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The characterization of weighted local hardy spaces on domains and its application*

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Abstract: In this paper, we give the four equivalent characterizations for the weighted local hardy spaces on Lipschitz domains. Also, we give their application for the harmonic function defined in bounded Lipschitz domains.

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INTRODUCTION

Weighted hardy spaces $H^p(\mathrm{R}^n,\mathrm{d}w)$ ($n\geq1$) and their atomic characterization were studied by Stromberg and Torchinsky (1989) and references therein, where w is in Muckenhoupt's class $A_p(\mathrm{R}^n)$. In Chang *et al.*(1993), Miyachi (1990) and Semmes (1994), the authors defined hardy spaces on domains without weights. In this paper, we will give the four equivalent characterizations of weighted local hardy spaces on Lipschitz domains. We also present a simple proof of the atomic characterization of $H_r^p(\Omega)$ of Miyachi [e.g. Semmes (1994)].

We say a nonnegative $w\in A_p(\mathrm{R}^n)$ ($1<p<\infty$) if for every cube $Q\subset\mathrm{R}^n$, we have

$$(\int_Q w(x)\mathrm{d}x)(\int_Q w(x)^{-1/(p-1)}\mathrm{d}x)^{p-1}\leq C|Q|^p. \tag{1}$$

For the case of p=1, $w\in A_1(\mathrm{R}^n)$ if $|Q|^{-1}\int_Q w(x)\mathrm{d}x \leq C\,\mathrm{ess\,inf}_{x\in Q}\, w(x)$ for every cube

$$Q\subset\mathrm{R}^n. \tag{1′}$$

Similarly, we say a nonnegative function $w\in A_p(\Omega)$ ($1<p<\infty$) or $w\in A_1(\Omega)$ if Eq.(1) or Eq.(1′) holds for every cube $Q\subset\Omega$. The least constant in Eq.(1) or Eq.(1′) is called the constant of $w\in A_p$ for $1\leq p<\infty$, denoted by $C(w)$.

Denote $A_\infty(\mathrm{R}^n):=\bigcup_{p\geq1}A_p(\mathrm{R}^n)$ and $A_\infty(\Omega):=\bigcup_{p\geq1}A_p(\Omega)$. If $w\in A_p(\mathrm{R}^n)$ with $1<p<\infty$, then $w\in A_r(\mathrm{R}^n)$ for every $r>p$ and $w\in A_q(\mathrm{R}^n)$ for some $1<q<p$. We thus use $q_w:=\inf\{q>1: w\in A_q(\mathrm{R}^n)\}$ to denote the critical index of w (Lee and Lin, 2002), and define a weighted measure $w(E):=\int_E w(x)\mathrm{d}x$ for measurable set $E\subset\mathrm{R}^n$.

Let $\varphi\in C_0^\infty(\mathrm{R}^n)$ with $\int_{\mathrm{R}^n}\varphi(x)\mathrm{d}x=1$. Define $\varphi_t(x):=t^{-n}\varphi(x/t)$ for $t>0$. For $f\in C_0^\infty(\mathrm{R}^n)'$, let $f^+(x):=\sup_{0<t<1}|f*\varphi_t(x)|$, then for $0<p\leq1$ and $w\in A_\infty(\mathrm{R}^n)$,

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$h^p(\mathrm{R}^n,\mathrm{d}w)$ consists of those tempered distributions $f \in C_0^\infty(\mathrm{R}^n)'$ with

$$\|f\|_{h^p(\mathrm{R}^n,\mathrm{d}w)} := (\int_{\mathrm{R}^n} |f^+(x)|^p\, w(x)\mathrm{d}x)^{1/p} < \infty .$$

Recall the definitions of the atom and the atomic decomposition of the weighted hardy spaces $h^p(\mathrm{R}^n,\mathrm{d}w)$ (Stromberg and Torchinsky, 1989).

Definition 1 On R^n, let $0<p\le 1\le q\le\infty$ and $p\ne q$ such that $w\in A_q(\mathrm{R}^n)$ with critical index q_w. Set $[\cdot]$ the integer function. For $s\in\mathbf{N}$ with $s\ge N(w):=[n(q_w/p-1)]$, a real-valued function a is called (p,q,s,w)-atom if (i) $a\in L^q(\mathrm{d}w)$ is supported in a cube Q, (ii) $\|a\|_{L^q_\omega} \le w(Q)^{-1/q-1/p}$, (iii) if $l_Q<1$, then $\int_{\mathrm{R}^n} a(x)x^\alpha \mathrm{d}x = 0$ for every multi-index α with $|\alpha|\le s$, or if $l_Q\ge 1$, then a_Q does not have any vanishing moment.

Theorem A Let $w\in A_\infty(\mathrm{R}^n)$ and $0<p\le 1$.

