Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

# The analytical solutions for orthotropic cantilever beams (I): Subjected to surface forces*

JIANG Ai-min (江爱民)[†1,2], DING Hao-jiang (丁皓江)[1]

([1]Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China)

([2]West Branch of Zhejiang University of Technology, Quzhou 324006, China)

[†]E-mail: jam@vip.sina.com

Received Sept. 20, 2004; revision accepted Oct. 4, 2004

**Abstract:** This paper first gives the general solution of two-dimensional orthotropic media expressed with two harmonic displacement functions by using the governing equations. Then, based on the general solution in the case of distinct eigenvalues, a series of beam problems, including the problem of cantilever beam under uniform loads, cantilever beam with axial load and bending moment at the free end, cantilever beam under the first, second, third and fourth power of *x* tangential loads, is solved by the superposition principle and the trial-and-error methods.

**Key words:** General solution, Orthotropic media, Cantilever beams, Analytical solutions
**doi:**10.1631/jzus.2005.A0126 **Document code:** A **CLC number:** O343.2

## INTRODUCTION

The problem of cantilever beams subjected to uniform loads is a classic one in elasticity studies. Timoshenko and Goodier (1970) presented a solution for an isotropic cantilever beam subjected to uniform load and cross load at free end. Lekhnitskii (1969) obtained analytical solutions for an orthotropic cantilever beam subjected to cross load at free end and uniform load on the upper surface. The solutions for constant body force cases were also presented in the above two books. To the authors' knowledge, no literature about the corresponding solution of orthotropic cantilever beam with variable body forces had been published yet. The problems of density functionally graded media can be transformed into those ones with variable body forces. In order to solve the problems of variable body forces, we should first analyze the solution for cantilever beam with axial load and bending moment at free end, and under the normal and tangential loads on the upper and bottom surfaces.

In this paper, we will consider the orthotropic plane problems. The general solution of two-dimensional orthotropic media expressed with two harmonic displacement functions is given at first by use of the governing equations. Then, based on the general solution in the case of distinct eigenvalues, a series of beam problems, including cantilever beam under uniform loads, cantilever beam with axial load and bending moment at the free end, cantilever beam under the first, second, third and fourth power of *x* tangential loads, is solved by the trial-and-error methods.

Analytical solutions for various problems are obtained by the superposition principle.

## GENERAL SOLUTION FOR THE PLANE PROBLEM OF ORTHOTROPIC SOLID

For the plane problems of orthotropic media, the

*Project (Nos. 10432030, 10472102) supported by the National Natural Science Foundation of China

displacements $u_i$ are assumed to be independent of $y$ for the plane-strain case. The basic equations for two-dimensional orthotropic solid in $xoz$ coordinates can be simplified as follows:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}+f_x=0,\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+f_z=0 \tag{1}$$

$$\sigma_x=c_{11}\frac{\partial u}{\partial x}+c_{13}\frac{\partial w}{\partial z},\ \tau_{xz}=c_{55}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),$$
$$\sigma_z=c_{13}\frac{\partial u}{\partial x}+c_{33}\frac{\partial w}{\partial z} \tag{2}$$

where $\sigma_x$ ($\sigma_z$, $\tau_{xz}$) and $u$($w$) are the components of stress and displacement, respectively; $f_x$ and $f_z$ are body force; $c_{ij}$ are the elastic constants.

Governing Eq.(1) can be expressed in terms of $u$ and $w$ by virtue of Eq.(2) as follows

$$\left(c_{11}\frac{\partial^2}{\partial x^2}+c_{55}\frac{\partial^2}{\partial z^2}\right)u+(c_{13}+c_{55})\frac{\partial^2 w}{\partial x\partial z}+f_x=0 \tag{3}$$

$$(c_{13}+c_{55})\frac{\partial^2 u}{\partial x\partial z}+\left(c_{55}\frac{\partial^2}{\partial x^2}+c_{33}\frac{\partial^2}{\partial z^2}\right)w+f_z=0 \tag{4}$$

Ding *et al.*(1997a; 1997b) derived the general solution for piezoelectric plane problem without body forces, in which all physical quantities are expressed in three harmonic functions. With the method and the strict differential operator theorem presented in Ding *et al.*(1997a; 1997b), the general solution of two-dimensional orthotropic media without body forces in the case of distinct eigenvalues can be easily derived and expressed in two harmonic functions as follows

$$u=\sum_{j=1}^{2}\frac{\partial\psi_j}{\partial x},\ w=\sum_{j=1}^{2}s_jk_j\frac{\partial\psi_j}{\partial z_j},\ \sigma_x=\sum_{j=1}^{2}\omega_{2j}\frac{\partial^2\psi_j}{\partial z_j^2},$$
$$\sigma_z=\sum_{j=1}^{2}\omega_{1j}\frac{\partial^2\psi_j}{\partial z_j^2},\ \tau_{xz}=\sum_{j=1}^{2}s_j\omega_{1j}\frac{\partial^2\psi_j}{\partial x\partial z_j} \tag{5}$$

