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PH-spline approximation for Bézier curve and rendering offset*

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Abstract: In this paper, a G^1, C^1, C^2 PH-spline is employed as an approximation for a given Bézier curve within error bound and further renders offset which can be regarded as an approximate offset to the Bézier curve. The errors between PH-spline and the Bézier curve, the offset to PH-spline and the offset to the given Bézier curve are also estimated. A new algorithm for constructing offset to the Bézier curve is proposed.

Keywords: PH-spline, Bézier curve, Offset, Approximation, Error
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INTRODUCTION

Offsets are used in many industrial applications, such as tool paths in numerical-control (NC) machining, planning paths for mobile robots and in CAD/CAM fields. The parametric representation of curve in CAGD is based on employing polynomial or rational function. Plane curve and its offsets are usually defined by parametric forms such as $\boldsymbol{r}(t) = (x(t), y(t))$; offsets are $\boldsymbol{r}_{\mathrm{d}}(t)=\boldsymbol{r}(t)\pm d\boldsymbol{n}(t)$, $\boldsymbol{n}(t)$ is normal vector of $\boldsymbol{r}(t)$, d is distance along $\boldsymbol{n}(t)$. But the generation of offset curves is not a simple task because $\boldsymbol{n}(t)$ in general has no rational expression. So far, authors have presented some remarkable approaches which can be categorized as follows:

1. Approximation. Klass (1983) constructed a cubic spline as an approximate offset to another cubic segment, where both curves can be defined in term of their endpoints and tangents at these points. Tiller and Hanson (1984) first calculated the offset lines for the original B-spline. The control points for the offset curve can be obtained from the points of intersection of offset lines. Coquillar (1987) constructed each control vertex for offset curve by offsetting a control vertex of the original curve along the normal direction at the point where the curve is closest to the original vertex. Hoschek and Wissel (1988) introduced a method for merging and splitting Bézier curve segments of different degrees that is based on the square error that sum between the original and approximation is minimized. Bercovier and Jacobi (1994) used a method based on a min-max problem which describes approximation and differential geometric characteristics under some constraints to render offset to the Bézier curve. Li (1998) employed Legendre series as approximation for the offset to the original polynomial curve.

2. Accuracy. Farouki (1990; 1994) advocated PH curve whose offset has rational polynomial form. Moon and Farouki (2001) analyzed how to construct PH curve according to the average cubic curve. Choi and Han (1999) gave a simple proce-

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dure to obtain offset curve in terms of the MPH curve. Lü (1995) proved the condition of exact offset to the Bézier curve and developed explicit representation for offset.

The uniform accuracy algorithm has shortcoming for ignoring the difference between the PH curve and the designed polynomial curve. The approximation algorithm yields high degree approximation curve with improved accuracy. In this paper, a new method combining the previous two algorithms will be presented.

PH-SPLINE APPROXIMATION FOR THE BÉZIER CURVE

Definition 1 Let $\boldsymbol{r}(t)$ = ($x(t)$, $y(t)$) be a given polynomial curve, $\boldsymbol{r}(t)$ is called a Pythagorean-hopograph curve (PH curve) if there exists a polynomial such that

$$x'^2(t) + y'^2(t) = \sigma^2(t).$$

PH curve is distinguished from polynomial parametric curve in general by a desirable property that the offset to the PH curve can be expressed in rational polynomial form.

The basic strategy of PH-spline approximation for Bézier curve

We shall employ the complex representation for planar curve $\boldsymbol{r}(t)$=($x(t)$, $y(t)$), wherein a plane curve is regarded as a complex-valued function $\boldsymbol{r}(t)$ =$x(t)$+i$y(t)$,where the bold character i represents $\sqrt{-1}$. Now a problem appears: For a given Bézier curve of degree n (n>2), $\boldsymbol{b}(t)=\sum_{j=0}^{n}\boldsymbol{b}_j B_j^n(t)$, where $\boldsymbol{b}_j$ are control points. To solve the problem of how to obtain a proper approximation for $\boldsymbol{b}(t)$ whose expression is compatible with the current CAD/CAM system, it is desirable to employ a PH-spline curve approximation for the given Bézier curve within a given error like arc spline because of the special property of the PH curve. First, $\boldsymbol{b}(t)$ is subdivided into N+1 segments of Bézier curves after N knots are inserted into the parameter interval [0,1] for t, namely

