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Wave propagation of the traffic flow dynamic model based on wavefront expansion*

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Abstract: This paper discusses propagation of perturbations along traffic flow modeled by a modified second-order macroscopic model through the wavefront expansion technique. The coefficients in this expansion satisfy a sequence of transport equations that can be solved analytically. One of these analytic solutions yields information about wavefront shock. Numerical simulations based on a Padé approximation of this expansion were done at the end of this paper and results showed that propagation of perturbations at traffic flow speed conforms to the theoretical analysis results.

Keywords: Perturbations, Macroscopic traffic flow model, Wavefront expansion
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INTRODUCTION

The mathematical modeling of traffic flow plays an important role in simulating the behavior of traffic systems or designing traffic control strategies (Papageorgiou *et al*., 1990a; 1990b). For the past few decades, studies on this subject have been made by many researchers. Among them the macroscopic traffic flow theory developed first by Lighthill and Witham (1955) and Richard (1956) (the LWR theory for short) was most suitable for correct description of traffic flow. In this theory, vehicles in traffic flow are considered as particles in fluid; further, the behavior of traffic flow is modeled by the method of hydrodynamics and formulated by hyperbolic partial differential equations. As an essential problem, propagation of perturbations along an equilibrium traffic flow governed by these equations has come to be an interesting issue (Zhang, 2000; Jiang and Wu, 2003). Since perturbations propagate in the form of waves, most wave methods can be used to study them. For example, studies on small perturbations on a traffic flow model can be done by the linearized method (Whitham, 1974). However, large perturbations are still less discussed in related literature. In this paper we will employ a wavefront-based method (Moodie *et al*., 1991) to study this problem and test the theoretical results by using a numerical method.

The rest of this paper is organized as follows. In Section 2 we will introduce a modified macroscopic model for traffic flow and Section 3 discusses solutions of the model equations in the form of wavefront expansion. Numerical results based on a Padé approximation of the expansion are shown in Section 4. Section 5 gives conclusions.

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MACROSCOPIC TRAFFIC FLOW MODELS

The macroscopic traffic flow theory is used to study traffic flow by collective variables such as traffic flow rate $q(x, t)$, traffic speed $u(x, t)$ and traffic density $\rho(x, t)$, all of which are functions of space (x) and time (t). The most well-known LWR model is formulated by employing the conservation equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial q_e}{\partial x}=0, \quad (1)$$

where $q_e=\rho u_e(\rho)$ is the flow rate of the traffic flow under the equilibrium conditions, $u_e(\rho)$ is an equilibrium speed-density relationship satisfying $u_e'(\rho)\le 0$. In this paper we shall use the formulation proposed by Greenshields (Gazis, 1974):

$$u_e=u_f-\frac{u_f}{\rho_{jam}}\rho \quad (2)$$

where u_f denotes the free flow speed and ρ_{jam} is the jam density.

The LWR model is simple and can be used to solve a variety of simple traffic flow problems analytically by the characteristic method (Whitham, 1974). However, it does not describe the traffic flow dynamics adequately because it does not consider acceleration and inertia effects. In order to improve this, the Payne-type models introducing a momentum equation in addition to the conservation equation are found, which can have the general hyperbolic form:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \quad (3)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\tau}(u-u_e(\rho))-\frac{C^2(\rho)}{\rho}\frac{\partial\rho}{\partial x} \quad (4)$$

where ρ is the traffic density and u is the traffic speed. The parameter τ denotes the relaxation time, usually treated as a constant but can be a function of vehicle density. $C(\rho)$ is a function of density, and is related to propagation of a perturbation. More information on the forms of $C(\rho)$ can be found in Payne (1971), Zhang (1998) and Lyrintzis *et al.*(1994).

