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Existence of solutions and positive solutions to a fourth-order two-point BVP with second derivative

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Abstract: Several existence theorems were established for a nonlinear fourth-order two-point boundary value problem with second derivative by using Leray-Schauder fixed point theorem, equivalent norm and technique on system of integral equations. The main conditions of our results are local. In other words, the existence of the solution can be determined by considering the "height" of the nonlinear term on a bounded set. This class of problems usually describes the equilibrium state of an elastic beam which is simply supported at both ends.

Key words: Nonlinear fourth-order equation, Two-point boundary value problem, Solution and positive solution, Existence, Fixed point theorem
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INTRODUCTION

In this paper we shall consider the existence of solutions and positive solutions for the following fourth-order two-point boundary value problem

$$(P)\ \begin{cases} u^{(4)}(t) = f(t,u(t),u''(t)), & 0 \le t \le 1, \\ u(0) = A, u(1) = B, u''(0) = C, u''(1) = D. \end{cases}$$

Throughout this paper, we always assume that f is continuous on its domain and denote $R = (-\infty, +\infty)$, $R_+ = [0,+\infty)$, $R_- = (-\infty,0]$, $N = \{1, 2, \cdots\}$.

Fourth-order boundary value problem (P) usually describes the equilibrium state of an elastic beam which is simply supported at both ends. The source of nonlinearity comes from a nonlinear lateral constraint. Therefore, the solvability of the nonlinear problem (P) is useful to the stability analysis and numerical method of corresponding elastic beam.

The problem (P) had been widely studied by many authors (Del Peno and Manasevich, 1991; Ma *et al.*, 1997; and references therein).

Usmani (1979) said that the problem (P) has a solution provided $f(t,u,v)=g(t)u+h(t)$, $(t,u,v)\in[0,1]\times R\times R$ and $\sup_{0\le t\le 1} |g(t)|<\pi^4$.

Aftabizadeh (1986) showed that the problem (P) has a solution provided f is a bounded function.

Yang (1988) proved that the problem (P) has a solution provided f satisfies the linear growth condition

$$|f(t,u,v)| \le a|u|+b|v|+c,\ (t,u,v)\in[0,1]\times R\times R,$$

where a, b, c are positive constants and $a/\pi^4+b/\pi^2<1$.

Del Peno and Manasevich (1991) verified that the problem (P) has a solution provided f satisfies the following linear growth conditions:

1. The pair (α, β) satisfies $\alpha/(n\pi)^4+\beta/(n\pi)^2\neq 1$,

$n\in N$.

2. There exist positive constants a, b, c such that $a\max_{n\in N}|n^4\pi^4-\alpha-\beta n^2\pi^2|^{-1}+b\max_{n\in N} n^2\pi^2|n^4\pi^4-\alpha-\beta n^2\pi^2|^{-1}<1$, $|f(t,u,v)-(\alpha u-\beta v)|\le a|u|+b|v|+c$, $(t,u,v)\in[0,1]\times R\times R$.

Ma *et al.*(1997) investigated the problem (P) with $A=B=C=D=0$. They proved that the problem (P) has a solution by using the method of lower and upper solutions. But, they required that the function $f(t,u,v)$ is nondecreasing in u and nonincreasing in v on $[0,1]\times R\times R$.

It is easy to see that the conditions of the above-mentioned results deal with the properties of the nonlinear term f on whole domain $[0,1]\times R\times R$. We call such condition the overall condition. For example, linear growth condition and monotonicity condition are overall.

In many real problems, nonlinear terms are polynomial function. It is well known that, if f is a polynomial of degree n ($n\ge2$) in u and/or in v, then f does not satisfy any linear growth conditions and is not monotonic in most cases. Therefore, we need to find a new method.