$$\boldsymbol{b}^{(i)}(t)=\sum_{j=0}^{n}\boldsymbol{b}_j^{(i)}B_j^n(t) \qquad i=1\ldots N+1$$

where $\boldsymbol{b}_j^{(i)}=b_{xj}^{(i)}+\mathrm{i}\,b_{yj}^{(i)}$. Then for each $\boldsymbol{b}^{(i)}(t)$, let the endpoint conditions of that be as follows: (1). endpoints $\boldsymbol{b}_0^{(i)}$, $\boldsymbol{b}_n^{(i)}$ and their directions; (2). endpoint derivatives $\boldsymbol{d}_0^{(i)}$, $\boldsymbol{d}_1^{(i)}$; (3). endpoint second derivatives $\boldsymbol{B}_0^{(i)}$, $\boldsymbol{B}_1^{(i)}$. The ith PH curve of degree $2p$+1

$$\boldsymbol{p}^{(i)}(t)=\sum_{j=0}^{2p+1}\boldsymbol{p}_j^{(i)}B_j^{2p+1}(t) \qquad i=1\ldots N+1$$

with hodograph

$$\boldsymbol{p}^{(i)\prime}(t)=(\sum_{j=0}^{p}\boldsymbol{\omega}_j^{(i)}B_j^p(t))^2$$

where $\boldsymbol{p}_j^{(i)}=p_{xj}^{(i)}+\mathrm{i}\,p_{yj}^{(i)}$, $\boldsymbol{\omega}_j^{(i)}=\omega_{xj}^{(i)}+\mathrm{i}\,\omega_{yj}^{(i)}$, can be constructed according to different endpoint conditions as stated above. See Fig.1. Finally, a total PH-spline $\boldsymbol{p}(t)=\bigcup_{i=1}^{N+1}\boldsymbol{p}^{(i)}(t)$ will be rendered when the N+1 corresponding PH curves are generated. We regard $\boldsymbol{p}(t)$ as an approximation for $\boldsymbol{b}(t)$. In the next sections, three kinds of PH-splines will be respectively presented in detail.

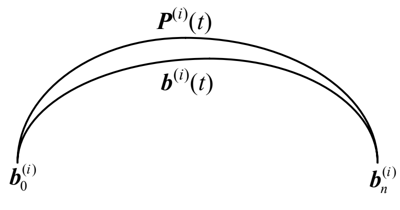

Fig.1 The relationship of $b^{(i)}(t)$ and $P^{(i)}(t)$

Constructing G^1 PH-spline

PH cubic curve, the simplest curve in geometric structure among PH class, will be used to construct G^1 PH-spline. A valuable theorem proven by Farouki (1994) is given by:

Theorem 1 For a cubic $z(t)$ with Bézier control points $z_0 \ldots z_3$, let $L_{i-1}=|z_i - z_{i-1}|$ be the lengths of the control-polygon legs, and θ_1, θ_2 be the control polygon angles at the interior vertices z_1 and z_2. Then the conditions $L_1=\sqrt{L_0 L_2}$ and $\theta_1=\theta_2$ are sufficient and necessary for $z(t)$ to be a PH curve.

With the aid of Theorem 1, another theorem on constructing PH cubic curve in practice can be obtained.

Theorem 2 Let ***A***, ***B*** be two given endpoints, L be the length of ***AB***, α, β be the angles between AB and the direction of point ***A***, ***AB*** and the direction of point B respectively, α, β be no more than 90 degrees, see Fig.2a. There exists a unique characteristic convex quadrilateral generating a PH cubic curve.

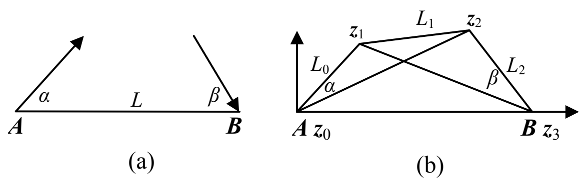

Fig.2 (a) Initial conditions; (b) Polygon for PH cubic curve

Proof A right angle coordinate is established, see Fig.2b. Suppose k_1, k_2, $k_{z_0z_2}$, $k_{z_1z_3}$ be respectively slopes of z_1z_2, z_2z_3, z_0z_2 and z_1z_3.

