and Point Processes. Springer-Verlag, Berlin.

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