For each $f\in h^p(\mathrm{R}^n,\mathrm{d}w)$, there exist $\{a_Q\}$ of $(p,\infty,N(w),w)$-atoms and $\{\lambda_Q\}$ of real numbers with $\sum_Q |\lambda_Q|^p \le c\|f\|^p_{h^p(\mathrm{R}^n,\mathrm{d}w)}$ such that $f=\sum \lambda_Q a_Q$ in $h^p(\mathrm{R}^n,\mathrm{d}w)$.

We define the weighted hardy spaces on Lipschitz domains via the restriction as below,

$$h_r^p(\Omega,\mathrm{d}w) := \{f\in C_0^\infty(\Omega)' : \exists F\in h^p(\mathrm{R}^n,\mathrm{d}w), \text{ s.t. } F|_\Omega = f\}$$

with the norm $\|f\|_{h_r^p(\Omega,\mathrm{d}w)} := \inf\{\|F\|_{h^p(\mathrm{R}^n,\mathrm{d}w)} : F|_\Omega = f\}$. When $w(x)\equiv 1$ in Ω, the spaces $h_r^p(\Omega,\mathrm{d}w)$ are denoted by $h_r^p(\Omega)$, which had been extensively studied in Chang *et al.*(1993), Miyachi (1990) and Semmes (1994).

To study the weighted hardy spaces on Lipschitz domains, we should introduce the weighted atom on domains and present an extension theorem for $w\in A_p(\Omega)$.

Definition 2 Let $w\in A_\infty(\Omega)$ and Q be a cube with $2Q\subset\Omega$, and suppose that $a(x)$ satisfies (i) and (ii) of Definition 1,

(i) if $4Q\subset\Omega$ then a_Q satisfies (iii) of Definition 1. This type of atom is called by an interior (p,q,s,w)-atom, which we denote by $Q\in\mathcal{I}$, or (ii) if $4Q\cap\partial\Omega\ne\varnothing$, then a_Q has no moment condition. This type of atom is called a boundary (p,q,s,w)-atom, which we denote by $Q\in\mathcal{B}$.

Lemma 3 Let Ω be a bounded Lipschitz domain, then for every $w\in A_\infty(\Omega)$, there exists an extension weight $W\in A_p(\mathrm{R}^n)$ with $W(x)|_\Omega = w(x)$. Moreover, the constant of $C(W)$ only depends on $C(w)$, the Lipschitz constant of Ω and n.

Lemma 3 will be shown in Section 2. Now we introduce our main theorem.

Theorem 4 Let Ω be a bounded Lipschitz domain and $w\in A_\infty(\Omega)$, assume that $f\in C_0^\infty(\Omega)'$, then the following are equivalent:

(i) Restriction: $f\in h_r^p(\Omega,\mathrm{d}w)$;

(ii) Maximal function:

$$f_\Omega^+(x) := \sup_{0<t<\delta(x)/c_0} |\varphi_t * f(x)| \in L^p(\Omega,\mathrm{d}w)$$

for some $c_0>1$, where $\varphi\in C_0^\infty(B(0,1))$ with $\int\varphi(x)\mathrm{d}x = 1$.

(iii) Grand Maximal:

$$f_\Omega^*(x) := \sup_{0<t<\delta(x)/c_0}\ \sup_{\varphi\in\mathcal{F}_t(x)} |\langle f,\varphi\rangle| \in L^p(\Omega,\mathrm{d}w),$$

where $\mathcal{F}_t(x) := \{\varphi\in C_0^\infty(\Omega);\ \mathrm{supp}\,\varphi\subset B(x,t),\ |\partial^\alpha\varphi| \le t^{-|\alpha|-n}$ for every $\alpha\}$, and for some $c_0>1$;

(iv) Atomic Decomposition:

f has an atomic decomposition $f=\sum_{Q_i\in\mathcal{I}} \lambda_{Q_i} a_{Q_i} \times \sum_{Q_b\in\mathcal{B}} \lambda_{Q_b} a_{Q_b}$ in $C_0^\infty(\Omega)'$, where a_{Q_i}'s are the interior (p,q,s,w)-atoms, a_{Q_i}'s are the boundary (p,q,s,w)-atoms, and $\{\lambda_{Q_i}\}, \{\lambda_{Q_b}\}$ satisfy

$$\sum_{Q_i\in\mathcal{I}} |\lambda_{Q_i}|^p + \sum_{Q_b\in\mathcal{B}} |\lambda_{Q_b}|^p < \infty .$$

When $w(x)\equiv 1$, (Restriction) $\Leftrightarrow$ (Atomic Decomposition) was studied in Chang *et al.*(1993) and (Maximal) $\Leftrightarrow$ (Grand Maximal) $\Leftrightarrow$ (Atomic Decomposition) in Miyachi (1990).

For any domain Ω, we let $\mathrm{dist}(x, \partial\Omega)$ be the distance from x to $\partial\Omega$. We know that there exists a function δ defined in Ω, such that

(i) $c_1\mathrm{dist}(x,\partial\Omega)\le\delta(x)\le c_2\mathrm{dist}(x,\partial\Omega)$ for any $x\in\Omega$;

(ii) $\delta(x)\in C^\infty(\Omega)$ and

$$\left|\partial^\alpha\delta(x)\right|\le c_\alpha\mathrm{dist}(x,\partial\Omega)^{1-|\alpha|}$$

for all α. Where c_1, c_2, c_α are independent of Ω (Stein, 1979).