Because $L_1=\sqrt{L_0L_2}$ and $\angle z_1z_2z_3=\angle z_0z_1z_2$, we have $\dfrac{k_{z_0z_2}-k_1}{1+k_{z_0z_2}k_1}=\dfrac{k_{z_1z_3}-k_2}{1+k_{z_1z_3}k_2}$. Let $\gamma=\dfrac{\alpha-\beta}{2}$. After tedious calculation, the following equation is deduced

$$pL_0^2+qL_0+r=0 \qquad (1)$$

$$\begin{cases} p=(\sin\alpha-\mathrm{tg}\gamma\cos\alpha)[(\mathrm{tg}\beta-\mathrm{tg}\gamma)(\cos\alpha+\mathrm{tg}\beta\sin\alpha) \\ \qquad -(1+\mathrm{tg}\beta\mathrm{tg}\gamma)(\sin\alpha-\mathrm{tg}\beta\cos\alpha)] \\ q=L[(\mathrm{tg}\gamma\cos\alpha-\sin\alpha)(2\mathrm{tg}\beta-\mathrm{tg}\gamma+\mathrm{tg}\gamma\mathrm{tg}^2\beta) \\ \qquad +(1+\mathrm{tg}^2\gamma)(\mathrm{tg}\beta\cos\alpha-\sin\alpha)\mathrm{tg}\beta] \\ r=-\mathrm{tg}^2\beta(1+\mathrm{tg}^2\gamma)L^2 \end{cases}$$

L_0 can be solved from Eq.(1). Moreover

$$L_1=\frac{-L_0\cos\gamma+\sqrt{L_0^2\cos^2\gamma-4\cos\beta L_0(L_0\cos\alpha-L)}}{2\cos\beta},$$

$$L_2=\frac{L_1^2}{L_0}.$$

Now we come back to construct the PH-spline. For each subdivided segment, let $\boldsymbol{b}_0^{(i)}, \boldsymbol{b}_n^{(i)}$ be two given endpoints and their directions be those of tangents of $\boldsymbol{b}^{(i)}(t)$ at the two endpoints. According to Theorem 2, the ith PH cubic curve can be generated. Correspondingly, the PH cubic spline is G^1 continuity because the adjacent PH curves have the same tangent directions at common endpoints.

Constructing C^1 PH-spline

In this section, we shall interpolate given first-order Hermite data by PH quintic curves. For each subdivision, let the corresponding PH quintic curve and its hodograph be $\boldsymbol{p}^{(i)}(t)=\sum_{j=0}^{5}\boldsymbol{p}_j^{(i)}B_j^5(t)$ and $\boldsymbol{p}^{(i)\prime}(t)=[\boldsymbol{\omega}_0^{(i)}(1-t)^2+2\boldsymbol{\omega}_1^{(i)}t(1-t)+\boldsymbol{\omega}_2^{(i)}t^2]^2$ respectively, where $\boldsymbol{\omega}_0^{(i)}, \boldsymbol{\omega}_1^{(i)}, \boldsymbol{\omega}_2^{(i)}$ are three unknown complex coefficients. From the interpolation conditions

$$\boldsymbol{p}^{(i)\prime}(0)=\boldsymbol{d}_0^{(i)}, \quad \boldsymbol{p}^{(i)\prime}(1)=\boldsymbol{d}_1^{(i)},$$

$$\int_0^1 \boldsymbol{p}^{(i)\prime}(t)\mathrm{d}t=\boldsymbol{b}_n^{(i)}-\boldsymbol{b}_0^{(i)}=\nabla\boldsymbol{p}^{(i)},$$

a group of the complex coefficients can be solved as follows:

$$\boldsymbol{\omega}_0^{(i)}=\sqrt{\boldsymbol{d}_0^{(i)}}, \quad \boldsymbol{\omega}_2^{(i)}=\sqrt{\boldsymbol{d}_1^{(i)}},$$

$$\boldsymbol{\omega}_1^{(i)}=\frac{-3(\boldsymbol{\omega}_0^{(i)}+\boldsymbol{\omega}_2^{(i)})}{4} + \frac{\sqrt{120\nabla\boldsymbol{p}^{(i)}-15(\boldsymbol{\omega}_0^{(i)\,2}+\boldsymbol{\omega}_2^{(i)\,2})+10\boldsymbol{\omega}_0^{(i)}\boldsymbol{\omega}_2^{(i)}}}{4}$$