This paper aims to provide some sufficient conditions for the existence of a solution (Theorems 1−2) and positive solution (Theorems 3−4) to the problem (P). In Theorems 1−4, the main condition has the form

$$\max\{|f(t,u,v)|:\cdots\} \text{ or } \max\{f(t,u,v):\cdots\}.$$

In geometry, the form expresses the "height" of the nonlinear term f on a bounded subset of $[0,1]\times R\times R$. We call such condition the localization condition. The localization condition deals with only the "height" of the nonlinear term f on a bounded set, and is independent of the growth of f outside the bounded set. Our results show that the problem (P) may have a solution provided the "height" of f on a bounded set is appropriate. It is easy to understand that the localization condition is favorable for the problem (P) where f is a polynomial function. This localization idea comes from our papers (Yao and Bai, 1999; Yao, 2002; 2003). Our main backups are Leray-Schauder fixed point theorem, equivalent norm and technique on system of integral equations.

Localization conditions and overall conditions are two different classes of conditions. At the end of this paper, we shall explain the relation between the two classes of conditions.

MAIN RESULTS

Let $C[0,1]$ be the Banach space endowed with max norm $\|u\|=\max_{0\le t\le 1}|u(t)|$. Let $\eta=\max\{|A|,|B|,|C|,|D|\}$, the function $u\in C[0,1]$ is said to be concave, if

$$u(\lambda t_1+(1-\lambda)t_2)\ge\lambda u(t_1)+(1-\lambda)u(t_2),\ \lambda,t_1,t_2\in[0,1].$$

In Theorem 1 and Theorem 2, we shall consider the existence of solution.

Theorem 1 Assume that f: $[0,1]\times R\times R\to R$ and $m>1$, $1/m\le k\le 8(m-1)/m$. If there exists $\rho>0$ such that

$$\max\{|f(t,u,v)|:t\in[0,1],\ |u|\le m\eta+\rho, |v|\le k(m\eta+\rho)\}$$
$$\le 8[(km-1)\eta+k\rho],$$

then the problem (P) has at least one solution $u^*\in C^4[0,1]$ satisfying $\|u^*\|\le m\eta+\rho$ and $\|(u^*)''\|\le k(m\eta+\rho)$.

Theorem 2 Assume that f: $[0,1]\times R\times R\to R$ and $A=B=C=D=0$. If there exists $\rho>0$ such that

$$\max\{|f(t,u,v)|:t\in[0,1],\ |u|\le\rho,\ |v|\le 8\rho\}\le 64\rho,$$

then the problem (P) has at least one solution $u^*\in C^4[0,1]$ satisfying $\|u^*\|\le\rho$ and $\|(u^*)''\|\le 8\rho$.

In many real problems, the problem will be significant only when the solution is positive. In Theorem 3 and Theorem 4, we shall consider the existence of positive solution.

Theorem 3 Assume that f: $[0,1]\times R_+\times R_-\to R_+$ and $A\ge0$, $B\ge0$, $C\le0$, $D\le0$, $m>1$, $1/m\le k\le 8(m-1)/m$. If there exists $\rho>0$ such that

$$\max\{f(t,u,v):0\le t\le 1, 0\le u\le m\eta+\rho, -k(m\eta+\rho)\le v\le 0\}$$
$$\le 8[(km-1)\eta+k\rho],$$

then the problem (P) has at least one solution u^*

$\in C^4$ [0,1] satisfying $\|u^*\| \le m\eta + \rho$ and $\|(u^*)''\| \le k(m\eta+\rho)$ and u^* is a nonnegative concave function. Moreover, if one of the following conditions is satisfied: (1) $A+B>0$; (2) $C+D<0$; (3) $f(t,0,0)\not\equiv 0$, $0\le t\le 1$, then $u^*(t)>0$, $0<t<1$.

Theorem 4 Assume that f: $[0,1]\times R_+\times R_-\to R_+$ and $A = B = C = D = 0$. If there exists $\rho>0$ such that

$$\max\{f(t,u,v):0\le t\le 1, 0\le u\le \rho, -8\rho\le v\le 0\}\le 64\rho,$$

then the problem (P) has at least one solution $u^*\in C^4$ [0,1] satisfying $\|u^*\|\le\rho$ and $\|(u^*)''\|\le 8\rho$ and u^* is a nonnegative concave function. Moreover, if $f(t,0,0)\not\equiv 0$, $0\le t\le 1$, then $u^*(t)>0$, $0<t<1$.