As an application of Theorem 4, we have

Proposition 5 Let u be harmonic in a bounded Lipschitz domain Ω, $0<p\le1$ and $-1<s<1$, then $\delta^s u\in h_r^p(\Omega)$ iff $u\in h_r^p(\Omega,\delta^{sp}\mathrm{d}x)$ iff $u\in L^p(\Omega,\delta^{sp}\mathrm{d}x)$.

PROOF OF THE MAIN THEOREM

We will use the following notations. If Q is a cube whose sides are parallel to the axis and C is a positive constant, then x_Q, l_Q denote the center, sidelength of Q, and CQ denote $Q(x_Q,Cl_Q)$. We denote χ_E by a characteristic function of the set E.

We recall a Whitney decomposition. That is, for an open set $\Omega\subset\mathrm{R}^n$, there exist the constants c_1, c_2, which are independent of Ω. (If there is no other claim, we always choose $c_1=1$, $c_2=4$), such that the set of dyadic cubes $W:=\{Q_k\}$ satisfies

$$\bigcup_{Q_k\in W}\bar{Q}_k=\Omega, c_1\mathrm{diam}(Q_k)\le\mathrm{dist}(Q_k,\Omega^c)\le c_2\mathrm{diam}(Q_k), \tag{2}$$

The Q_k are mutually disjoint and nonoverlap.

Proof of Lemma 3 We complete the proof by two steps.

Step 1: Ω is a Lipschitz graph. That is, $\exists\,\psi\in\mathrm{Lip}(1)$, $M>0$, s.t. $|\psi(x')-\psi(y')|\le M|x'-y'|$ for $x'\in\mathrm{R}^{n-1}$, and $\Omega=\{x{:}x_n>\psi(x')\}$ (Stein, 1979).

To prove it, we use the methods of reflection, i.e., given $w\in A_p(\Omega)$, we define $W(x)=w(x)$ if $x\in\Omega$ or $=w(x',-x_n+\psi(x'))$ if $(x', x_n)\in(\bar{\Omega})^c$.

Now if $Q\subset\Omega$ or $Q\subset\Omega^c$, by the definition of $w(x)\in A_p(\Omega)$ and Lipschitz graph, Eq.(1) or Eq.(1′) is obviously true. If $Q\cap\partial\Omega\ne\varnothing$ because $w(x)\in A_p(\Omega)$ has doubling property, i.e., $\exists C=C(w)$, s.t. $w(2P)\le Cw(P)$ for every cube $2P\subset\Omega$, we know $W(2Q\cap\Omega)\le C(w)W(Q\cap\Omega)$ if $|Q\cap\Omega|^{-}\ge|Q|/4$ or $W(2Q\cap\Omega^c)\le C(w)W(Q\cap\Omega^c)$ if $|Q\cap\Omega^c|\ge|Q|/4$. Similarly the above properties are true for $w'(x):=w(x)^{-1/(p-1)}$ by the fact $w'(x)\in A_{p'}(\Omega)$ for $1/p'+1/p=1$.

Step 2: Ω is a bounded Lipschitz domain. By Step 1 and the definition of bounded Lipschitz domain (Stein, 1979), we can obtain that $w(x)\in A_p(\Omega)$ has a slight extension $w_1(x)\in A_p$ defined in a Lipschitz domain Ω_1, where $\Omega\subset\Omega_1$ and $\mathrm{dist}(\partial\Omega,\partial\Omega_1)\ge C_0$ for some small constant $C_0>0$. For Ω_1, there exists a Lipschitz domain Ω_2 with $\Omega_1\subset\Omega_2$ and $\mathrm{dist}(\partial\Omega_1,\partial\Omega_2)\ge C_0$ for the same constant $C_0>0$, such that $w_2(x)\in A_p(\Omega_2)$. After finite N steps as above, we know that $w\in A_p(\Omega)$ has an extension $w_N(x)\in A_p$, defined in Ω_N with $\Omega\subset\Omega_N$; and moreover, there exists a cube Q_0 with $\Omega\subset Q_0\subset\Omega_N$.