Then the control points of $\boldsymbol{p}^{(i)}(t)$ are expressed as

$$\boldsymbol{p}_0^{(i)}=\boldsymbol{b}_0^{(i)},\ \boldsymbol{p}_1^{(i)}=\boldsymbol{p}_0^{(i)}+\frac{1}{5}\boldsymbol{\omega}_0^{(i)\,2},$$
$$\boldsymbol{p}_2^{(i)}=\boldsymbol{p}_1^{(i)}+\frac{1}{5}\boldsymbol{\omega}_0^{(i)}\boldsymbol{\omega}_1^{(i)},\ \boldsymbol{p}_3^{(i)}=\boldsymbol{p}_2^{(i)}+\frac{2}{15}\boldsymbol{\omega}_1^{(i)\,2}+\frac{1}{15}\boldsymbol{\omega}_0^{(i)}\boldsymbol{\omega}_2^{(i)}$$
$$\boldsymbol{p}_4^{(i)}=\boldsymbol{p}_5^{(i)}-\frac{1}{5}\boldsymbol{\omega}_2^{(i)\,2},\ \boldsymbol{p}_5^{(i)}=\boldsymbol{b}_n^{(i)}$$

This kind of PH-spline can guarantee C^1 continuity as any two adjacent PH curves share the same tangent vector at a common endpoint.

Constructing C^2 PH-spline

In this section, the PH curve of degree seven will be used to render a C^2 PH-spline. Suppose $\boldsymbol{p}^{(i)}(t)$ and its hodograph to be respectively

$$\boldsymbol{p}^{(i)}(t)=\sum_{j=0}^{7}\boldsymbol{p}_j^{(i)}B_j^7(t)\quad\text{and}\quad\boldsymbol{p}^{(i)\prime}(t)=(\sum_{j=0}^{3}\boldsymbol{\omega}_j^{(i)}B_j^3(t))^2.$$

where $\boldsymbol{\omega}_0^{(i)}, \boldsymbol{\omega}_1^{(i)}, \boldsymbol{\omega}_2^{(i)}, \boldsymbol{\omega}_3^{(i)}$ are complex coefficients. For each subdivision, now consider $\boldsymbol{p}^{(i)}(t)$ interpolating the data: endpoints, endpoint derivatives and endpoint second derivatives of $\boldsymbol{b}^{(i)}(t)$, we come up with the following theorem.

Theorem 3 The control points of $\boldsymbol{p}^{(i)}(t)$ may be expressed as

$$\boldsymbol{p}_1^{(i)}=\boldsymbol{p}_0^{(i)}+\frac{1}{7}\boldsymbol{\omega}_0^{(i)\,2},\ \boldsymbol{p}_2^{(i)}=\boldsymbol{p}_1^{(i)}+\frac{1}{7}\boldsymbol{\omega}_0^{(i)}\boldsymbol{\omega}_1^{(i)},$$
$$\boldsymbol{p}_3^{(i)}=\boldsymbol{p}_2^{(i)}+\frac{3}{35}\boldsymbol{\omega}_1^{(i)\,2}+\frac{2}{35}\boldsymbol{\omega}_0^{(i)}\boldsymbol{\omega}_2^{(i)},$$
$$\boldsymbol{p}_4^{(i)}=\boldsymbol{p}_3^{(i)}+\frac{1}{70}\boldsymbol{\omega}_0^{(i)}\boldsymbol{\omega}_3^{(i)}+\frac{9}{70}\boldsymbol{\omega}_1^{(i)}\boldsymbol{\omega}_2^{(i)},$$
$$\boldsymbol{p}_5^{(i)}=\boldsymbol{p}_6^{(i)}-\frac{1}{7}\boldsymbol{\omega}_2^{(i)}\boldsymbol{\omega}_3^{(i)},\ \boldsymbol{p}_6^{(i)}=\boldsymbol{p}_7^{(i)}-\frac{1}{7}\boldsymbol{\omega}_3^{(i)\,2}.$$

where

$$\boldsymbol{\omega}_0^{(i)}=\pm\sqrt{\boldsymbol{d}_0^{(i)}},\ \boldsymbol{\omega}_1^{(i)}=\boldsymbol{\omega}_0^{(i)}+\frac{\boldsymbol{B}_0^{(i)}}{6\boldsymbol{\omega}_0^{(i)}},$$
$$\boldsymbol{\omega}_2^{(i)}=\boldsymbol{\omega}_3^{(i)}-\frac{\boldsymbol{B}_1^{(i)}}{6\boldsymbol{\omega}_0^{(i)}},\ \boldsymbol{\omega}_3^{(i)}=\pm\sqrt{\boldsymbol{d}_1^{(i)}}. \quad (2)$$

In fact, there are only two solutions to Eq.(2), $(+\boldsymbol{\omega}_0^{(i)},+\boldsymbol{\omega}_3^{(i)})$ is used to compute control points. Note that $\boldsymbol{p}^{(i)}(t)$ is constructed by C^2 Hermite interpolations, it is obvious for this kind of spline to be C^2 continuity.