In a real problem, we may choose the parameters m, k in Theorem 1 and 3 by the "height" of nonlinear term $f(t,u,v)$.

PROOFS OF MAIN RESULTS

Proof of Theorem 1 Consider the Banach space $C[0,1]\times C[0,1]$ with norm $\|(u,v)\| = \max\{\|u\|, k^{-1}\|v\|\}$.

Let $v(t) = u''(t), 0\le t\le 1$. Then the problem ($P$) is equivalent to the system of ordinary differential equations

$$(S_1)\quad \begin{cases} u''(t) - v(t) = 0, & \\ v''(t) - f(t,u(t),v(t)) = 0, & 0\le t\le 1, \\ u(0) = A, u(1) = B, v(0) = C, v(1) = D. \end{cases}$$

The system (S_1), in turn, is equivalent to the system of integral equations

$$(S_2)\quad \begin{cases} u(t) = (B-A)t + A - \int_0^1 G(t,s)v(s)\mathrm{d}s, \\ v(t) = (D-C)t + C - \int_0^1 G(t,s)f(t,u(s),v(s))\mathrm{d}s, \end{cases}$$

where $G(t, s)$ is the Green function

$$G(t,s) = \begin{cases} t(1-s), & 0\le t\le s\le 1, \\ s(1-t), & 0\le s\le t\le 1. \end{cases}$$

It is easy to see that $G(0, s)=G(1, s)=0$ and $G(t, s)\ge 0$, $(t, s)\in[0,1]\times[0,1]$. After a direct computation, we get

$$\max_{0\le t\le 1}\int_0^1 G(t,s)\mathrm{d}s = \frac{1}{2}\max_{0\le t\le 1} t(1-t) = \frac{1}{8}.$$

Let $T(u,v) = (\varphi(u,v),\psi(u,v))$, where

$$\varphi(u,v)(t) = (B-A)t + A - \int_0^1 G(t,s)v(s)\mathrm{d}s,$$
$$\psi(u,v)(t) = (D-C)t + C - \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s.$$

Then the system (S_2) is equivalent to the fixed point equation

$$(E)\quad T(u,v) = (u,v),\quad (u,v)\in C[0,1]\times C[0,1].$$

By using Arzela-Ascoli theorem we can prove that $T: C[0,1]\times C[0,1]\to C[0,1]\times C[0,1]$ is completely continuous.

Let

$$V_{m\eta+\rho} = \{(u,v)\in C[0,1]\times C[0,1]: \|(u,v)\|\le m\eta+\rho\}.$$

Then $V_{m\eta+\rho}$ is a bounded convex closed set in $C[0,1]\times C[0,1]$.

If $(u,v)\in V_{m\eta+\rho}$, then $\|u\|\le m\eta+\rho$ and $\|v\|\le k(m\eta+\rho)$. So,

$$|u(t)|\le m\eta+\rho,\quad |v(t)|\le k(m\eta+\rho),\quad 0\le t\le 1,$$
$$|f(t,u(t),v(t))|\le 8[(km-1)\eta + k\rho],\quad 0\le t\le 1.$$