So we can define $W(x)$ on R^n by the period extension, i.e. $W(x+k):=w_N(x)$ for $\forall x\in Q_0$, $\forall k\in\mathcal{Z}^n$. Now we prove Eq.(1) or Eq.(1′) for every $Q\subset\mathrm{R}^n$. If $Q\subset Q_k=Q_0+k=\{y+k{:}y\in Q_0\}$ for same $k\in\mathcal{Z}^n$, Eq.(1) or Eq.(1′) is obviously true. For general cube Q, by the fact the number $\#\{Q_k\cap Q\ne\varnothing\}\le(l_Q/l_{Q_0})^n$, we have

$$\begin{aligned}|Q|^{-1}\int_Q W(x)\mathrm{d}x&=|Q|^{-1}\sum_{k:Q_k\cap Q\ne\varnothing}\int_{Q_k\cap Q}W(x)\mathrm{d}x\\&\le|Q|^{-1}\sum_{k:Q_k\cap Q\ne\varnothing}\int_Q w_N(x)\mathrm{d}x\\&\le C(w)|Q|^{-1}\int_{Q_0}w_N(x)\mathrm{d}x(l_Q/l_{Q_0})^n\\&\le C(w)|Q_0|^{-1}\int_{Q_0}w_N(x)\mathrm{d}x.\end{aligned}$$

Similarly we have

$$|Q|^{-1}\int_Q W(x)^{-1/(p-1)}\mathrm{d}x\le C(w)|Q|^{-1}\int_Q w_N(x)^{-1/(p-1)}\mathrm{d}x\,.$$

By the above analysis, we complete the proof of Step 2.

Proof of Theorem 4 We complete the proof by showing (Restriction) $\Leftrightarrow$ (Atomic Decomposition) $\Rightarrow$ (Grand Maximal) $\Leftrightarrow$ (Maximal) $\Rightarrow$ (Restriction).

The case (Restriction) $\Rightarrow$ (Atomic Decomposition): This proof is similar to that of Chang *et al.*(1993) and so we omit the details.

The case (Atomic Decomposition) $\Rightarrow$ (Restriction): Assume that f has the atomic decomposition $f=\sum_{Q_i\in\mathcal{I}}\lambda_{Q_i}a_{Q_i}\sum_{Q_b\in\mathcal{B}}\lambda_{Q_b}a_{Q_b}$, we should construct a function $F=h^p(\mathrm{R}^n,\mathrm{d}w)$ with $F|_\Omega=f$. Note that for every $w(x)\in A_p(\Omega)$, there exists an extension $W(x)\in A_p(\mathrm{R}^n)$ (see Lemma 3).

If $4Q_i\subset Q$ or $l_{Q_b}=1$ we set $A_{Q_i}(x):=a_{Q_i}(x)$ or $A_{Q_b}(x):=a_{Q_b}(x)$.

Now we treat a_{Q_b} with $l_{Q_b}<1$.

For every $N\in\mathbf{N}\cup\{0\}$, there exist $\{\phi_\alpha\}\subset C_0^\infty(B(0,1))$ such that $\int_{\mathrm{R}^n}x^\beta\phi_\alpha(x)\mathrm{d}x=\delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}=0$ if $\alpha\neq\beta$, or $=1$ if $\alpha=\beta$ for all $|\alpha|,|\beta|\leq N$.

Using the fact that Ω is a Lipschitz domain, $\forall Q_b$ (so $l_{Q_b}\approx\mathrm{dist}(Q_b,\Omega)$), $\exists Q_b'$, Q_b^e, s.t. $Q_b\subset Q_b'$, $Q_b^e\subset(\bar{\Omega})^c\cap Q_b'$ with $l(Q_b^e)\approx l(Q_b)\approx l(Q_b')$.

So for $N(w)=[n(q_w/p-1)_+]$, we define the function

$$A_{Q_b}(x):=a_{Q_b}(x)-\sum_\alpha b_\alpha\phi_\alpha((x-x_{Q_b^e})/l(Q_b^e)).$$

To satisfy $\int A_{Q_b}(x)x^\alpha\mathrm{d}x=0$, $0\leq|\alpha|\leq N(w)$, we take $b_a=\int a_{Q_b}(x)(x-x_{Q_b^e})^\alpha\,\mathrm{d}x/(l(Q_b^e))^{\alpha+n}$.

So we can obtain that A_{Q_b} is (p,q,s,w)-atom in R^n by the following estimate

$$\begin{aligned}\|A_{Q_b}\|_{L_w^q}&\leq\|a_{Q_b}\|_{L_w^q}+\sum_\alpha|c_\alpha|\|\varphi_\alpha\|_{L_w^q}\\&\leq w(Q_b)^{1/q-1/p}+\sum_\alpha l_{Q_b}^{-n-|\alpha|}\left|\int a_{Q_b}(x)(x-x_{Q_b^e})^\alpha\,\mathrm{d}x\right|w(Q_b)^{1/q}\\&\leq w(Q_b)^{1/q-1/p}+|Q_b|^{-1}\int|a_{Q_b}|\mathrm{d}x w(Q_b)^{1/q}\\&\leq C(w)w(Q_b)^{1/q-1/p}.\end{aligned}$$

So $F=\sum_{Q_i\in\mathcal{I}}\lambda_{Q_i}a_{Q_i}\sum_{Q_b\in\mathcal{B}}\lambda_{Q_b}a_{Q_b}$, is an extension of f.