The Payne-type models take into account nonequilibrium effects by replacing the equilibrium speed with a momentum Eq.(4) and improve the simple LWR model. The left hand side of Eq.(4) represents the acceleration. The first term on the right-hand side of Eq.(4) is relaxation term describing the tendency of traffic flow to relax to an equilibrium speed, and the second term accounts for speed adjustment of a traffic flow in anticipation of changing traffic conditions ahead. The most important problem in these models is that a characteristic speed is greater than the macroscopic traffic flow speed (Daganzo, 1995), which means that the future conditions of a traffic variable are partly determined by what is happening behind it. This is not desirable in real-life. Zhang (2002) proposed that a possible way to eliminate this characteristic is considering replacing the density gradient on the right side of Eq.(4) by the speed gradient. Zhang' model is as follows:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \quad (5)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-C(\rho)\frac{\partial u}{\partial x} \quad (6)$$

where $C(\rho)=\rho u_e'(\rho)$ is the relative speed of perturbation propagating against the traffic flow.

Results showed that this model not only captures certain phenomena that elude the LWR model, but also completely eradicates phenomena propagating faster than the traffic flow. Nevertheless there is no relaxation term in Eq.(6), which means that the traffic flow is always under equilibrium conditions or never transits from one state to another.

Considering all above discussions, we prefer a modified second-order macroscopic traffic flow model such as:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0,$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\tau}(u-u_e(\rho))-C(\rho)\frac{\partial u}{\partial x} \quad (7)$$

Here $C(\rho)=\rho u_e'(\rho)$. Model Eq.(7) adds the relaxation term in the momentum Eq.(6) and has properties similar to those of Zhang's model. In this paper, we only focus on the propagation of some kinds of large perturbations in this model by using the wavefront-based method.

First we obtain a hyperbolic system of partial differential equations in terms of the vector $\boldsymbol{U}=\begin{pmatrix}\rho\\ u\end{pmatrix}$:

$$\boldsymbol{U}_t+\boldsymbol{A}(\boldsymbol{U})\boldsymbol{U}_x+\boldsymbol{b}(\boldsymbol{U})=\boldsymbol{0} \quad (8)$$

where $\boldsymbol{A}(\boldsymbol{U})=\begin{pmatrix}u & \rho\\ 0 & u+\rho u_e'(\rho)\end{pmatrix}$, $\boldsymbol{b}(\boldsymbol{U})=\begin{pmatrix}0\\ \dfrac{u-u_e}{\tau}\end{pmatrix}$.

It is obvious that the two characteristic speeds of the new model Eq.(7), u and $u+\rho u_e'(\rho)$, are always not greater than the macroscopic traffic flow speed.

WAVEFRONT EXPANSIONS

Considering the boundary value problem of Eq.(8), we assume that an equilibrium traffic flow with density ρ_0 and speed $u_0=u_e(\rho_0)$ is disturbed at the boundary x=0 by a prescribed speed change $\tilde{u}(t)$ with Taylor series expansion about t=0+ of the form

$$\tilde{u}(t)=\sum_{n=1}^{\infty}\tilde{u}_n t^n \quad (9)$$

where we assume that $\tilde{u}(0)=0$.

At t=0 a wavefront Γ leaves x=0 and propagates along the traffic flow. Γ can be denoted as

$$\Gamma:\ t=T(x),\ T(0)=0 \quad (10)$$

The speed at which the wavefront propagates is $s=\dfrac{1}{T'}$.

In the following discussion we will be interested in the behavior of $\boldsymbol{U}$ in the neighborhood of Γ.