Error estimation between Bézier and PH-spline under L_2

Theorem 4 The error between the original Bézier curve $\boldsymbol{b}(t)$ and its corresponding PH-spline $\boldsymbol{p}(t)$ after inserting N knots into the interval [0,1] for t under L_2 is expressed as

$$\varepsilon_{\mathrm{a}}=\sum_{i=1}^{N+1}(\sum_{l=0}^{2k}B(l+1,2k+1-l)\Delta p_l^{(i)*})^{\frac{1}{2}} \quad (3)$$

Where B is the Beta function, $k=\max(2p+1, n)$,

$$\Delta p_l^{(i)*}=\sum_{j=\max(0,l-k)}^{\min(k,l)}(\Delta p_{xj}^{(i)*}\Delta p_{xl-j}^{(i)*}+\Delta p_{yj}^{(i)*}\Delta p_{yl-j}^{(i)*})C_k^jC_k^{l-j},$$
$$\boldsymbol{p}_j^{(i)*}=p_{xj}^{(i)*}+\mathrm{i}p_{yj}^{(i)*},\ \boldsymbol{b}_j^{(i)*}=b_{xj}^{(i)*}+\mathrm{i}b_{yj}^{(i)*},\quad j=0\cdots k$$
$$\Delta p_{xj}^{(i)*}=p_{xj}^{(i)*}-b_{xj}^{(i)*},\ \Delta p_{yj}^{(i)*}=p_{yj}^{(i)*}-b_{yj}^{(i)*},\quad j=0\cdots k$$

$\boldsymbol{p}_j^{(i)*}$ and $\boldsymbol{b}_j^{(i)*}$ are the control points of the curves respectively when the degree of $\boldsymbol{p}^{(i)}(t)$ or $\boldsymbol{b}^{(i)}(t)$ is elevated to k.

Criteria for determining inserted knots

The simplest way to obtain knots is by dividing the interval [0,1] for t equally. However, the ideal PH-spline should be such kind of spline that has fewest segments within a given error. Now let the error between the ith subdivided Bézier curve $\boldsymbol{b}^{(i)}(k,t)$ and its corresponding PH curve $\boldsymbol{p}^{(i)}(k,t)$ after k knots are inserted is

$$\varepsilon_{\mathrm{a}}^{(i)}(k)=\left\|\boldsymbol{p}^{(i)}(k,t)-\boldsymbol{b}^{(i)}(k,t)\right\|_2.$$

The procedure of procuring optimal knots is illustrated as follows (Fig.3):