The Case (Atomic Decomposition) $\Rightarrow$ (Grand Maximal): Let $f=\sum_Q\lambda_Q a_Q(x)$ with $\sum_Q|\lambda_Q|^P<\infty$ be the atomic decomposition. We need estimate $\|f_\Omega^*\|_{L^q(\Omega,\mathrm{d}w)}$, and so we need only check $\|(a_Q)_\Omega^*\|_{L^q(w,\Omega)}\leq C$ for some constant $C=(n,w,p,q)$.

We choose $\varphi\in C_0^\infty(B(0,1))$ with $\int\varphi(x)\mathrm{d}x=1$.

Case (i): a_Q is an interior atom with $l_Q<1$. The estimate of $\|(a_Q)_\Omega^*\|_{L^q(w,\Omega)}$ is similar to that in Stein (1993) and so the details are omitted.

Case (ii): a_Q is an interior atom with $l_Q<1$. This case is trivial.

Case (iii): a_Q is a boundary atom.

For $0<t<\delta(x)/c_0$ and $|x-x_Q|\leq4c_0\sqrt{n}\,l_Q/(c_0-1)$, for $c_0>1$, obviously we have the inequality

$$(a_Q)_\Omega^*(x)\leq Cw(Q)^{-1/p}.$$

For $0<t<\delta(x)/c_0$ and $|x-x_Q|\leq4c_0\sqrt{n}\,l_Q/(c_0-1)$, we only need check $\varphi_t*a_Q(x)=0$. Denote $y_2\in B(x,t)$, $y_1\in Q$. If $\delta(x)<4c_0\sqrt{n}\,l_Q/(c_0-1)$, then

$$\begin{aligned}&|y_2-y_1|\geq|x-x_Q|-|y_2-x|-|y_1-x_Q|\\&\geq4c_0\sqrt{n}\,l_Q/(c_0-1)-\frac{1}{c_0}4c_0\sqrt{n}\,l_Q/(c_0-1)-4\sqrt{n}\,l_Q>0,\end{aligned}$$

and so we have $\varphi_t*a_Q(x)=0$. If $\delta(x)>4c_0\sqrt{n}\,l_Q/(c_0-1)$, then $\delta(y_2)>(c_0-1)\delta(x)/c_0>4\sqrt{n}\,l_Q$, and so $\varphi_t*a_Q(x)=0$ too.

And so we complete the proof of this case.

The case (Grand Maximal) $\Leftrightarrow$ (Maximal): It is trivial.

The case (Maximal) $\Rightarrow$ (Grand Maximal): This case can be easily obtained from the inequality $f_\Omega^*(x)\leq CM_{\tau/(\tau+a)}[f^+]^e(x)$ for all $x\in\Omega$ (see Section 2 in Miyachi (1987)), where

(i) $f^e(x)$ is defined by $f^e(x)=f(x)$ if $x\in\Omega$ or $=0$ if $x\in(\bar{\Omega})^c$;

(ii) $M_q(f)(x)=\sup_{t>0}(|B(x,t)|^{-1}\int_{B(x,t)}|f(y)|^q\,\mathrm{d}y)^{1/q}$ for $q>0$; and

(iii) τ, $\alpha>0$ with $\tau/(\tau+\alpha)<p$.

The case (Maximal) $\Rightarrow$ (Restriction): Using the above fact (Atomic Decomposition) $\Leftrightarrow$ (Restriction), we prove that every f with $f_\Omega^+(x)\in L^p(\Omega,\mathrm{d}w)$ has an atomic decomposition.

Though it can be similarly proved by the methods of Miyachi (1990), we give another simple proof in here, that is, we should construct a distribution $F\in h^p(\mathrm{R}^n,\mathrm{d}w)$, with $F|_\Omega=f$ in the $C_0^\infty(\Omega)'$ and

$$\|F\|_{h^p(\mathrm{R}^n,\mathrm{d}w)}\le c\|f_\Omega^+\|_{L^p(\Omega,\mathrm{d}w)}.$$

Let $\{Q\}$ be the Whitney decomposition with $\Omega=\cup Q$ as described in Chang *et al.*(1993) and $\{\varphi_Q\}$ be a partition of unity i.e., $\varphi_Q\in C_0^\infty(2Q)$, $0\le\varphi_Q(x)\le1$, $\varphi_Q(x)\equiv1$ in Q, and $\sum_Q\varphi_Q(x)=1$ ($\forall x\in\Omega$). We define the polynomial P_Q as follows. For every $Q\in\mathscr{W}$, $P_Q\in\mathscr{P}_{[s]}$ is the unique element such that $\langle f\varphi_Q-P_Q\chi_{2Q},P\rangle=0$ for every $P\in\mathscr{P}_{[s]}$, where $\mathscr{P}_{[s]}$ denotes the set of polynomials on R^n of order not exceeding $[s]$.