From this, we get that

$$\begin{aligned} \|\varphi(u,v)\| &= \max_{0\le t\le 1}\left|(B-A)t + A - \int_0^1 G(t,s)v(s)\mathrm{d}s\right| \\ &\le \max_{0\le t\le 1}|(B-A)t+A| + \max_{0\le t\le 1}\int_0^1 G(t,s)|v(s)|\mathrm{d}s \\ &\le \max\{|A|,|B|\} + k(m\eta+\rho)\max_{0\le t\le 1}\int_0^1 G(t,s)\mathrm{d}s \\ &\le (1+\frac{1}{8}km)\eta + \frac{1}{8}k\rho \le m\eta + \frac{m-1}{m}\rho; \end{aligned}$$

$$\|\psi(u,v)\| = \max_{0\le t\le 1}\left|(D-C)t + C - \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s\right|$$

$$\le \max_{0\le t\le 1}|(D-C)t+C|+\max_{0\le t\le 1}\int_0^1 G(t,s)|f(s,u(s),v(s))|\mathrm{d}s$$
$$\le \max\{|C|,|D|\}+8[(km-1)\eta+k\rho]\max_{0\le t\le 1}\int_0^1 G(t,s)\mathrm{d}s$$
$$\le \eta+(km-1)\eta+k\rho=k(m\eta+\rho).$$

Thus,

$$\|(\varphi(u,v),\psi(u,v))\|=\max\{\|\varphi(u,v)\|,k^{-1}\|\psi(u,v)\|\}\le m\eta+\rho.$$

It follows that $T: V_{m\eta+\rho}\to V_{m\eta+\rho}$.

By Leray-Schauder fixed point theorem, we assert that the operator T has a fixed point $(u^*, v^*)\in V_{m\eta+\rho}$. In other words, the problem (P) has one solution $u^*\in C^4[0,1]$ satisfying $\|u^*\|\le m\eta+\rho$ and $\|(u^*)''\|\le k(m\eta+\rho)$.

Proof of Theorem 2 Consider the Banach space $C[0,1]\times C[0,1]$ with norm $\|(u,v)\|=\max\{\|u\|, \frac{1}{8}\|v\|\}$. Similar to proof of Theorem 1, the problem (P) is equivalent to the fixed point equation

$$T(u,v)=(u,v),\quad (u,v)\in C[0,1]\times C[0,1],$$

where

$$T(u,v)=(\varphi(u,v),\psi(u,v)),$$
$$\varphi(u,v)(t)=-\int_0^1 G(t,s)v(s)\mathrm{d}s,$$
$$\psi(u,v)(t)=-\int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s.$$

Let $V_\rho=\{(u,v)\in C[0,1]\times C[0,1]:\|(u,v)\|\le\rho\}$. It is easy to prove that $T:V_\rho\to V_\rho$ is completely continuous. By Leray-Schauder fixed point theorem, we assert that the problem (P) has one solution $u^*\in C^4[0,1]$ satisfying $\|u^*\|\le\rho$ and $\|(u^*)''\|\le 8\rho$.

Proof of Theorem 3 Let

$$f_1(t,u,v)=\begin{cases} f(t,u,v), & (t,u,v)\in[0,1]\times R_+\times R_-,\\ f(t,u,0), & (t,u,v)\in[0,1]\times R_+\times R_+,\end{cases}$$
$$f_2(t,u,v)=\begin{cases} f_1(t,u,v), & (t,u,v)\in[0,1]\times R_+\times R,\\ f_1(t,0,v), & (t,u,v)\in[0,1]\times R_-\times R.\end{cases}$$

Then $f_2:[0,1]\times R\times R\to R_+$ is continuous and

$$\max\{|f_2(t,u,v)|:t\in[0,1],|u|\le m\eta+\rho,|v|\le k(m\eta+\rho)\}$$
$$=\max\{f(t,u,v):0\le t\le 1, 0\le u\le m\eta+\rho,$$
$$-k(m\eta+\rho)\le v\le 0\}$$
$$\le 8[(mk-1)\eta+k\rho].$$

By Theorem 1, the problem

$$\begin{cases} u^{(4)}(t)=f_2(t,u(t),u''(t)), & 0\le t\le 1,\\ u(0)=A, u(1)=B, u''(0)=C, u''(1)=D,\end{cases}$$

has one solution $u^*\in C^4[0,1]$ satisfying $\|u^*\|\le m\eta+\rho$ and $\|(u^*)''\|\le k(m\eta+\rho)$.

