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Quasilinear singularly perturbed problem with boundary perturbation*

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Abstract: A class of quasilinear singularly perturbed problems with boundary perturbation is considered. Under suitable conditions, using theory of differential inequalities we studied the asymptotic behavior of the solution for the boundary value problem.

Key words: Quasilinear problem, Singular perturbation, Boundary perturbation
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INTRODUCTION

The nonlinear singularly perturbed problem is a very attractive study subject in international academic circles (De Jager and Jiang, 1996). During the past decade many approximate methods have been refined and developed, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales. Many scholars (O'Malley Jr., 2000; Butuzov *et al.*, 2001; Kelley, 2001; Hamouda, 2002; Bell and Deng, 2003; Adams *et al.*, 2003) conducted a great deal of research on the problem. Using differential inequalities and other methods, Mo *et al.* also considered a class of singularly perturbed nonlinear boundary value problems for the ordinary differential equation (Mo, 1993; 1999), the reaction diffusion equations (Mo, 1989; 2001a; Mo and Feng, 2001), the boundary value problems of elliptic equation (Mo and Shao, 2001; Mo and Ouyang, 2001), the initial boundary value problems of hyperbolic equation (Mo, 2001b), the shock layer solution of nonlinear equation for singularly perturbed problem (Mo and Wang, 2002a; Mo *et al.*, 2003) and the problems of biomathematics (Mo and Wang, 2002b). In this work, using a special method, we studied a class of singularly perturbed quasilinear problem with boundary perturbation.

Now we consider the following problem:

$$\varepsilon y'' = f(x,y)y' + g(x,y), \qquad (1)$$

$$y = A(\varepsilon),\ x = r(\varepsilon), \qquad (2)$$

$$y = B(\varepsilon),\ x = 1 - r(\varepsilon), \qquad (3)$$

where ε is a small positive parameter, $r(0)=0$. Eqs.(1)−(3) involve a class of quasilinear singularly perturbed problem with boundary perturbation. We shall construct the asymptotic expansion of the solution and discuss its asymptotic behavior.

We need the following hypotheses:

[H_1] f, g, A, B and $r \geq 0$ are sufficiently smooth

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functions with regard to their variables in corresponding ranges.

[H2] $f(x, y)\geq k>0$ where k is a constant.

CONSTRUCTING THE FORMAL ASYMPTOTIC SOLUTION

We now construct the formal asymptotic expansion for the solution of Eqs.(1)–(3).

Eqs.(1)–(3) reduces to

$$f(x,Y)Y' + g(x,Y) = 0, \tag{4}$$

$$Y = A(0),\ x = 0. \tag{5}$$

Obviously, there is a sufficiently smooth solution $Y_0(x)$ for Eqs.(4) and (5).

Let the formal expansion of the outer solution $Y(x,\varepsilon)$ for the original problem Eqs.(1)–(3) be

$$Y(x,\varepsilon) \sim \sum_{i=0}^{\infty} Y_i(x)\varepsilon^i . \tag{6}$$

Substituting Eq.(6) into Eq.(1), expanding f, g in ε, grouping coefficients of the same powers of ε and equating the corresponding terms for the two sides of the equation respectively, for i=1,2… we obtain

$$f(x,Y_0)Y_i' + [f_y(x,Y_0)Y_0' + g_y(x,Y_0)]Y_i = F_i + (Y_{i-1})'', \tag{7}$$

where F_i, i=1,2,…, are successively determined functions of Y_j, $j\leq i-1$.

Substituting Eq.(6) into Eq.(2), considering the perturbed boundary $x=r(\varepsilon)$ near x=0, we have also respectively (Kevorkian and Cole, 1996)

$$Y_i = A_i - \sum_{j=0}^{i} Y_{j(i-j)},\quad x = 0, \tag{8}$$

where

$$A_i = \frac{1}{i\,!}[\frac{\mathrm{d}^i A}{\mathrm{d}\varepsilon^i}]_{\varepsilon=0},\quad i = 1,2,\cdots,$$

$$Y_{jk} = \frac{1}{k\,!}\ [\frac{\mathrm{d}^k Y_j(r(\varepsilon))}{\mathrm{d}\varepsilon^k}]_{\varepsilon=0},\ j,k = 0,1,\cdots.$$

From the linear problems Eqs.(7), (8), we can solve Y_i successively. From Eq.(6), we obtain the outer solution $Y(x,\varepsilon)$ for the original problem. But it may not satisfy the condition Eq.(3), so that we need to construct the boundary layer corrective term Z near x=1.