Define

$$g(x):=f(x)-\sum_Q\chi_{2Q}(x)P_Q(x)\ \text{ if }\ x\in\Omega$$
$$\text{or}\quad =0 \qquad\qquad \text{if}\ \ x\notin\Omega, \tag{3}$$

We should prove that $g\in h^p(\mathrm{R}^n,\mathrm{d}w)$ with

$$\|g\|_{h^p(\mathrm{R}^n,\mathrm{d}w)}\le c\|f_\Omega^+\|_{L^p(\Omega,\mathrm{d}w)},$$

and $R(x):=\sum_Q\chi_{2Q}(x)P_Q(x)$ is the sum of a series of constant and boundary atoms. Using the fact (Atomic Decomposition) $\Leftrightarrow$ (Restriction), $R(x)$ has an extension $R'(x)\in h^p(\mathrm{R}^n,\mathrm{d}w)$ with

$$\|R'\|_{h^p(\mathrm{R}^n,\mathrm{d}w)}\le c\|f_\Omega^+\|_{L^p(\Omega,\mathrm{d}w)}.$$

Therefore, $F(x)=g(x)+R'(x)$ is an extension of f.

So we only consider g and R. We first prove $g\in h^p(\mathrm{R}^n,\mathrm{d}w)$.

Recall that for $f\in C_0^\infty(\Omega)'$, $s>0$ and $\psi\in C_0^\infty(2Q)$. We have

$$|\langle f,\psi\rangle|\le C|Q|\sum_{|\alpha|\le[s]+1}\sup_y|\partial_y^\alpha\psi|(x_Q+l_Qy)|\inf_{x\in2Q}f_\Omega^*(x), \tag{4}$$

$$\|P_Q\chi_{2Q}\|_{L^\infty}\le C\inf_{x\in2Q}f_\Omega^*(x). \tag{5}$$

Eqs.(4)–(5) can be found in Sections 2 and 3 in Miyachi (1990). Now we continue to prove $g\in h^p(\mathrm{R}^n,\mathrm{d}w)$. To show it, there are two cases considered, i.e., $x\in\Omega$ and $x\notin\Omega$. Also we will choose the constants in Whitney decomposition Eq.(2) as below

$$c_1>5\sqrt{n},\ c_2=2(1+c_1). \tag{2$'$}$$

Case I: $x\in\Omega$. If $t<\delta(x)/c_0$, using Eqs.(3) and (5), we can easily obtain

$$|\varphi_t*g(x)|\le f_\Omega^+(x)+C\sum_Q\inf_{x\in2Q}f_\Omega^*(x), \tag{6}$$

If $t\ge\delta(x)/c_0$, to estimate the term $\varphi_t*g(x)$, we should consider the following formula. For $x\in\mathrm{R}^n$, We have

$$\begin{aligned}
|\varphi_t*g(x)|&=\left|\sum_Q\int(f(y)\varphi_Q(y)-P_Q(y)x_{2Q}(y))\varphi_t(x-y)\mathrm{d}y\right|\\
&=\left|\sum_Q\int(f(y)\varphi_Q(y)-P_Q(y)x_{2Q}(y))(\varphi_t(x-y)\right.\\
&\qquad\left.-\sum_{|\alpha|\le[s]}\partial^\alpha(\varphi_t)(x-x_Q)(y-x_Q)^\alpha)\mathrm{d}y\right|\\
&\le\left|\sum_{Q:2Q\cap B(x.t)\ne\varnothing}\int f(y)\varphi_Q(y)\sum_{|\alpha|=[s]+1}\partial^\alpha(\varphi_t)(x-\xi)(y-x_Q)^\alpha\mathrm{d}y\right|\\
&\quad+\left|\sum_{Q:2Q\cap B(x,t)\ne\varnothing}\int\sum_Q\chi_{2Q}(y)P_Q(y)\right.\\
&\quad\left.\times\sum_{|\alpha|=[s]+1}\partial^\alpha(\varphi_t)(x-\xi)(y-x_Q)^\alpha\mathrm{d}y\right|\\
&:=A_t(x)+B_t(x).
\end{aligned}\tag{7}$$

Now we estimate $A_t(x)$ and $B_t(x)$ for $x\in\Omega$ or $x\in\Omega^c$.

If $x\in\Omega$ and $t>\delta(x)/c_0$, we are only interested in the cubes Q with $2Q\cap B(x,t)\neq\varnothing$. Let $y\in 2Q\cap B(x,t)$.

If $|x-x_Q|>4\sqrt{n}\,l_Q$, then $t>|x-y|\geq|x-x_Q|-|y-x_Q|>|x-x_Q|-2\sqrt{n}\,l_Q$, and so we have

$$t>|x-x_Q|/2>2\sqrt{n}\,l_Q. \tag{8}$$

If $|x-x_Q|\leq 4\sqrt{n}\,l_Q$, for every $x'\in\partial\Omega$, by Eq.(2′) as above, we have