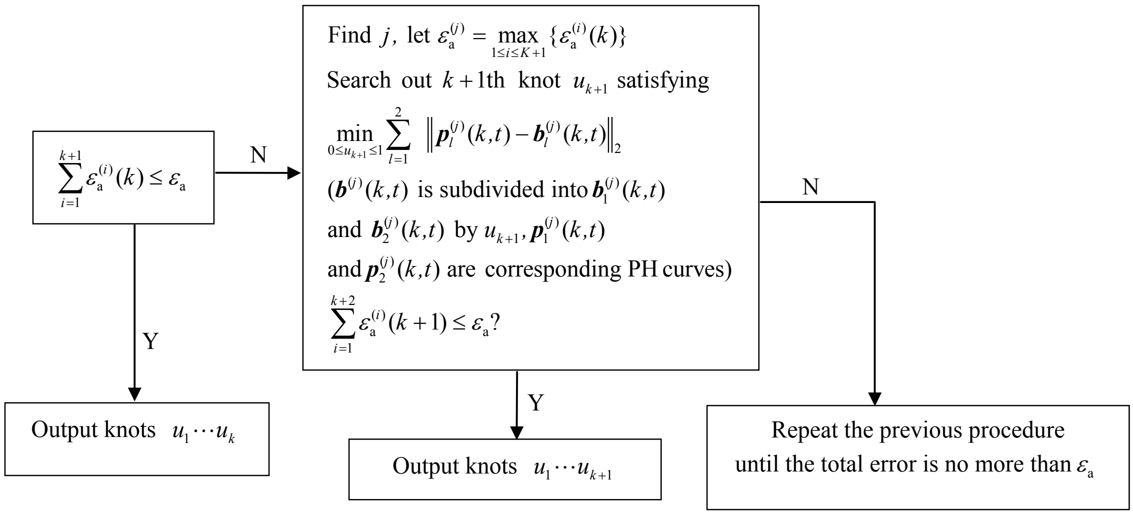

Fig.3 Procedure of procuring optimal knots

RENDERING OFFSET TO PH-SPLINE

Representation for offset

Theorem 5 Two offset curves at each (signed) distanced, defined as

$$\boldsymbol{p}_{\mathrm{d}}^{(i)}(t)=\boldsymbol{p}^{(i)}(t)\pm d\boldsymbol{n}^{(i)}(t)$$

where $\boldsymbol{n}^{(i)}(t)$ is the unit normal to $\boldsymbol{p}^{(i)}(t)$, can be represented in rational polynomial as

$$\boldsymbol{p}_{\mathrm{d}}^{(i)}(t)=\frac{\sum_{l=0}^{4p+1}\boldsymbol{p}_l^{(i)**}B_l^{4p+1}(t)}{\sigma^{(i)}(t)}$$

where

$$\boldsymbol{p}_l^{(i)**}=p_{xl}^{(i)**}+\mathrm{i}p_{yl}^{(i)**},\ \sigma^{(i)}(t)=\sum_{k=0}^{2p}\omega_k^{(i)*}B_k^{2p}(t)$$

$$\omega_k^{(i)*}=\sum_{j=\max(0,k-p)}^{\min(p,k)}\frac{(\omega_{xj}^{(i)}\omega_{xk-j}^{(i)}+\omega_{yj}^{(i)}\omega_{yk-j}^{(i)})C_p^jC_p^{k-j}}{C_{2p}^k}\tag{4}$$

$$p_{xl}^{(i)**}=\sum_{k=\max(0,l-2p-1)}^{\min(2p,l)}\frac{(p_{xl-k}^{(i)}\omega_k^{(i)*}\mp(2p+1)\Delta p_{yk}^{(i)}d)C_{2p}^kC_{2p+1}^{l-k}}{C_{4p+1}^k}$$

$$p_{yl}^{(i)**}=\sum_{k=\max(0,l-2p-1)}^{\min(2p,l)}\frac{(p_{yl-k}^{(i)}\omega_k^{(i)*}\pm(2p+1)\Delta p_{xk}^{(i)}d)C_{2p}^kC_{2p+1}^{l-k}}{C_{4p+1}^k}$$

Δ denotes the difference in x or y of $\boldsymbol{p}_k^{(i)}$.

Thus the total offset $\boldsymbol{p}_{\mathrm{d}}(t)$ to a PH-spline $\boldsymbol{p}(t)$ is given by $\boldsymbol{p}_{\mathrm{d}}(t)=\bigcup_{i=1}^{N+1}\boldsymbol{p}_{\mathrm{d}}^{(i)}(t)$.

Error estimation between offset to the original Bézier curve and offset to PH-spline curve

Theorem 6 The error between offset to original Bézier curve $\boldsymbol{b}(t)$ of degree n and offset to its corresponding PH-spline curve $\boldsymbol{p}(t)$ of degree $2p+1$ after $\boldsymbol{b}(t)$ is subdivided into $N+1$ segments has the following estimation:

$$\varepsilon_{\mathrm{d}}\le\varepsilon_{\mathrm{a}}+\sqrt{2}d\sum_{i=1}^{N+1}\sqrt{m^{(i)}}\tag{5}$$

where $m^{(i)}=\int_0^1 f^{(i)}(t)\mathrm{d}t$,

$$f^{(i)}(t)=\frac{\sum_{k=0}^{2p+n-1}r_k^{(i)*}B_k^{2p+n-1}(t)}{\sum_{k=0}^{2p+n-1}q_k^{(i)*}B_k^{2p+n-1}(t)},\ Lb_j^{(i)}=\left|\boldsymbol{b}_{j+1}^{(i)}-\boldsymbol{b}_j^{(i)}\right|,$$

$$p_k^{(i)*}=\sum_{j=\max\{0,k-(n-1)\}}^{\min\{2p,k\}}\frac{(2p+1)(\Delta b_{xk-j}^{(i)}\Delta p_{xj}^{(i)}+\Delta b_{yk-j}^{(i)}\Delta p_{yj}^{(i)})C_{n-1}^{k-j}C_{2p}^j}{C_{2p+n-1}^k},$$

