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A minimal axiom group for rough set based on quasi-ordering*

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Abstract: Rough set axiomatization is one aspect of rough set study to characterize rough set theory using dependable and minimal axiom groups. Thus, rough set theory can be studied by logic and axiom system methods. The classic rough set theory is based on equivalent relation, but rough set theory based on reflexive and transitive relation (called quasi-ordering) has wide applications in the real world. To characterize topological rough set theory, an axiom group named RT, consisting of 4 axioms, is proposed. It is proved that the axiom group reliability in characterizing rough set theory based on similar relation is reasonable. Simultaneously, the minimization of the axiom group, which requires that each axiom is an equation and each is independent, is proved. The axiom group is helpful for researching rough set theory by logic and axiom system methods.

Key words: Rough set theory, Quasi-ordering, Axioms, Minimization
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INTRODUCTION

As a new effective mathematical tool to deal with vagueness and uncertainty, the rough set theory was first proposed by Pawlak (1991). On the basis of classification, rough set theory looks at knowledge as a partition over data using equivalence relation, and has been successfully applied in many fields such as machine learning, pattern recognition, decision support and data mining (Pawlak *et al*., 1995).

Lower approximation and upper approximation are two basic concepts in rough set theory. Pawlak (1991) derived many interesting properties of upper and lower approximations, while some researchers studied the reverse problem. Namely, can we characterize the notion of rough sets in terms of those properties? Lin and Liu (1994) studied this problem from the viewpoint of topology and proposed an axiom group consisting of six axioms of rough set and presented the concepts of Rough Set Axiom Group and Axiomatic Rough Set Theory. Zhu and He (2000) and Sun *et al*.(2002) discussed the redundancy of rough set axiom group. But these rough set axiom groups are used to characterize the classic rough set which is defined by equivalence relation. In fact, we can find that there exists indiscernibility relations which are not symmetric, although it is usually supposed that indiscernibility relations are symmetric, which means that if we cannot discern x from y, then we cannot discern y from x either. But indiscernibility relations may be directional. For example, if a person x speaks English and Chinese, and a person

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y speaks English, Finnish and Chinese, then x cannot discern y from himself by the property "knowledge of languages" since y can communicate with x in any languages that x speaks. On the other hand, y can discern x from himself by asking a simple question in Finnish, for example. So, the rough set based on reflexive and transitive relation must be studied. Since reflexive and transitive relation is also called quasi-ordering, the rough set based on reflexive and transitive relation is also called rough set based on quasi-ordering.

To characterize rough set based on quasi-ordering, an axiom group named RT, consisting of 4 axioms, is proposed. The validity of the axiom group, which shows that characterizing of rough set theory based on quasi-ordering is rational, is proved. Simultaneously, the minimization of the axiom group, which requires that each axiom is an equation and each equation is independent, is proved. The axiom group is helpful for researching rough set based on quasi-ordering by logic and axiom system methods.

BASIC NOTATIONS AND DEFINITIONS

Pawlak classic rough set

Let $U=\{u_1, u_2, \ldots, u_n\}$ denote a finite and non-empty set of objects called the universe, and let $R\subseteq U\times U$ denote an equivalence relation on U. The pair (U, R) is called an approximation space. The equivalence classes partitioned by R are called elementary sets of (U, R).

The equivalence relation and the induced equivalence classes may be regarded as the available information or knowledge about the objects under consideration. Given an arbitrary set $X\subseteq U$, it may be impossible to describe X precisely using the equivalence classes of R. That is, the available information is not sufficient to give a precise representation of X. In this case, one may characterize X by a pair of lower and upper approximations:

$$\text{L\_APP}_R(X) = \bigcup_{[x]_R\subseteq X} [x]_R \quad (1)$$

$$\text{U\_APP}_R(X) = \bigcup_{[x]_R\cap X\neq\phi} [x]_R \quad (2)$$

where $[x]_R$ is the equivalence class containing x. The lower approximation L_APP(X) is the union of all the elementary sets which are subsets of X. It is the largest composed set contained in X. The upper approximation U_APP(X) is the union of all the elementary sets which have a non-empty intersection with X. It is the smallest composed set containing X.