$$|x'-x|\geq|x'-x_Q|-|x_Q-x|\geq 5\sqrt{n}l_Q-4\sqrt{n}l_Q=\sqrt{n}l_Q,$$

By $t>\delta(x)/c_0$, we have

$$t>\sqrt{n}l_Q/c_0. \tag{9}$$

For $A_t(x)$, by Eq.(4) and Eqs.(7)–(9), we have the estimate

$$\begin{aligned}A_t(x)&=\left|\sum_{Q:2Q\cap B(x,t)\neq\varnothing}\int f(y)\varphi_Q(y)\right.\\&\quad\left.\times\sum_{|\alpha|=[s]+1}\partial^\alpha(\varphi_t)(x-\xi)(y-x_Q)^\alpha\,\mathrm{d}y\right|\\&\leq C\sum_{Q:2Q\cap B(x,t)\neq\varnothing}\inf_{y\in 2Q}f_\Omega^*(y)|Q|(l_Q)^{[s]+1}/t^{n+[s]+1}\\&\leq C\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)/(1+|x_Q-x|/l_Q)^{n+[s]+1}.\end{aligned}$$

So for every $x\in\Omega$ and $0<t\leq 1$, by Eq.(6), we have

$$A_t(x)\leq C\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)(1+|x_Q-x|/l_Q)^{(n+[s]+1)p}+f_\Omega^+(x).$$

Then by the inequality $|a+b|^p\leq|a|^p+|b|^p$ $(0<p\leq 1)$, we can obtain

$$A_t(x)^p\leq c\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)^p(1+|x_Q-x|/l_Q)^{(n+[s]+1)p}+f_\Omega^+(x)^p \tag{10}$$

Now, for $x\in\Omega$, $0<t\leq 1$ by Eq.(5) and Eqs.(7)–(9), similarly we have the estimate

$$\begin{aligned}B_t(x)&\leq C\sum_{Q:2Q\cap B(x,t)\neq\varnothing}\inf_{y\in 2Q}f_\Omega^*(y)|Q|(l_Q)^{[s]+1}/t^{n+[s]+1}\\&\leq C\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)(1+|x_Q-x|/l_Q)^{n+[s]+1}.\end{aligned}$$

And we similarly obtain

$$B_t(x)^p\leq C\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)^p(1+|x_Q-x|/l_Q)^{(n+[s]+1)p}. \tag{11}$$

Case II: $x\notin\Omega$. As above considered, we are interested in the cubes Q with $2Q\cap B(x,t)\neq\varnothing$. By the support of $\varphi_t(x-y)$, $\varphi_Q(y)$ and Eq.(2′), we can get

$$t\geq|x-y|\geq|x-x_Q|-|y-x_Q|\geq|x-x_Q|-2\sqrt{n}\,l_Q>|x-x_Q|/2,$$

similar to the analysis of the case $x\in\Omega$, we have the estimate

$$A_t(x)\leq c\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)(1+|x-x_Q|/l_Q)^{n+[s]+1}. \tag{12}$$

Similarly we have

$$B_t(x)^p\leq C\sum_Q\inf_{y\in 2Q}f_\Omega^*(y)^p(1+|x-x_Q|/l_Q)^{(n+[s]+1)p} \tag{13}$$

Remember that if $w\in A_\infty(\mathrm{R}^n)$ with the critical index q_w, then

$$\int_{\mathrm{R}^n}(1+|x-x_Q|/l_Q)^{(n+[s]+1)p}w(x)\mathrm{d}x\leq C(w)w(Q)$$

for every cube $Q\subset\mathrm{R}^n$ and $(n+[s]+1)p>n$.

Combining Eqs.(10)–(13) the above estimate, we obtain

$$\begin{aligned}\int_{\mathrm{R}^n}g^+(x)^p w(x)\mathrm{d}x&\leq\int_\Omega g^+(x)^p w(x)\mathrm{d}x+\int_{\Omega^c}g^+(x)^p w(x)\mathrm{d}x\\&\leq C\int_\Omega\sup_{0<t\leq 1}(A_t(x)^p+B_t(x)^p)w(x)\mathrm{d}x\\&\quad+\int_{\Omega^c}\sup_{0<t\leq 1}(A_t(x)^p+B_t(x)^p)w(x)\mathrm{d}x\end{aligned}$$

$$\leq C\int_{\Omega}\sum_{Q}\inf_{y\in 2Q} f_{\Omega}^{*}(y)^{p}(1+|x_{Q}-x|/l_{Q})^{(n+[s]+1)p}w(x)\mathrm{d}x$$
$$+\int_{\Omega} f_{\Omega}^{+}(x)^{p}w(x)\mathrm{d}x$$
$$+C\int_{\Omega^{c}}\sum_{Q}\inf_{y\in 2Q} f_{\Omega}^{*}(y)^{p}(1+|x-x_{Q}|/l_{Q})^{(n+[s]+1)p}w(x)\mathrm{d}x$$
$$\leq C\sum_{Q}\inf_{y\in 2Q} f_{\Omega}^{*}(y)^{p}(\int_{\Omega}+\int_{\Omega^{c}}(1+|x-x_{Q}|/l_{Q})^{(n+[s]+1)p}$$
$$\times w(x)\mathrm{d}x+\int_{\Omega} f_{\Omega}^{*}(x)^{p}w(x)\mathrm{d}x$$
$$\leq C\sum_{Q}\inf_{y\in 2Q} f_{\Omega}^{*}(y)^{p}w(Q)$$
$$\leq C\int_{\Omega} f_{\Omega}^{*}(y)^{p}w(y)\mathrm{d}y. \quad (14)$$