Introducing a new variable $\xi \mathrel{\hat=} t-T(x)$, and the parameter $|\xi|$ which is a measure of the distance from the wavefront ξ=0. The solution of Eq.(8) in the form of wavefront expansion is

$$\rho=\rho_0,\ u=u_0 \quad \text{for } \xi\le 0 \quad (11)$$

$$\rho(x,t)=\rho_0+\sum_{n=1}^{\infty}\rho_n(x)\xi^n,\quad u(x,t)=u_0+\sum_{n=1}^{\infty}u_n(x)\xi^n \quad \text{for } \xi\ge 0 \quad (12)$$

The amplitude coefficients in these two expansions are related to jumps in time derivatives as follows:

$$\rho_n(x)=\frac{1}{n!}\left(\frac{\partial^n\rho}{\partial\xi^n}\right)\Big|_{\xi=0^+} \quad (13)$$

Furthermore we write power series expansions in ξ for $\boldsymbol{A}$ and $\boldsymbol{b}$ in Eq.(8) as:

$$\boldsymbol{A}=\sum_{n=0}^{\infty}\boldsymbol{A}_n(x)\xi^n,\ \boldsymbol{b}=\sum_{n=0}^{\infty}\boldsymbol{b}_n(x)\xi^n,\ \text{for } \xi\ge 0 \quad (14)$$

where

$$\boldsymbol{A}_n=\begin{pmatrix}u_n & \rho_n\\ 0 & u_n-\dfrac{u_f}{\rho_{jam}}\rho_n\end{pmatrix},\ n\ge 0 \quad (15)$$

$$\boldsymbol{b}_0=\boldsymbol{0},\ \boldsymbol{b}_n=\begin{pmatrix}0\\ \dfrac{1}{\tau}u_n+\dfrac{u_f}{\tau\rho_{jam}}\rho_n\end{pmatrix},\ n\ge 1 \quad (16)$$

At the boundary, the amplitude coefficients $u_n(x)$ satisfy the condition

$$u_n(0)=\tilde{u}_n,\ n\ge 1 \quad (17)$$

Substituting Eq.(12) and Eqs.(14)−(16) into Eq.(8) and equating coefficients of like powers of ξ we can get

$$\rho_1 - T'\rho_0 u_1 - T' u_0 \rho_1 = 0 \tag{18}$$

$$u_1 - (u_0 - \frac{u_f}{\rho_{jam}}\rho_0) T' u_1 = 0 \tag{19}$$

for n=0, and

$$(n+1)(1 - T'u_0)\rho_{n+1} - T'(n+1)\rho_0 u_{n+1} + \rho_0 u_n' + u_0 \rho_n' + \sum_{k=1}^{n}(u_k Q_{n-k} + \rho_k R_{n-k}) = 0 \tag{20}$$

$$(n+1)[1 - (u_0 - \frac{u_f}{\rho_{jam}}\rho_0)T']u_{n+1} + (u_0 - \frac{u_f}{\rho_{jam}}\rho_0)u_n' + \frac{1}{\tau}u_n + \frac{u_f}{\tau\rho_{jam}}\rho_n + \sum_{k=1}^{n}(u_k - \frac{u_f}{\rho_{jam}}\rho_k)R_{n-k} = 0 \tag{21}$$

for n≥1, where

$$Q_n = \rho_n' - (n+1)\rho_{n+1}T', \quad R_n = u_n' - (n+1)u_{n+1}T' \tag{22}$$

The pair of Eqs.(18), (19) have a nontrivial solution $\begin{pmatrix}\rho_1 \\ u_1\end{pmatrix}$ if and only if

$$T' = \frac{1}{u_0} \text{ or } T' = \frac{1}{u_0 - \frac{u_f}{\rho_{jam}}\rho_0} \tag{23}$$

In this paper we will choose

$$T' = \frac{1}{u_0 - \frac{u_f}{\rho_{jam}}\rho_0}, \tag{24}$$

that is, the speed of propagation of the wavefront Γ is

$$s = u_0 - \frac{u_f}{\rho_{jam}}\rho_0 \tag{25}$$

Using Eq.(24) in Eq.(18) gives

$$\rho_1 = -\frac{\rho_{jam}}{u_f}u_1 \tag{26}$$

Let n=1 in Eq.(21), with Eq.(24) and Eq.(26), we obtain the differential equation

$$u_1' - \frac{2}{(u_0 - \frac{u_f}{\rho_{jam}}\rho_0)^2}u_1^2 = 0 \tag{27}$$

Since Eq.(27) involves only u_1, it can be solved exactly. The solution to Eq.(27) is

$$u_1(x) = \frac{\tilde{u}_1}{1 - \frac{2}{(u_0 - \frac{u_f}{\rho_{jam}}\rho_0)^2}\tilde{u}_1 x} \tag{28}$$

