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A characteristic condition of finite nilpotent group*

LI Yang-ming (李样明)

(*Department of Mathematics, Zhejiang University, Hangzhou 310016, China*)

(*Department of Mathematics, Guangdong College of Education, Guangzhou 510310, China*)

E-mail: liyangming@gdei.edu.cn

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Abstract: This paper gives a characteristic condition of finite nilpotent group under the assumption that all minimal subgroups of G are well-suited in G.

Key words: Z-permutable subgroup, Nilpotent group, The generalized Fitting subgroup, Hypercenter subgroup

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INTRODUCTION

In this paper, all groups considered are finite; G means a finite group.

We use conventional notions and notations, as in Huppert (1968). Recall that a minimal subgroup of a finite group is a subgroup of prime order. For the group of even order, it is helpful to also consider the cyclic subgroup of order 4. Two subgroups H and K of a group G are said to permute if $HK=KH$. It is easily seen that H and K permute if and only if the set of HK is a subgroup of G. We know that a normal subgroup of G permutes with every subgroup of G. So Ore (1937) extended normal subgroup to quasinormal subgroup, a subgroup of G is called quasinormal subgroup of G if it permutes with every subgroup of G. Kegel (1962) went further to define Π-quasinormal subgroup, a subgroup of G is Π-quasinormal in G if it permutes with every Sylow subgroup of G. Recently, Asaad and Heliel (2003) extended Π-quasinormality to a new embedding property, namely the Z-permutability. Z is called a complete set of Sylow subgroups of G if for each prime $p\in\Pi(G)$ (the set of distinct primes dividing $|G|$), Z contains exactly one Sylow p-subgroup of G, G_p say. A subgroup of G is said to be Z-permutable in G if it permutes with every member of Z.

A number of authors had considered how minimal subgroups could be embedded in a nilpotent group or a p-nilpotent group. Huppert (1968) proved that if G is a group of odd order and all minimal subgroups of G lie in the center of G, then G is nilpotent. An extension of his result is the following statement: If for an odd prime p, every subgroup of order p lies in the center of G, then G is p-nilpotent. If all the elements of G of order 2 or 4 lie in the center of G, then G is 2-nilpotent (Huppert, 1968). Recently the result was generalized as follows: Let N be a normal subgroup of a group G such that G/N is nilpotent. Suppose every element of order 4 of $F^*(N)$ is c-supplemented in G, then G is nilpotent if and only if every element of prime order of $F^*(N)$ is contained in the hypercenter $Z_\infty(G)$ of G (Wang *et al*., 2003). All the results mentioned above were also extended with formation theory,

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such as in Asaad *et al.*(1996). In this paper, we want to get some results analogous to the above theorems by replacing the *c*-supplementation by *Z*-permutability. The main theorem is as follows:

Main Theorem Let F be a saturated formation such that $N\subseteq F$, where N is the class of all nilpotent groups. Let G be a group and Z a complete set of Sylow subgroups of G. Suppose every element of order 4 of $F^*(G^F\cap G_2)$ is Z-permutable in G, where $G_2\in Z$. Then G belongs to F if and only if $<x>$ lies in the F-hypercenter $Z_F(G)$ of G for every element x of $F^*(G^F\cap G_p)$ of prime order, for every $G_p\in Z$.

It is significant to mention first there are soluble group with Л-quasinormal (Z-permutable) subgroups which are not c-supplemented. Conversely, there are soluble groups with Л-quasinormal (Z-permutable) subgroups which are not c-supplemented subgroup; Secondly our results give the sufficient and necessary condition of nilpotent group, i.e., it is a characteristic condition of nilpotent (ref. Theorem 5).

For the definitions and terminology of formations, please refer to Finite soluble groups (Doerk and Hawkes, 1992).

Let Z be a complete set of Sylow subgroups of a group G. If $N\lhd G$, we shall denote by $Z\cap N$ the following set of subgroups of G:

$$Z\cap N=\{G_p\cap N:G_p\in Z\}.$$

An element x of a group G is said to be Л-quasinormal (Z-permutable) in G if $<x>$ is Л-quasinormal (Z-permutable) in G.

SOME LEMMAS

Lemma 1 (Kegel, 1962)

(1) A Л-quasinormal subgroup of G is subnormal in G;

(2) If $H\leq K\leq G$ and H is Л-quasinormal in G, then H is Л-quasinormal in K;

(3) If H is Л-quasinormal Hall subgroup of G, then $H\lhd G$;

(4) Let $K\lhd G$ and $K\leq H$. Then H is Л-quasinormal in G if and only if H/K is Л-quasinormal in G/K.

Lemma 2 Suppose G is a group and P is a normal p-subgroup of G contained in $Z_\infty(G)$, then $C_G(P)\geq O^P(G)$.