Rough set based on quasi-ordering

Some researchers (Yao *et al*., 1997; Yao, 1998b) generalized the classic rough set by replacing equivalent classes with neighborhood. Suppose $R\subseteq U\times U$ is a binary relation on U, $x\in U$, then R-neighborhood of x is defined as:

$$N_R(x) = \{y \mid (x, y)\in R\} \quad (3)$$

Based on Eq.(3), the lower and upper approximation of the objects set $X\subseteq U$ can be defined as:

$$\text{L\_APP}_R(X) = \{x \mid N_R(x)\subseteq X\} \quad (4)$$

$$\text{U\_APP}_R(X) = \{x \mid N_R(x)\cap X\neq\phi\} \quad (5)$$

Based on the above discussion on rough set based quasi-ordering, the relation R in Eqs.(3), (4) and (5) is supposed to be reflexive and transitive.

Minimum of rough set axiom groups

Definition 1 Rough Set Axiom Formulas are defined as follows:

(1) An arbitrary set $X\subseteq U$ is a rough set axiom formula;

(2) If α is a rough set axiom formula, then so are $\sim\alpha$, $L(\alpha)$ and $H(\alpha)$;

(3) If α, β are rough set axiom formulas, then so are $\alpha\cup\beta$ and $\alpha\cap\beta$;

(4) The only rough set axiom formulas are those obtainable by finite application of (1)–(3) in the above.

Remark The conception of Rough Set Axiom Formula (RSAF) is not used to describe rough set which is an opposite conception to definable (crisp) set. Normally, a Rough Set Axiom Formula is not a rough set. One can find that $L(\alpha)$ and $H(\alpha)$ are surely crisp sets in classic rough set theory. In fact, a rough set α is described by a pair of definable sets $< L(\alpha), H(\alpha)>$.

Definition 2 If α, β are rough set axiom formulas, then $\alpha\subseteq\beta$ and $\beta\subseteq\alpha$ are rough set inequalities.

Definition 3 The rough set axiom group satisfying the follow conditions is called minimal rough set axiom group:

(1) Each axiom in the axiom group is a rough set inequality;

(2) Each axiom in the axiom group is independent of others.

We adopt here the concept of minimization of rough set axiom group from Sun *et al.*(2002) from which three definitions above can be found.

A MINIMAL AXIOM GROUP OF ROUGH SET BASED ON QUASI-ORDERING

We now propose a minimal axiom group of rough set based on quasi-ordering, named Rough Set Axiom Group RT, as follows:

(RT1) $U\subseteq L(U)$
(RT2) $L(\sim X\cup Y)\subseteq\sim L(X)\cup L(Y)$
(RT3) $L(X)\subseteq X$
(RT4) $L(X)\subseteq L(L(X))$

Reliability of axiom group RT

Lemma 1 For the pair of rough operators L and H, $X\rightarrow(H(X), L(X))$, which satisfy the following axioms

(L1) $L(U)=U$
(L2) $L(X\cap Y)=LX\cap LY$
(H1) $H(\phi)=\phi$
(H2) $H(X\cup Y)=H(X)\cup H(Y)$
(LH) $H(X)=\sim L(\sim X)$
(M1) $X\subseteq H(X)$
(M2′) $H(H(X))\subseteq H(X)$

there exists a reflexive and transitive relation R on U satisfying $H(X)=R^{-}(X)$ and $L(X)=R_{-}(X)$.

Proof The lemma is easy to prove by Theorem 3 and Theorem 6 of Yao (1998a).

Lemma 2 (Yao, 1998a) Under the condition $L(U)=U$, we have

$$L(X\cap Y)=LX\cap LY \Leftrightarrow$$
$$L(U)=U \& L(\sim X\cup Y)\subseteq\sim L(X)\cup L(Y).$$

Theorem 1 For the rough operator $X\rightarrow L(X)$, which satisfy the axiom (RT1) through (RT4), there is a reflexive and transitive relation on U such that $L(X)= R_{-}(X)$.

Defining the dual operator of L by $H(X)=\sim L(\sim X)$, we have $H(X)= R^{-}(X)$.

Proof From axiom (RT1), we have

$$L(U)=U \quad (6)$$

That is axiom (L1) in Lemma 1. By (LH) and (L1), we get

$$H(\phi)=\phi \quad (7)$$

From axiom (RT2) and Eq.(6), together with Lemma 2, we have

$$L(X\cap Y)=L(X)\cap L(Y) \quad (8)$$

That is axiom (L2) in Lemma 1. By (LH) and (L2), we get

$$H(X\cup Y)=H(X)\cup H(Y) \quad (9)$$

From axiom (RT3), we have

$$X=\sim L(\sim X) \quad (10)$$

From (LH), we get

$$X\subseteq H(X) \quad (11)$$

In the same way, from axiom (RT4) and (LH) we have

$$H(H(X))\subseteq H(X) \tag{12}$$

From the proofs above, we know that

{(RT1), (RT2), (RT3), (RT4), (LH)}⇒{(L1), (L2), (H1), (H2), (LH), (M1), (M2′)}

In fact, axiom (LH) is the mutual defining method between lower approximation operator and upper approximation operator. From Lemma 1, we get Theorem 1.