The solution to Eq.(27) shows whether a shock can form at the wavefront Γ. From Eq.(28) we can see that breaking will occur at the front if $\tilde{u}_1 > 0$; and that the breaking distance is

$$x = \frac{(u_0 - \frac{u_f}{\rho_{jam}}\rho_0)^2}{2\tilde{u}_1}. \tag{29}$$

ρ_1 is found from Eq.(26) and the coefficients ρ_n, u_n for n≥2 can be found from Eq.(20) and Eq.(21), respectively.

NUMERICAL RESULTS

To obtain numerical results from the wavefront expansions we use Padé approximant. The parameters for model Eq.(7) are taken from Table 1 (Treiber *et al.*, 1999).

Table 1 Parameters for model Eq.(7)

τ (h)	ρ_{jam} (veh/km)	u_f (km/h)
7/720	160	110

Considering an equilibrium traffic flow (with ρ_0=72.73 and u_0=$u_e(\rho_0)$=60) disturbed at the boundary x=0 by a speed change $\tilde{u}(t)=at\mathrm{e}^{-t}$, $t\geq0$, with a equaling 5 and −10 respectively. We write the Taylor series expansions about $t\geq0$ of $\tilde{u}^{(1)}(t)=5t\mathrm{e}^{-t}$ and $\tilde{u}^{(2)}(t)=-10t\mathrm{e}^{-t}$ as:

$$\tilde{u}^{(1)}(t)=\sum_{n=1}^{\infty}\tilde{u}_n^{(1)}t^n=5t-5t^2+\frac{5}{2!}t^3-\frac{5}{3!}t^4+... \qquad (30)$$

$$\tilde{u}^{(2)}(t)=\sum_{n=1}^{\infty}\tilde{u}_n^{(2)}t^n=-10t+10t^2-\frac{10}{2!}t^3+\frac{10}{3!}t^4+... \qquad (31)$$

that is, $\tilde{u}_n^{(1)}=(-1)^{n-1}\dfrac{5}{(n-1)!}$, $\tilde{u}_n^{(2)}=(-1)^{n}\dfrac{10}{(n-1)!}$, $n\geq1$. We will presume that the flow rate at the boundary is unchanged, i.e. $q(0, t)=\rho_0u_0$=4364 veh/h, $t\geq0$. At the boundary the curves of the traffic flow speed and density are shown in Fig.1 and Fig.2.

The wavefront is given by Γ: $t=T(x)=0.1x$, and the solution for $u_1(x)$ and $\rho_1(x)$ is:

$$u_1(x)=\frac{5}{1-0.1x},\ \rho_1(x)=-\frac{80}{11}\times\frac{1}{1-0.1x} \qquad (32)$$

for condition Eq.(30) and

$$u_1(x)=\frac{-10}{1+0.2x},\ \rho_1(x)=\frac{160}{11}\times\frac{1}{1+0.2x} \qquad (33)$$

for condition Eq.(31).

Since $\tilde{u}_1^{(1)}>0$, breaking occurs at the front and the breaking distance is x=10 km.