Proof Applying Satz 4.4 of Endliche Gruppen (Huppert, 1968).

The generalized Fitting subgroup $F^*(G)$ of G is an important subgroup of G and it is a natural generalization of $F(G)$. The definition and impotent properties can be found in Huppert and Blackburn (1982). We wound like to gather the following basic facts which we will use in our proof.

Lemma 3 (Li and Wang, 2003) Let G be a group and M a subgroup of G.

(1) If M is normal in G, then $F^*(M)\leq F^*(G)$;

(2) $F^*(G)\neq 1$ if $G\neq 1$; in fact, $F^*(G)/F(G)=\text{soc}(F(G)C_G(F(G))/F(G))$;

(3) $F^*(F^*(G))=F^*(G)\geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G)=F(G)$.

(4) $C_G(F(G))\leq F(G)$;

(5) Suppose K is a subgroup of G contained in $Z(G)$, then $F^*(G/K)=F^*(G)/K$.

Lemma 4 (Asaad and Heliel, 2003) Let Z be a complete set of Sylow subgroups of G, U be a Z-permutable subgroup of G, and N a normal subgroup of G. Then:

(1) $Z\cap N$ is a complete set of Sylow subgroups of N.

(2) If $U\leq N$, then U is a $Z\cap N$-permutable subgroup of N.

Lemma 5 Let P be a normal 2-subgroup of a group G, and Z a complete set of Sylow subgroups of G. If every cyclic subgroup of order 4 of P is Z-permutable subgroup in G, then every cyclic subgroup of order 4 of P is Л-quasinormal in G.

Proof Let L be an arbitrary subgroup of P of order 4. Then LG_{p_i} is a subgroup of G for every $G_{p_i}\in Z$. Since $P\rhd G$, $L^{x^{-1}}G_{p_i}\leq G$. But $L\,G_{p_i}{}^x=(L^{x^{-1}}G_{p_i})^x$ is a subgroup of G, then L is Л-quasinormal in G.

Lemma 6 Suppose M, N are normal subgroups of G. If there exists a Sylow p-subgroup P of G such that every element of $M\cap P$ of order p lies in N, then every element of M of prime order lies in N.

Proof Since M is a normal subgroup of G, $M\cap P$ is a Sylow p-subgroup of M. By Sylow Theorem, for any element x of M of prime order, there exists $m\in M$ such that $x^m\in M\cap P$, so $x^m\in N$ by the hypotheses. Then $x\in N^{m^{-1}}=N$. Thus the lemma holds.

MAIN RESULTS

Theorem 1 Suppose G is a group, p is a fixed prime number. If every element of G of order p is contained in $Z_\infty(G)$. If p=2, in addition, suppose every cyclic subgroup of order 4 of G is Π-quasinormal, then G is p-nilpotent.

Proof Suppose that the theorem is false and let G be a counter-example of smallest order.

(a) The hypotheses are inherited by all proper subgroups, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

In fact, $\forall H < G$, K is a cyclic subgroup of H of order p (or 4 if p=2), then $K\leq Z_\infty(G)\cap H\leq Z_\infty(H)$. By Lemma 1(2), we know that the Π-quasinormality in G can imply the Π-quasinormality in H. Thus H satisfies the hypotheses of the theorem. The minimal choice of G implies that H is p-nilpotent, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. So, $G=PQ$, where $P\lhd G$ and Q is not normal in G (Huppert, 1968).

(b) p=2 and every element of order 4 is Π-quasinormal in G.

If not, then p>2, then $\exp(P)=p$ (Huppert, 1968). Thus $P\leq Z_\infty(G)$ by the hypotheses. Therefore $G=PQ=P\times Q$, then is nilpotent by Lemma 2, a contradiction. Thus (b) holds.

(c) $\forall a\in P\backslash\Phi(P)$, $o(a)$=4.

If not, there exists $a\in P\backslash\Phi(P)$, such that $o(a)$ =2. Denote $M=<a^G>\leq P$. Then $M\Phi(P)/\Phi(P)\lhd G/\Phi(P)$, we have that $P=M\Phi(P)=M\leq Z_\infty(G)$ as $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ (Huppert, 1968), a contradiction.

(d) Final contradiction.

$\forall x\in P\backslash\Phi(P)$, $o(x)$=4. Then $<x>$ is Π-quasinormalin G, so $<x>Q<G$, thus $<x>Q=<x>\times Q$ by (a). Therefore $<x>\subseteq N_G(Q)$, it follows that $P\subseteq N_G(Q)$, the final contradiction.