Lastly, we should prove that $R(w)=\sum_{Q}P_{Q}(x)\times\chi_{2Q}(x)$ is the sum of a series of constant and boundary atom. In fact, we set $\lambda_{Q}=w(Q)^{1/p}\|P_{Q}\|_{L^{\infty}}$ and $a_{Q}(x)=w(Q)^{-1/p}\|P_{Q}\|_{L^{\infty}}^{-1}P_{Q}(x)$.

Obviously $a_{Q}(x)$'s are boundary atoms. Moreover,

$$\sum_{Q}|\lambda_{Q}|^{p}=\sum_{Q}w(Q)\|P_{Q}\|_{L^{\infty}}^{p}\leq c\sum_{Q}w(Q)\inf_{y\in 2Q} f_{\Omega}^{*}(y)^{p}$$
$$\leq c\int_{\Omega} f_{\Omega}^{*}(y)^{p}w(y)\mathrm{d}y. \quad (15)$$

Using the fact (Restriction) ⇔ (Atomic Decomposition), we know that there exists a function $R'(x)\in h^{p}(\mathrm{R}^{n},\mathrm{d}w)$ with $R'(x)|_{\Omega}=R(x)$ and $\|R'\|_{h^{p}(\mathrm{R}^{n},\mathrm{d}w)}\leq c\|R\|_{h^{p}(\Omega,\mathrm{d}w)}$.

So the proof of (Maximal) ⇒ (Restriction) is completely by combining Eq.(14) with Eq.(15), and setting $F(x)=g(x)+R'(x)$.

Proof of Proposition 5 Note that when Ω is a bounded Lipschitz domain, for $0<p\leq 1$ and $-1<s<1$, we have $\delta(x)^{sp}\in A_{\infty}(\Omega)$ (Stein, 1993). To prove Proposition 5, we have two steps considered.

Step 1: $u\in h^{p}(\Omega,\delta^{sp}\mathrm{d}x) \Leftrightarrow u\in L^{p}(\Omega,\delta^{sp}\mathrm{d}x)$. Choose the nonnegative radical function $\varphi(x)\in B(0,1)$ with $\int\varphi(x)\mathrm{d}x=1$. Since u is harmonic, then for all $x\in\Omega$, $0<t<\delta(x)/2$, we have $\varphi_{t}*u(x)=u(x)$ and so $u^{+}(x)=|u(x)|$ for all $x\in\Omega$. This completes the proof of Step 1 by Theorem 4.

Step2: $\delta^{s}u\in h_{r}^{p}(\Omega,\mathrm{d}x)\Leftrightarrow u\in h^{p}(\Omega,\delta^{sp}\mathrm{d}x)$. For $0<t<\delta(x)/2$, we consider

$$|\varphi_{t}*(\delta^{s}u)(x)|=\left|\int\varphi_{t}(x-y)\delta^{s}(y)u(y)\mathrm{d}y\right|$$
$$=\left|\int\varphi_{t}(x-y)\delta^{s}(y)(\varphi_{t_{0}}*u)(y)\mathrm{d}y\right|(t_{0}=\delta(x)/8)$$
$$=\left|\int\int\varphi_{t}(x-y)\delta^{s}(y)\varphi_{t_{0}}(y-z)u(z)\mathrm{d}z\mathrm{d}y\right|$$
$$=\left|\delta(x)^{s}\int u(z)(\int\varphi_{t}(x-y)(\delta^{s}(y)/\delta(x))^{s}\right.$$
$$\left.\times\varphi_{t_{0}}(y-z)\mathrm{d}y)\mathrm{d}z\right|$$
$$\leq c\delta(x)^{s}u_{\Omega}^{*}(x).$$

This means $(\delta^{s}u)^{+}(x)\leq c\delta(x)^{s}u_{\Omega}^{*}(x)$, so we get $u\in h^{p}(\Omega,\delta^{sp}\mathrm{d}x)\Rightarrow\delta^{s}u\in h_{r}^{p}(\Omega)$.

Conversely, similarly we can prove $|\delta^{s}(x)u*\varphi_{t}(x)|\leq c(\delta^{s}u)_{\Omega}^{*}(x)$ for $0<t<\delta(x)/2$. That is, $|\delta^{s}(x)u^{+}(x)|\leq c(\delta^{s}u)_{\Omega}^{*}(x)$, so using Theorem 4, we get $\delta^{s}u\in h_{r}^{p}(\Omega)\Rightarrow u\in h^{p}(\Omega,\delta^{sp}\mathrm{d}x)$.

Combing Step 1 with Step 2, we complete the proof of Proposition 5.

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