We introduce a stretched variable (De Jager and Jiang, 1996):

$$\tau = \frac{1-x}{\varepsilon}.$$

And let the solution $y(x,\varepsilon)$ of original problem Eqs.(1)–(3) be

$$y = Y(x,\varepsilon) + Z(\tau,\varepsilon). \tag{9}$$

Substituting Eq.(9) into Eq.(1), Eq.(3), we have

$$\frac{1}{\varepsilon}Z_{\tau\tau} = \frac{1}{\varepsilon}f(1-\tau\varepsilon, Y+Z)Z_\tau + g(1-\tau\varepsilon, Y+Z) + [f(x,Y+Z) - f(x,Y)]Y' - g(x,Y)], \tag{10}$$

$$Z = B(\varepsilon) - Y(1-r(\varepsilon),\varepsilon),\quad \tau = r(\varepsilon)/\varepsilon. \tag{11}$$

Let

$$Z(\tau,\varepsilon) \sim \sum_{i=0}^{\infty} Z_i(\tau)\varepsilon^i. \tag{12}$$

Notice that the perturbed boundary $x=1-r(\varepsilon)$ near x=1 (Kevorkian and Cole, 1996), substituting Eqs.(9), (6) and (12) into Eqs.(10), (11), developing f, g, B and r in ε, grouping coefficients of the same powers of ε and equating the corresponding terms for the two sides of the equation respectively, for i=1,2,…, we obtain

$$\frac{\mathrm{d}^2 Z_i}{\mathrm{d}\tau^2} = f(1, Y_0 + Z_0)\frac{\mathrm{d}Z_i}{\mathrm{d}\tau} + H_i, \tag{13}$$

$$Z_i = B_i - Y_i(1) - \sum_{j=0}^{i} Z_{j(i-j)},\quad \tau = r(\varepsilon)/\varepsilon, \tag{14}$$

where H_i, i=1,2,…, are successively determined functions, and

$$B_i = \frac{1}{i!}[\frac{\mathrm{d}^i B}{\mathrm{d}\varepsilon^i}]_{\varepsilon=0}, \quad i = 0,1,2,\ldots,$$

$$Z_{jk} = \frac{1}{k!}[\frac{\mathrm{d}^k[Z_j((1-r(\varepsilon))/\varepsilon) + Y_j((1-r(\varepsilon))/\varepsilon)]}{\mathrm{d}\varepsilon^k}]_{\varepsilon=0},$$

j, k=0,1, ….

From the problems Eqs.(13), (14), we can obtain Z_i, i=1,2,…, and satisfy

$$Z_i = O(\exp(-k_i\tau)) = O(\exp(-k_i\frac{1-x}{\varepsilon})), \quad 0 < \varepsilon << 1,$$

where k_i, i=1,2,…, are constants and $k \ge k_{i-1} \ge k_i > 0$.

Then we can construct the following formal asymptotic expansion of the solution $y(x,\varepsilon)$ for the original problem Eqs.(1)–(3):

$$y(x,\varepsilon) \sim \sum_{i=0}^{\infty}[Y_i(x) + Z_i(\frac{1-x}{\varepsilon})]\varepsilon^i, \quad r(\varepsilon) \le x \le 1-r(\varepsilon), \quad 0 < \varepsilon << 1. \tag{15}$$

THE FINAL RESULT

Now we prove that Eq.(15) is a uniformly valid asymptotic expansion.

Theorem Under the hypotheses [H_1], [H_2], there exists a solution $y(x,\varepsilon)$ of the singularly perturbed problem Eqs.(1)–(3) for the quasilinear equations with boundary perturbation and the solution holds for the uniformly valid asymptotic expansion Eq.(15) on $r(\varepsilon) \le x \le 1-r(\varepsilon)$ for $0<\varepsilon<<1$.

Proof We first construct the auxiliary functions α and β:

$$\alpha = W_m - \frac{\delta}{l}(2\mathrm{e}^{\lambda x} - 1)\varepsilon^m, \tag{16}$$

$$\beta = W_m + \frac{\delta}{l}(2\mathrm{e}^{\lambda x} - 1)\varepsilon^m, \tag{17}$$

where δ is a large enough positive constant, which will be decided below, and

$$W_m \equiv \sum_{i=0}^{m}[Y_i + Z_i]\varepsilon^i,$$

while l is a certain positive constant such that

$$|[f_x]W_m'| + |[g_x]W_m| \le l,$$

and λ is a negative real root satisfying equation $\varepsilon\lambda^2 + k\lambda + l = 0$ for small enough ε.