Theorem 2 Suppose N is a normal subgroup of a group G such that G/N is p-nilpotent, where p is a fixed prime number. Suppose every element of N of order p is contained in $Z_\infty(G)$. If p=2, in addition, suppose every cyclic subgroup of order 4 of N is Π-quasinormal in G, then G is p-nilpotent.

Proof Assume that the theorem is false and let G be a counterexample of minimal order, then we have:

(a) The hypotheses are inherited by all proper subgroups, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

In fact, $\forall K<G$, since G/N is p-nilpotent, $K/K\cap N\cong KN/N$ is also p-nilpotent. The element of order p of $K\cap N$ is contained in $Z_\infty(G)\cap K\leq Z_\infty(K)$, the cyclic subgroup of order 4 of $K\cap N$ is Π-quasinormal in G, then is Π-quasinormal in K by Lemma 1. Thus K, $K\cap N$ satisfy the hypotheses of the theorem, so K is p-nilpotent, therefore G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then $G=PQ$, $P\lhd G$, Q is not normal in G (Huppert, 1968).

(b) $G/P\cap N$ is p-nilpotent.

Since $G/P\cong Q$ is nilpotent, G/N is p-nilpotent and $G/(P\cap N)\leq G/P\times G/N$, therefore $G/(P\cap N)$ is p-nilpotent.

(c) $P\leq N$.

If not, then $P\cap N<P$. So $Q(P\cap N)<QP=G$. Thus $Q(P\cap N)$ is nilpotent by (a), $Q(P\cap N)=Q\times(P\cap N)$. Since $G/P\cap N=P/P\cap N\cdot Q(P\cap N)/P\cap N$, it follows that $Q(P\cap N)/P\cap N\lhd G/P\cap N$ by (b). So Q *char* $Q(P\cap N)\lhd G$. Therefore, $G=P\times Q$, a contradiction.

(d) Final contradiction.

If p>2, then $\exp(P)=p$ by (a). Thus $P=P\cap N\leq Z_\infty(G)$, then that $G=P\times Q$ (Huppert, 1968), a contradiction.

If p=2, since $P\lhd G$, so all elements of order 2 of G are contained in P, i.e., contained in N. Thus every element of order 2 of G lies in $Z_\infty(G)$, every cyclic subgroup of order 4 is Π-quasinormal in G. Applying Theorem 1, we have that G is 2-nilpotent, a contradiction, completing the proof.

Since a group G if nilpotent if and only if G is p-nilpotent, $\forall p\in\Pi(G)$. By Theorem 2, we have:

Theorem 3 Suppose N is a normal subgroup of a group G such that G/N is nilpotent. Then G is nilpotent if and only if every element of N of prime order is contained in $Z_\infty(G)$, every cyclic subgroup of order 4 of N is Π-quasinormal in G.

Revising the proof of Theorem 3.3 of Wang *et al.*(2003), we can minimize the number of restricted elements in Theorem 3.

Theorem 4 Suppose N is a normal subgroup of a group G such that G/N is nilpotent, then G is nilpotent if and only if every element of $F^*(N)$ of order 4 is Π-quasinormal in G and every element of $F^*(N)$ of prime order is contained in the hypercenter $Z_\infty(G)$ of G.

Theorem 5 Let Z be a complete set of Sylow subgroups of a group G and N a normal subgroup of G such that G/N is nilpotent. Then G is nilpotent if and only if every element of $F^*(N)\cap G_2$ of order 4 is Z-permutable in G, and every element of $F^*(N)\cap G_p$ of prime order is contained in the hypercenter $Z_\infty(G)$ of G, for any $G_p\in Z$.

By Lemma 6, it is easy to see Theorem 5 is equivalent to the following:

Theorem 5' Let Z be a complete set of Sylow subgroups of a group G, N is a normal subgroup of G such that G/N is nilpotent, then G is nilpotent if and only if every element of $F^*(N)\cap G_2$ of order 4 is Z-permutable in G, every element of $F^*(N)$ of prime order is contained in the hypercenter $Z_\infty(G)$ of G.

Proof The necessity is the same as that in Theorem 4, we only need to prove the converse is true.

Let G be a counterexample of minimal order, then we have:

(1) Every proper normal subgroup of G is nilpotent.

If M is a maximal normal subgroup of G, we have that $M/M\cap N$ is nilpotent, $F^*(M\cap N)$ is contained in $F^*(N)$ and $Z_\infty(G)\cap M$ is contained in $Z_\infty(M)$, so every element of $F^*(M\cap N)$ of prime order is contained in the hypercenter $Z_\infty(M)$, and every element of $F^*(N)\cap(G_2\cap N)$ of order 4 is Z-permutable in G by hypotheses, thus is $Z\cap M$-permutable in M by Lemma 4(2), so M, $M\cap N$ satisfies the hypotheses of the theorem. The minimal choice of G implies that M is nilpotent.