Since $C\le 0$, $D\le 0$, we have

$(D-C)t+C=Dt+C(1-t)\le 0$, $0\le t\le 1$.

Since $f_2(t,u^*(t),(u^*)''(t))\ge 0, 0\le t\le 1$, we see that, for $0\le t\le 1$,

$$(u^*)''(t)=(D-C)t+C-\int_0^1 G(t,s)f(s,u^*(s),(u^*)''(s))\mathrm{d}s\le 0.$$

Since $u^*(0)=A\ge 0$, $u^*(1)=B\ge 0$, we assert that $u^*(t)$ is a nonnegative concave function on [0,1]. On this score,

$$f_2(t,u^*(t),(u^*)''(t))=f(t,u^*(t),(u^*)''(t)),\quad 0\le t\le 1.$$

Thus, u^* is a solution of the problem (P).

If $C+D<0$, then $Dt+C(1-t)<0$, $0<t<1$. It follows $(u^*)''(t)<0, 0<t<1$. So, $u^*(t)\not\equiv 0$ and $u^*(t)>0$, $0<t<1$.

If $A+B>0$, then, for $0<t<1$,

$$u^*(t)=(B-A)t+A-\int_0^1 G(t,s)(u^*)''(s)\mathrm{d}s\ge Bt+A(1-t)>0.$$

If $A=B=C=D=0$ and $f(t,0,0)\not\equiv 0$, $0\le t\le 1$, then the zero function is not the solution of problem (P). Thus, $u^*(t)\not\equiv 0$ and $u^*(t)>0$, $0<t<1$.

Proof of Theorem 4 The conclusion can derived from Theorem 2 in the same way as that for Theorem 3.

REMARKS

Now, we explain the relation between localization condition and overall condition. In a few words, the two classes of conditions do not include each other.

Example 5 Consider the fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t)=6u^3(t)-3tu^2(t)u''(t)+\frac{1}{2}t^2u(t)(u'')^2(t)+2, \\ \qquad\qquad 0\le t\le 1, \\ u(0)=u(1)=u''(0)=u''(1)=0. \end{cases}$$

Here, $f(t,u,v)=6u^3-3tu^2v+\frac{1}{2}uv^2+2$,

$f:[0,1]\times R_+\times R_-\to R_+, \eta=0$.

Choose $\rho=1$.

Obviously,

$$\max\{f(t,u,v):0\le t\le 1, 0\le u\le 1, -8\le v\le 0\}=64.$$

By Theorem 4, the problem has at least a positive solution $u^*\in C^4[0,1]$ satisfying $\|u^*\|\le 1$ and $\|(u^*)''\|\le 8$. Clearly, the conclusion cannot be derived from the papers mentioned in Introduction.

Example 6 Consider the fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t)=\frac{2}{3}\pi^4u(t)-\frac{1}{4}\pi^2u''(t)+t(1-t), \quad 0\le t\le 1, \\ u(0)=u(1)=u''(0)=u''(1)=0. \end{cases}$$

Here, $f(t,u,v)=\frac{2}{3}\pi^4u-\frac{1}{4}\pi^2v+t(1-t)$, $\eta=0$.

Obviously,

$$|f(t,u,v)|\le\frac{2}{3}\pi^4|u|+\frac{1}{4}\pi^2|v|+1, (t,u,v)\in[0,1]\times R\times R,$$

and $\frac{2}{3}\pi^4/\pi^4+\frac{1}{4}\pi^2/\pi^2<1$. By Yang (1988), the problem has at least a nontrivial solution $u^*\in C^4[0,1]$. But, for any $\rho>0$,

$$\max\{|f(t,u,v)|:0\le t\le 1, |u|\le\rho, |v|\le 8\rho\}$$
$$\ge(\frac{2}{3}\pi^4+\frac{1}{4}\pi^2)\rho>64\rho.$$

The conclusion cannot be derived from the results of this paper.

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