Minimum of axiom group RT

Theorem 2 Rough set axiom group RT is minimal rough set axiom group.

Proof It is obvious that each axiom in RT is a rough set inequality, so axiom group RT satisfies the first condition in Definition 3.

We then prove RT also satisfies the second condition in Definition 3 as follows:

(1) Suppose that $U=\{a,b,c\}$ is the universe. $L(U)=\{a,b\}$, $L(\{a,b\})=\{a,b\}$, $L(\{a,c\})=\{a\}$, $L(\{b,c\})=\{b\}$, $L(\{a\})=\{a\}$, $L(\{b\})=\{b\}$, $L(\{c\})=\{c\}$, $L(\phi)=\phi$. Then we get the following validation (Table 1) showing that axiom (RT1) is independent of (RT2), (RT3) and (RT4).

In Table 1, the underlined items show contradiction of (RT1). For example, when $X=\{a,b,c\}$, we can find $L(U)=L(\{a,b,c\})=\{a,b\}$. Hence, we know that $U\not\subset L(U)$ which implies that axiom (RT1) is not satisfied. And at the same time, axioms (RT2), (RT3) and (RT4) are satisfied. So we know that (RT1) is independent of (RT2), (RT3) and (RT4). The meaning and function of the validation tables following are the same as here, and we just give the validation tables without further interpretation in the rest part of the paper.

(2) Suppose that $U=\{a,b,c\}$ is the universe. $L(U)=U$, $L(\{a,b\})=\{a,b\}$, $L(\{a,c\})=\{a\}$, $L(\{b, c\})=\{b\}$, $L(\{a\})=\{a\}$, $L(\{b\})=\{b\}$, $L(\{c\})=\{c\}$, $L(\phi)=\phi$.

Table 1 Validation table showing that (RT1) is independent of (RT2), (RT3) and (RT4)

$L(\sim X\cup Y)$, $\sim L(X)\cup L(Y)$		Y								$L(X)$	$L(L(X))$
		U	{a,b}	{a,c}	{b,c}	{a}	{b}	{c}	ϕ		
X	<u>U</u>	{a,b},{a,b}	{a,b},{a,b}	{a},{a}	{b},{b}	{a},{a}	{b},{b}	{c},{c}	ϕ,ϕ	<u>{a,b}</u>	{a,b}
	{a,b}	{a,b},U	{a,b},U	{a},{a,c}	{b},{b,c}	{a},{a,c}	{b},{b,c}	{c},{c}	{c},{c}	{a,b}	{a,b}
	{a,c}	{a,b},U	{a,b},U	{a,b},U	{b},{b,c}	{a,b},U	{b},{b,c}	{b},{b,c}	{b},{b,c}	{a}	{a}
	{b,c}	{a,b},U	{a,b},U	{a},{a,c}	{a,b},U	{a},{a,c}	{a,b},U	{a},{a,c}	{a},{a,c}	{b}	{b}
	{a}	{a,b},U	{a,b},U	{a,b},U	{b},{b,c}	{a,b},U	{b},{b,c}	{b},{b,c}	{b},{b,c}	{a}	{a}
	{b}	{a,b},U	{a,b},U	{a},{a,c}	{a,b},U	{a},{a,c}	{a,b},U	{a},{a,c}	{a},{a,c}	{b}	{b}
	{c}	{a,b},{a,b}	{a,b},{a,b}	{a,b},{a,b}	{a,b},{a,b}	{a,b},{a,b}	{a,b},{a,b}	{a,b},U	{a,b},{a,b}	{c}	{c}
	ϕ	{a,b},U	{a,b},U	{a,b},U	{a,b},U	{a,b},U	{a,b},U	{a,b},U	{a,b},U	ϕ	ϕ

Table 2 Validation table showing that (RT2) is independent of (RT1), (RT3) and (RT4)