Obviously, we have

$$\alpha \le \beta, \ r(\varepsilon) \le x \le 1-r(\varepsilon). \tag{18}$$

And there are positive constants $r_0>0$, M_1 and M_2 for $x=r(\varepsilon)$ near x=0, such that

$$\begin{aligned} \alpha &= W_m - \frac{\delta}{l}(2\mathrm{e}^{\lambda r(\varepsilon)} - 1)\varepsilon^m \\ &= \sum_{i=0}^{m} Y_i\varepsilon^i + \sum_{i=0}^{m} Z_i\varepsilon^i - \frac{\delta\, r_0}{l}\varepsilon^m \\ &\le A(0) + \sum_{i=1}^{m} A_i\varepsilon^i + M_1\varepsilon^m - \frac{\delta\, r_0}{l}\varepsilon^m \\ &\le A(\varepsilon) + (M_1 + M_2 - \frac{\delta\, r_0}{l})\varepsilon^m. \end{aligned}$$

Thus selecting $\delta \ge (M_1+M_2)l/r_0$, we have

$$\alpha \le A(\varepsilon), \quad x = r(\varepsilon). \tag{19}$$

and

$$\beta \ge A(\varepsilon), \quad x = r(\varepsilon). \tag{20}$$

Analogously, we can prove that

$$\alpha \le A(\varepsilon) \le \beta, \quad x = 1-r(\varepsilon). \tag{21}$$

Now we prove that

$$\varepsilon\alpha'' - f(x,\alpha)\alpha' - g(x,\alpha) \ge 0, \ r(\varepsilon) \le x \le 1-r(\varepsilon), \tag{22}$$

$$\varepsilon\beta'' - f(x,\beta)\beta' - g(x,\beta) \le 0, \ r(\varepsilon) \le x \le 1-r(\varepsilon). \tag{23}$$

From the hypotheses, for small enough ε, there is a positive constants M_3 such that

$$\begin{aligned}
&\varepsilon\alpha'' - f(x,\alpha)\alpha' - g(x,\alpha) \\
&= \varepsilon W_m'' - f(x,W_m)W_m' - g(x,W_m) \\
&+[f(x,W_m) - f(x,W_m - \frac{\delta}{l}(2\mathrm{e}^{\lambda x} - 1))]W_m' \\
&+[g(x,W_m) - g(x,W_m - \frac{\delta}{l}(2\mathrm{e}^{\lambda x} - 1)\varepsilon^m)] \\
&\geq -[f(x,Y_0)Y_0' + g(x,Y_0)] \\
&+\sum_{i=1}^{m}[(Y_{i-1})'' - f_y(x,Y_0)Y_i' - g_y(x,Y_0)Y_i + F_i]\varepsilon^i \\
&+\sum_{i=0}^{m}[\frac{\mathrm{d}^2 Z_i}{\mathrm{d}\tau^2} - f(1,Y_0 + Z_0)\frac{\mathrm{d}Z_i}{\mathrm{d}\tau} - H_i]\varepsilon^i \\
&-2(\varepsilon\lambda^2 + k\lambda + l)\frac{\delta}{l}\mathrm{e}^{\lambda x}\varepsilon^m - M_3\varepsilon^m + \delta\varepsilon^m \\
&\geq (\delta - M_3)\varepsilon^m .
\end{aligned}$$

Selecting $\delta \geq M_3$, then we proved Eq.(22).

Analogously, we can prove Eq.(23).

Thus from Eqs.(18)–(23) and the Nagumo theorem (Chang and Howes, 1984), for small enough ε, we obtain

$$\alpha(x,\varepsilon) \leq y(x,\varepsilon) \leq \beta(x,\varepsilon),$$
$$r(\varepsilon) \leq x \leq 1 - r(\varepsilon) .$$

From Eq.(16) and Eq.(17), we have

$$y(x,\varepsilon) = \sum_{i=0}^{m}[Y_i(x) + Z_i(\frac{1-x}{\varepsilon})]\varepsilon^i + O(\varepsilon^m),$$
$$r(\varepsilon) \leq x \leq 1 - r(\varepsilon),\ 0 < \varepsilon << 1.$$

The proof of the theorem is completed.

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