Eqs.(21) and (22) can be solved successively analytically in terms of quadrature. The formulas will become complicated expressions soon. Since we will only use the higher order amplitude functions for purely numerical work, the unwieldy process is of no advantage. Therefore, we solve the

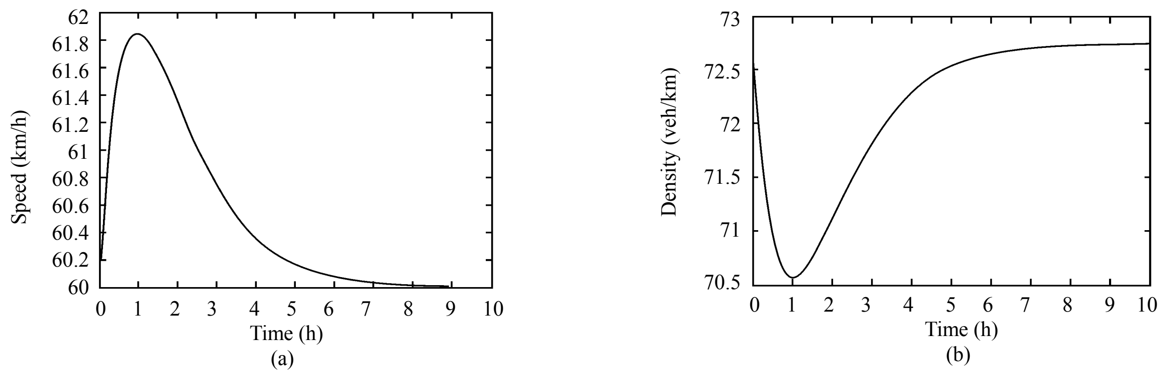

Fig.1 Variation of traffic speed and density at boundary for boundary condition Eq.(30)
(a) traffic flow speed at boundary x=0; (b) traffic flow density at boundary x=0

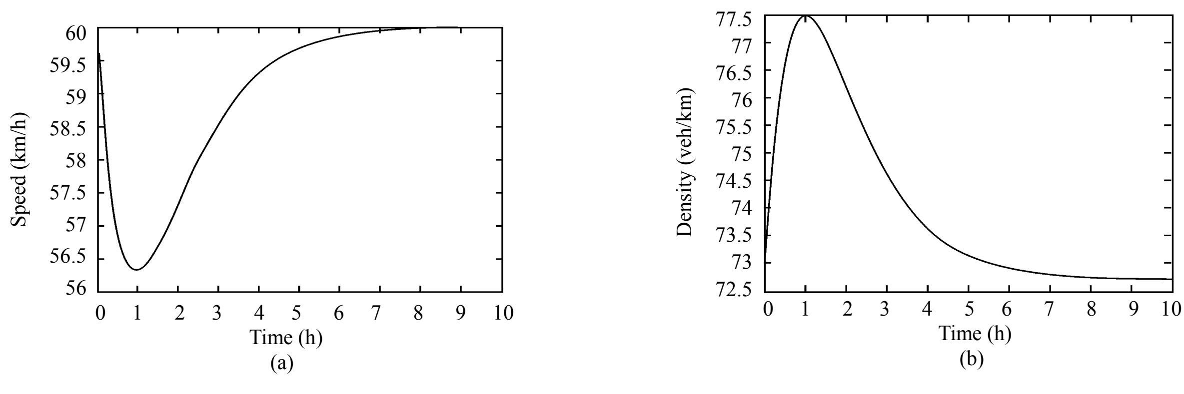

Fig.2 Variation of traffic speed and density at boundary for boundary condition Eq.(31)
(a) traffic flow speed at boundary x=0; (b) traffic flow density at boundary x=0

sequence of differential equations numerically.

We will employ the method used in Moodie *et al.*(1991). We discretize the region $x \geq 0$ by choosing a mesh width $h>0$, and find the values of $u_n(x)$ and $\rho_n(x)$ for $n=1, \ldots, N$ at the points $x_i=(i-1)h$, $i=1, \ldots, I$. At each step we find u_n, ρ_n at x_{i+1} by means of the power series:

$$u_n(x_{i+1}) = \sum_{j=1}^{\infty} u_{n,j}^{(i)} h^{j-1}, \ \rho_n(x_{i+1}) = \sum_{j=1}^{\infty} \rho_{n,j}^{(i)} h^{j-1} \quad (34)$$

At each step we use $J-n+1$ terms. The integer J means the number of terms used in the series Eq.(34) to computer u_n and ρ_n. Here we choose $J=2N$.