$L(\sim X\cup Y)$, $\sim L(X)\cup L(Y)$		Y								$L(X)$	$L(L(X))$
		U	{a,b}	{a,c}	{b,c}	{a}	{b}	{c}	ϕ		
X	U	U,U	{a,b},{a,b}	{a},{a}	{b},{b}	{a},{a}	{b},{b}	{c},{c}	ϕ,ϕ	{a,b}	{a,b}
	{a,b}	U,U	U,U	{a},{a,c}	{b},{b,c}	{a},{a,c}	{b},{b,c}	{c},{c}	{c},{c}	{a,b}	{a,b}
	{a,c}	U,U	{a,b},U	U,U	{b},{b,c}	{a,b},U	{b},{b,c}	{b},{b,c}	{b},{b,c}	{a}	{a}
	{b,c}	U,U	{a,b},U	{a},{a,c}	U,U	{a},{a,c}	{a,b},U	{a},{a,c}	{a},{a,c}	{b}	{b}
	{a}	U,U	U,U	U,U	{b},{b,c}	U,U	{b},{b,c}	{b},{b,c}	{b},{b,c}	{a}	{a}
	{b}	U,U	{a,b},U	{a},{a,c}	U,U	{a},{a,c}	U,U	{a},{a,c}	{a},{a,c}	{b}	{b}
	{c}	U,U	{a,b},{a,b}	<u>U,{a,b}</u>	<u>U,{a,b}</u>	{a,b},{a,b}	{a,b},{a,b}	U,U	{a,b},{a,b}	{c}	{c}
	ϕ	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	ϕ	ϕ

Then we get the following validation (Table 2) showing that axiom (RT2) is independent of (RT1), (RT3) and (RT4).

(3) Suppose that $U=\{a,b,c\}$ is the universe. $L(U)=L(\{a,b\})=L(\{a,c\})=U$, $L(\{b,c\})=\{b,c\}$, $L(\{a\})=U$, $L(\{b\})=L(\{c\})=\{b,c\}$, $L(\phi)=\{b,c\}$. Then we get the following validation (Table 3) showing that axiom (RT3) is independent of (RT1), (RT2) and (RT4).

(4) Let $U=\{a,b,c,d\}$ be the universe. Suppose $L(U)=U$, $L(\{a,b,c\})=\{a,b\}$, $L(\{a,b,d\})=\{a\}$, $L(\{a,c,d\})=\{a\}$, $L(\{b,c,d\})=\{b\}$, $L(\{a,b\})=\{a\}$, $L(\{a,c\})=\{a\}$, $L(\{a,d\})=\{a\}$, $L(\{b,c\})=\{b\}$, $L(\{b,d\})=\phi$, $L(\{c,d\})=\phi$, $L(\{a\})=\{a\}$, $L(\{b\})=\phi$, $L(\{c\})=\phi$, $L(\{d\})=\phi$, $L(\phi)=\phi$, then we get the following validation (Table 4) showing that axiom (RT4) is independent of (RT1), (RT2) and (RT3).

From the steps (1)~(4) above, we know that each axiom in axiom group RT is independent of others. So the second condition in Definition 3 is satisfied and Theorem 2 is proved.

CONCLUSION

Rough set axiomatization is an aspect of rough set study aimed at characterizing rough set theory using dependable and minimal axiom groups. Thus,

Table 3 Validation table showing that (RT3) is independent of (RT1), (RT2) and (RT4)

$L(\sim X\cup Y)$, $\sim L(X)\cup L(Y)$		Y								$L(X)$	$L(L(X))$
		U	{a,b}	{a,c}	{b,c}	{a}	{b}	{c}	ϕ		
X	U	U,U	U,U	U,U	{b,c},{b,c}	U,U	{b,c},{b,c}	{b,c},{b,c}	{b,c},{b,c}	U	U
	{a, b}	U,U	U,U	U,U	{b,c},{b,c}	U,U	{b,c},{b,c}	{b,c},{b,c}	{b,c},{b,c}	U	U
	{a, c}	U,U	U,U	U,U	{b,c},{b,c}	U,U	{b,c},{b,c}	{b,c},{b,c}	{b,c},{b,c}	U	U
	{b, c}	U,U	U,U	U,U	U,U	U,U	U,U	{b,c},{b,c}	U,U	{b,c}	{b,c}
	{a}	U,U	U,U	U,U	{b,c},{b,c}	U,U	{b,c},{b,c}	{b,c},{b,c}	{b,c},{b,c}	U	U
	{b}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	{b,c}	{b,c}
	{c}	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	{b,c}	{b,c}
	ϕ	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	{b,c}	{b,c}