(2) $F^*(G)=G$.

If $F^*(G)<G$, then $F^*(G)$ is nilpotent by (1), in particular, $F^*(G)$ is solvable, so $F^*(G)=F(G)$ by Lemma 3. For the Sylow 2-subgroup P of $F^*(G)$, $P=O_2(G)\leq G_2$, we know that the cyclic subgroups of P of order 4 are Z-permutable subgroups in G by hypotheses, now Lemma 2.5 implies the cyclic subgroups of order 4 of P are Π-quasinormal in G. Applying Theorem 4, G is nilpotent, a contradiction.

(3) G is almost simple, i.e., $G/Z(G)$ is simple.

By (2), $G=F^*(G)=F(G)E(G)$, where $E(G)$ is layer of G. If $E(G)\leq F(G)$, then $G=F(G)$ is nilpotent, a contradiction. Thus assume $E(G)$ is not contained $F(G)$, then we can pick a component H of $E(G)$ (Huppert and Blackburn, 1982), and H is almost simple. By (2), $[H, G]=[H, F^*(G)]=[H, F(G)E(G)]=[H, E(G)]\leq H$, i.e., H is normal in G. If $H<G$, then H is solvable by (1), a contradiction. So $G=H$ is almost simple.

(4) $G^N=N=G$, and $Z_\infty(G)=Z(G)$.

If $G^N<G$, then G^N is nilpotent by (1), then G is solvable, contrary to (3), thus $G^N=G$, and $G^N\leq N$ implies that $N=G$. By Huppert (1968), we have $G^N\cap Z_\infty(G)\leq Z(G^N)$, so $Z_\infty(G)=Z(G)$.

(5) The final contradiction.

We know that G is a quasisimple group by (3). So $Z(G)$ is a subgroup of the Schur multiplier of $G/Z(G)$ (Gorenstein, 1982). Again by Table 4.1 in (Gorenstein, 1982), $Z(G)\leq R$ or $Z(G)\leq R\times P$. Therefore $\Pi(Z(G))$ contains at most two primes. Then every element of prime order of G lies in $Z_\infty(G)=Z(G)$ by hypotheses and (4), we conclude that Π(G) contains at most two primes, the well-known Burnside p^aq^b-theorem implies that G is solvable, contrary to (3), the final contradiction.

This completes the proof of the theorem.

With the similar the proof of Theorem 4.4 of Wang *et al.*(2003), we can extend Theorem 3 with formation theory.

Theorem 6 Let F be a saturated formation such that $N\subseteq F$. Let G be a group such that every element of G^F of order 4 is Π-quasinormal in G. Then G belongs to F if and only if $<x>$ lies in the

F-hypercenter $Z_F(G)$ of G for every element x of G^F of prime order.

Following the proof Theorem 4.5 of Wang *et al*. (2003), we have:

Theorem 7 Let F be a saturated formation such that $N \subseteq F$. Let G be a group such that every element of $F^*(G^F)$ of order 4 is $Л$-quasinormal in G. Then G belongs to F if and only if $<x>$ lies in the F-hypercenter $Z_F(G)$ of G for every element x of $F^*(G^F)$ of prime order.

By Lemma 5, the Main Theorem is equivalent to the following, so we prove it to end this paper.

Equivalent form of Main Theorem Let F be a saturated formation such that $N \subseteq F$. Let G be a group and Z a complete set of Sylow subgroups of G. Suppose very element of $F^*(G^F) \cap G_2$ of order 4 is Z-permutable in G, where $G_2 \in Z$. Then G belongs to F if and only if $<x>$ lies in the F-hypercenter $Z_F(G)$ of G for every element x of $F^*(G^F)$ of prime order.

Proof If $G \in F$, then $Z_F(G)=G$ and we are done. So we only need to prove that the converse is true.

Since $Z_F(G) \cap G^F \leq Z(G^F) \leq Z_\infty(G^F)$ (Doerk and Hawkes, 1992), by the hypotheses, every element of $F^*(G^F)$ of prime order lies in $Z_\infty(G^F)$. Every element of $F^*(G^F) \cap G_2$ of order 4 is Z-permutable in G, thus is $Z \cap G^F$-permutable in G^F by Lemma 4. Applying Theorem 3 for G^F, we get G^F is nilpotent. So $F^*(G^F)=F(G^F)=G^F$. Thus the Sylow 2-subgroup $G^F \cap G_2$ of G^F is normal in G. By hypotheses and Lemma 5, very element of $F^*(G^F) \cap G_2$ of order 4 is $Л$-quasinormal in G. Since every element of G^F of prime order lies in $Z_F(G)$ by hypotheses, now Theorem 7 implies that $G \in F$. These complete the proof of Theorem.

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