All numerical results obtained from wavefront expansions use a [4/5] Padé approximant. We employ the numerical scheme with $N=10$, $I=10$ and $h=0.5$ to find the values of the amplitude functions $u_n(x_i)$, $\rho_n(x_i)$ for $n=1, \ldots, N$; $i=1, \ldots, I$. Results are shown in Fig.3 and Fig.4.

CONCLUSION

Wavefront expansions can be used to study wave propagations. The coefficients in these expansions satisfy a sequence of transport equations, with the solution of the lowest one giving information about breaking at the wavefront. By employing Padé approximant, these methods can compute valid results for much great times after the arrival of the wavefront.

In this paper we used a wavefront expansion to study propagation of perturbations along an equilibrium traffic flow governed by a modified macroscopic traffic flow model. The transport equations obtained from the expansion provide interesting information about wave propagation. For our problem, the shock occurs at the wavefront for posi-

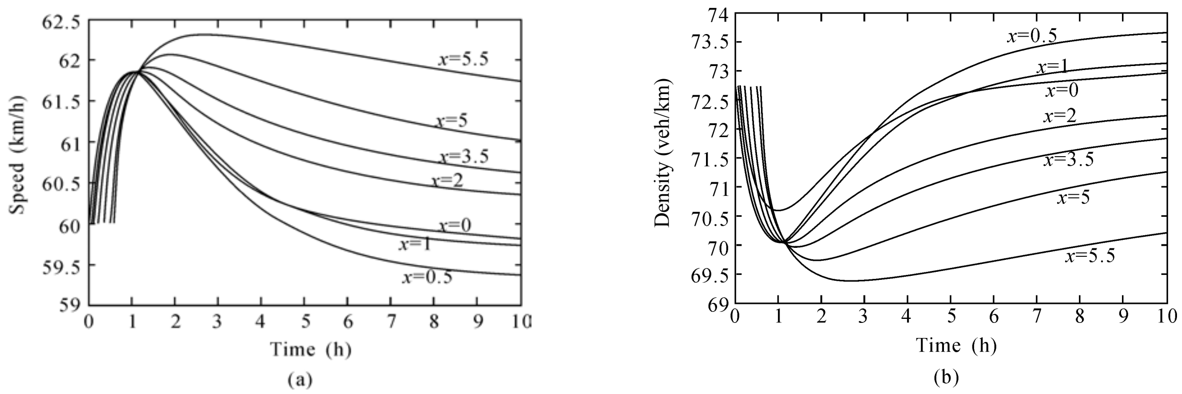

Fig.3 Variation of traffic speed and density for boundary condition Eq.(30)
(a) traffic speed at x=0, 0.5, 1, 2, 3.5, 5, 5.5; (b) traffic density at x=0, 0.5, 1, 2, 3.5, 5, 5.5