Δ is the difference in x or y of $\boldsymbol{b}_j^{(i)}$ or $\boldsymbol{p}_j^{(i)}$,

$$q_k^{(i)*}=\sum_{j=\max\{0,k-(n-1)\}}^{\min\{2p,k\}}\frac{Lb_{k-j}^{(i)}C_{n-1}^{k-j}C_{2p}^{j}\omega_j^{(i)*}}{C_{2p+n-1}^{k}},$$

$\omega_j^{(i)*}$ as same as Eq.(4), $r_k^{(i)*}=q_k^{(i)*}-p_k^{(i)*}$.

Proof In fact, we notice from the inequality

$$\left\|\boldsymbol{p}_{\mathrm{d}}^{(i)}(t)-\boldsymbol{b}_{\mathrm{d}}^{(i)}(t)\right\|_2\le$$
$$\varepsilon_{\mathrm{a}}^{(i)}+\sqrt{2}d[\int_0^1(1-\frac{b_x^{(i)}{'}(t)p_x^{(i)}{'}(t)+b_y^{(i)}{'}(t)p_y^{(i)}{'}(t)}{\sqrt{b_x^{(i)}{'^2}+b_y^{(i)}{'^2}}\sqrt{p_x^{(i)}{'^2}+p_y^{(i)}{'^2}}})\mathrm{d}t]^{\frac{1}{2}}$$

that because

$$\Delta b_{xj}^{(i)}\Delta b_{xk}^{(i)}+\Delta b_{yj}^{(i)}\Delta b_{yk}^{(i)}\le\sqrt{\Delta b_{xj}^{(i)\,2}+\Delta b_{yj}^{(i)\,2}}\sqrt{\Delta b_{xk}^{(i)\,2}+\Delta b_{yk}^{(i)\,2}},$$

we have the relationship

$$\sqrt{b_x^{(i)\,\prime 2}(t)+b_y^{(i)\,\prime 2}(t)}\le n\sum_{j=0}^{n-1}\sqrt{\Delta b_{xj}^{(i)\,2}+\Delta b_{yj}^{(i)\,2}}B_j^{n-1}(t)$$

After a series of computations, the following conclusion can be obtained:

$$\left\|\boldsymbol{p}_{\mathrm{d}}^{(i)}(t)-\boldsymbol{b}_{\mathrm{d}}^{(i)}(t)\right\|_2\le\varepsilon_{\mathrm{a}}^{(i)}+\sqrt{2}d(\int_0^1 f^{(i)}(t)\mathrm{d}t)^{\frac{1}{2}}$$

Corollary 1 The error ε_{d} can further be represented as follows:

$$\varepsilon_{\mathrm{d}}\le\varepsilon_{\mathrm{a}}+\sqrt{2}d\sum_{i=1}^{N+1}\sqrt{\max_{1\le l\le n+2p}\left|\frac{v_l^{(i)}}{u_l^{(i)}}\right|}\tag{6}$$

where

$$u_l^{(i)},v_l^{(i)}=\sum_{k=\max(0,l-2p-n)}^{\min(1,l)}\frac{R_{l-k}^{(i)}C_1^kC_{n+2p-1}^{l-k}}{C_{n+2p}^{l}},$$

$R_{l-k}^{(i)}=q_{l-k}^{(i)*}$ for $u_l^{(i)}$, $R_{l-k}^{(i)}=r_{l-k}^{(i)*}$ for $v_l^{(i)}$.

ALGORITHM FOR GENERATING APPROXIMATION FOR OFFSET TO BÉZIER CURVE

Input Bézier curve $\boldsymbol{b}(t)$, distance d, approximation error $\varepsilon_{\mathrm{a}}^*$ and offset error $\varepsilon_{\mathrm{d}}^*$. The Bézier curve should be subdivided at the points with relatively larger curvature.

Initial conditions If $\|\boldsymbol{p}(t)-\boldsymbol{b}(t)\|_2\le\varepsilon_{\mathrm{a}}^*$ and $\|\boldsymbol{p}_{\mathrm{d}}(t)-\boldsymbol{b}_{\mathrm{d}}(t)\|_2\le\varepsilon_{\mathrm{d}}^*$, then output $\boldsymbol{p}(t)$.