Table 4 Validation table showing that (RT4) is independent of (RT1), (RT2) and (RT3)

$L(\sim X\cup Y)$, $\sim L(X)\cup L(Y)$		Y																$L(X)$	$L(L(X))$
		U	{a,b,c}	{a,b,d}	{a,c,d}	{b,c,d}	{a,b}	{a,c}	{a,d}	{b,c}	{b,d}	{c,d}	{a}	{b}	{c}	{d}	ϕ		
X	U	U,U	ab,ab	a,a	a,a	b,b	a,a	a,a	a,a	b,b	ϕ,ϕ	ϕ,ϕ	a,a	ϕ,ϕ	ϕ,ϕ	ϕ,ϕ	ϕ,ϕ	U	U
	{a,b,c}	U,U	U,U	a,acd	a,acd	b,bcd	a,acd	a,acd	a,acd	b,bcd	ϕ,cd	ϕ,cd	a,acd	ϕ,cd	ϕ,cd	ϕ,cd	ϕ,cd	ab	a
	{a,b,d}	U,U	ab,U	U,U	a,U	b,bcd	ab,U	a,U	a,U	b,bcd	b,bcd	ϕ,bcd	a,U	b,bcd	ϕ,bcd	ϕ,bcd	ϕ,bcd	a	a
	{a,c,d}	U,U	ab,U	a,U	U,U	b,bcd	a,U	ab,U	a,U	b,bcd	ϕ,bcd	b,bcd	a,U	ϕ,bcd	b,bcd	ϕ,bcd	ϕ,bcd	a	a
	{b,c,d}	U,U	ab,U	a,acd	a,acd	U,U	a,acd	a,acd	a,acd	ab,U	a,acd	a,acd	a,acd	a,acd	a,acd	a,acd	a,acd	b	ϕ
	{a,b}	U,U	ab,U	U,U	U,U	b,bcd	ab,U	ab,U	U,U	b,bcd	b,bcd	b,bcd	ab,U	b,bcd	b,bcd	b,bcd	b,bcd	a	a
	{a,c}	U,U	U,U	a,U	U,U	b,bcd	a,U	U,U	a,U	b,bcd	ϕ,bcd	b,bcd	a,U	ϕ,bcd	b,bcd	ϕ,bcd	ϕ,bcd	a	a
	{a,d}	U,U	ab,U	U,U	U,U	b,bcd	ab,U	ab,U	U,U	b,bcd	b,bcd	b,bcd	ab,U	b,bcd	b,bcd	b,bcd	b,bcd	a	a
	{b,c}	U,U	U,U	a,acd	a,acd	U,U	a,acd	a,acd	a,acd	U,U	a,acd	a,acd	a,acd	a,acd	a,acd	a,acd	a,acd	b	ϕ
	{b,d}	U,U	ab,U	U,U	a,U	U,U	ab,U	a,U	a,U	ab,U	U,U	a,U	a,U	ab,U	ϕ,U	a,U	a,U	ϕ	ϕ
	{c,d}	U,U	ab,U	a,U	U,U	U,U	a,U	ab,U	a,U	ab,U	a,U	U,U	a,U	a,U	ab,U	a,U	a,U	ϕ	ϕ
	{a}	U,U	U,U	U,U	U,U	b,bcd	U,U	U,U	U,U	b,bcd	b,bcd	b,bcd	U,U	b,bcd	b,bcd	b,bcd	b,bcd	a	a
	{b}	U,U	U,U	U,U	a,U	U,U	U,U	a,U	a,U	U,U	U,U	a,U	a,U	U,U	a,U	a,U	a,U	ϕ	ϕ
	{c}	U,U	U,U	a,U	U,U	U,U	a,U	U,U	a,U	U,U	a,U	U,U	a,U	a,U	U,U	a,U	a,U	ϕ	ϕ
	{d}	U,U	ab,U	U,U	U,U	U,U	ab,U	ab,U	U,U	ab,U	U,U	U,U	ab,U	ab,U	ab,U	U,U	ab,U	ϕ	ϕ
	ϕ	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	U,U	ϕ	ϕ

rough set theory can be studied by logic and axiom system methods. To characterize the rough set based on quasi-ordering, an axiom group named RT, consisting of 4 axioms, is proposed. The validity of the axiom group, which shows that characterizing of rough set theory is reasonable, is proved. Simultaneously, the minimization of the axiom group, which requires that each axiom is an equation and each is independent, is proved. The axiom group is helpful for researching on rough set theory by logic and axiom system methods.

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