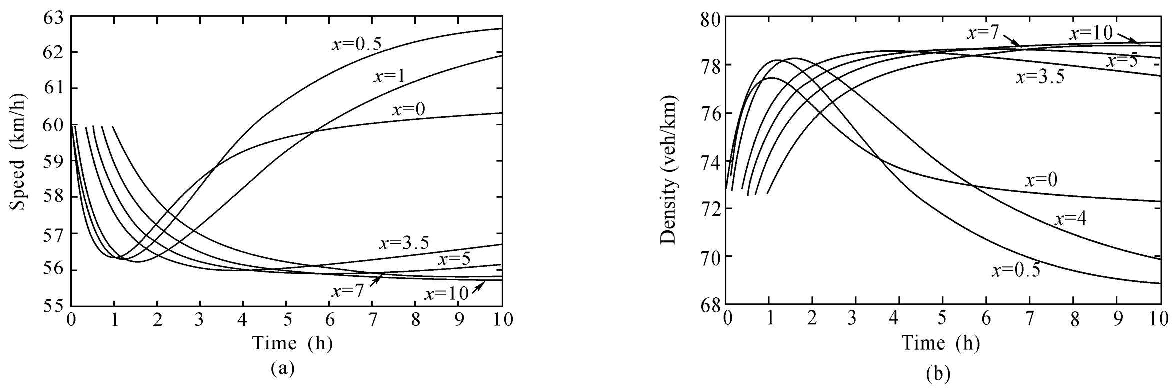

Fig.4 Variation of traffic speed and density for boundary condition Eq.(31)
(a) traffic speed at x=0, 0.5, 1, 3.5, 5, 7, 10; (b) traffic density at x=0, 0.5, 1, 3.5, 5, 7, 10

tive $\tilde{u}_1$ for speed change $\tilde{u}(t)=\sum_{n=1}^{\infty}\tilde{u}_n t^n,\ t\geq 0$. The numerical results of this situation are shown in Fig.3. As one can see, the traffic flow speed will be higher and higher with increasing space and time; at the same time the density becomes lower and lower. The speed and density will both rapidly become unreality. Fig.4 shows a contrary example, from which we can see that the variables of traffic flow tend to be stable in time and space. These two results confirm the theoretical analysis in Section 3.

References

Daganzo, C., 1995. Requiem for second-order approximations of traffic flow. *Transportation Research-B*, **29**(4):277-286.

Gazis, D.C. (Eds.), 1974. Traffic Science. John Wiley and Sons, p.43.

Jiang, R., Wu, Q.S., 2003. Study on propagation speed of small disturbance from a car-following approach. *Transportation Research-B*, **37**:85-99.

Lighthill, M.J., Whitham, G.B., 1955. On kinematic waves II: A theory of traffic flow on long crowded roads. *Proc. Royal Soc.*, **A229**:317-345.

Lyrintzis, A.S., Liu, G., Michalopoulos, P.G., 1994. Development and comparative evaluation of high-order traffic flow models. *Trans. Res. Rec.*, **1457**:174-183.

Moodie, T.B., He, Y.P., Barclay, D.W., 1991. Wavefront expansions and nonlinear hyperbolic waves. *Wave Motion*, **14**:347-367.

Papageorgious, M., Jean-Marc, B., Habib, H.S., 1990a. Modelling and real-time control of traffic flow on the southern part of Boulevard Peripherique in Paris: Part I: modelling. *Transportation Research-A*, **24**(5):345-359.

Papageorgious, M., Jean-Marc, B., Habib, H.S., 1990b. Modelling and real-time control of traffic flow on the southern part of Boulevard Peripherique in Paris: Part II: coordinated on-ramp metering. *Transportation Research-A*, **24**(5):361-370.

Payne, H.J., 1971. Models of freeway traffic and control. *Simulation Councils Proc*. *Series*: *Mathematical Model of Public Systems*, **1**(1):51-60.

Richards, P.I., 1956. Shockwaves on the highway. *Operations Research*, **4**:42-51.

Treiber, M., Hennecke, A., Helbing, D., 1999. Derivation, properties and simulation of a gas-kinetic-based, nonlocal traffic model. *Physical Review E*, **59**(1):239-253.

Whitham, G.B., 1974. Linear and Nonlinear Waves. John Wiley and Sons Inc., New York.

Zhang, H.M., 1998. A theory of nonequilibrium traffic flow. *Transportation Research-B*, **32**(7):485-498.

Zhang, H.M., 2000. Structural properties of solutions arising from a nonequilibrium traffic flow theory. *Transportation Research-B*, **34**:583-603.

Zhang, H.M., 2002. A nonequilibrium traffic model devoid of gas-like behavior. *Transportation Research-B*, **36**:275-290.