$i=1$.

Step 1 Subdivide $\boldsymbol{b}(t)$ into $\boldsymbol{b}^{(j)}(t)$ according to the criteria for determining inserted knots and construct $\boldsymbol{p}^{(j)}(t)$. $j=1\cdots i+1$.

Step 2 Estimate ε_{a} and ε_{d} according to Eqs.(3) and (5) or (6) respectively.

Step 3 If $\varepsilon_{\mathrm{a}}\le\varepsilon_{\mathrm{a}}^*$ and $\varepsilon_{\mathrm{d}}\le\varepsilon_{\mathrm{d}}^*$ then Step 4
else
$i=i+1$.

Step 4 Output $\boldsymbol{p}^{(j)}(t)$ and offset to PH-spline joined by $\boldsymbol{p}^{(j)}(t)$.

EXAMPLE

According to PH algorithm, we can procure offset easily. Table 1 shows the procedures for four Bézier curves, Figs.4a–4d show their corresponding offsets. Moreover, we know that the B-spline curve can be transformed into several Bézier curves according to Boehm theorem, thus the offset to the B-spline curve can also be constructed. Fig.4e shows a group of offsets with distance at 0.2, 0.4, 0.6, 0.8, 1 to a cubic B-spline whose knots and control points are respectively (0, 0, 0, 0, 1, 1, 1, 2, 2, 2, 2) and (1, 1), (2, 2), (3, 2), (4, 1), (5, 0), (6, 0), (7, 1); the offsets are rendered by a PH cubic spline, a PH quintic spline or a PH-spline of degree seven.

CONCLUSION

For realizing the remarkable advantages of PH curve in practical applications, an efficient and reliable algorithm for approximating the Bézier curve is given and then used to render offset to the

Table 1 Examples of PH-spline approximation for Bézier curve and rendering offset

Bézier curve and control points	PH-spline	Inserted knots	d	ε_a	ε_d	Fig.4
Cubic, (1,1), (3,4), (5,4), (6,1)	C^2	{0.2223,0.5113,0.7391}	0.5	0.000895	0.069	(a)
Quintic, (1,5), (1,6), (2,7), (3,7.5), (4,7), (5,5)	C^2	{0.5172}	0.5	0.0064	0.082	(b)
Cubic, (1,7), (4,1), (5,2), (7,7)	C^1	$\{\frac{i}{8}\}$	0.8	0.00705	0.076	(c)
Cubic, (2,5), (3,6), (5,7), (7,5)	G^1	$\{\frac{i}{9}\}$	0.8	0.00934	0.0868	(d)

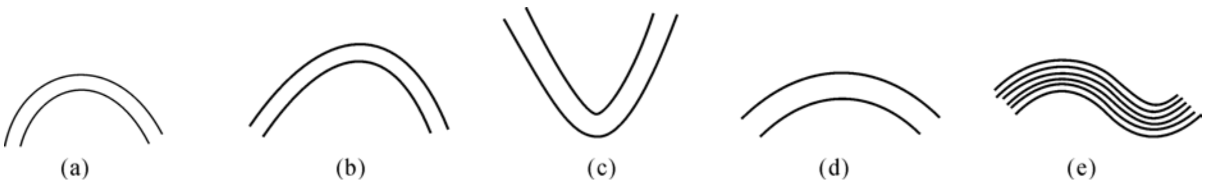

Fig.4 The approximate offsets to the four given Bézier curves shown in Table 1 and the given cubic B-spline curve
(a) An offset approximation for a quintic Bézier curve with a PH-spline of degree 7; (b) An offset approximation for a cubic Bézier curve with a PH-spline of degree 7; (c) An offset approximation for a cubic Bézier curve with a PH quintic spline; (d) An offset approximation for a cubic Bézier curve with a PH cubic spline; (e) An offset approximation for B-spline with a PH cubic spline

Bézier curve. The inserted knots can be efficiently controlled by error formulae. In this paper, several key contributions toward this end are described. Finally, an ideal approximation of the offset to the Bézier curve can be obtained. When a certain kind of PH-spline is selected, an approximation for offset to the Bézier curve with uniform rational polynomial of lower degree which is a very convenient piecewise representation in CAGD can be constructed. Compared with the original applications of PH curves, we hereby concentrate on solving the problem on offsets to the average polynomial curves.

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