where the functions $\psi_j$ satisfy the following equations:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_j^2}\right)\psi_j=0,\ (j=1,2) \tag{6}$$

where $z_j=s_jz$ ($j$=1,2) and $s_j^2$ are the two roots of the equation [we take Re($s_j$)>0]

$$a_1s^4-a_2s^2+a_3=0 \tag{7}$$

where

$$a_1=c_{33}c_{44},\ a_2=c_{11}c_{33}+c_{55}^2-(c_{13}+c_{55})^2,\ a_3=c_{11}c_{55} \tag{8a}$$

$$k_j=\frac{-c_{11}+c_{55}s_j^2}{-(c_{13}+c_{55})s_j^2},\ \omega_{1j}=c_{33}s_j^2k_j-c_{13},$$
$$\omega_{2j}=-s_j^2\omega_{1j},\qquad (j=1,2) \tag{8b}$$

The polynomials listed in Appendix A can be chosen as harmonic functions $\psi_j$ simply by replacing $z$ with $z_j$. In the next sections, we will consider three loads cases of cantilever beam shown in Fig.1, and derive the analytical solutions by using the general solution (5).

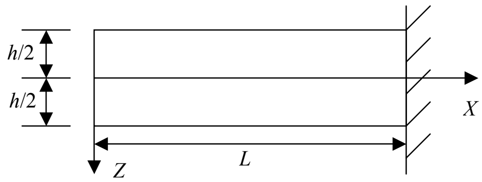


**Fig.1 The geometry and coordinate system of a cantilever beam**

## THREE SOLUTIONS FOR CANTILEVER BEAM WITHOUT BODY FORCES

### Cantilever beam under uniform loads on the upper and bottom surfaces

We introduce the displacement function as follows

$$\psi_j=(x^2-z_j^2)A_{2j}+(x^2z_j-\frac{1}{3}z_j^3)B_{3j}$$
$$+B_{5j}(x^4z_j-2x^2z_j^3+\frac{1}{5}z_j^5) \tag{9}$$

where $A_{2j}$, $B_{3j}$ and $B_{5j}$ ($j$=1,2) are unknown constants to be determined .

Substituting Eq.(9) into Eq.(5) leads to

$$u=\sum_{j=1}^{2}\left[2xA_{2j}+2xz_jB_{3j}+(4x^3z_j-4xz_j^3)B_{5j}\right] \tag{10a}$$

$$w=\sum_{j=1}^{2}s_jk_j[-2z_jA_{2j}+(x^2-z_j^2)B_{3j}+(x^4-6x^2z_j^2+z_j^4)B_{5j}] \tag{10b}$$

$$\sigma_z=\sum_{j=1}^{2}\omega_{1j}[-2A_{2j}-2z_jB_{3j}+(-12x^2z_j+4z_j^3)B_{5j}] \tag{10c}$$

$$\tau_{xz}=\sum_{j=1}^{2}s_j\omega_{1j}[2xB_{3j}+(4x^3-12xz_j^2)B_{5j}] \tag{10d}$$

$$\sigma_x=\sum_{j=1}^{2}\omega_{2j}[-2A_{2j}-2z_jB_{3j}+(-12x^2z_j+4z_j^3)B_{5j}] \tag{10e}$$

The boundary conditions are

$$z=\pm h/2:\sigma_z=\beta_1\pm C_1,\quad \tau_{xz}=0 \tag{11a}$$

$$x=0:\int_{-h/2}^{h/2}\sigma_x\mathrm{d}z=0,\int_{-h/2}^{h/2}\sigma_xz\mathrm{d}z=0,\int_{-h/2}^{+h/2}\tau_{xz}\mathrm{d}z=0 \tag{11b}$$

$$(x=L,z=0):u=0,w=0,\partial w/\partial x=0 \tag{11c}$$

Substituting Eqs.(10c), (10d) and (10e) into Eqs.(11a) and (11b) , we arrive at

$$\sum_{j=1}^{2}\omega_{1j}A_{2j}=-\beta_1/2,\ \sum_{j=1}^{2}\omega_{1j}(-hs_jB_{3j}+\frac{1}{2}h^3s_j^3B_{5j})=C_1 \tag{12}$$

$$\sum_{j=1}^{2}s_j\omega_{1j}B_{5j}=0,\qquad \sum_{j=1}^{2}\omega_{2j}A_{2j}=0 \tag{13}$$

$$\sum_{j=1}^{2}s_j\omega_{1j}(2B_{3j}-3h^2s_j^2B_{5j})=0 \tag{14}$$

$$\sum_{j=1}^{2}s_j\omega_{2j}(-10B_{3j}+3h^2s_j^2B_{5j})=0 \tag{15}$$

Then, the unknown constants $A_{2j}$, $B_{3j}$ and $B_{5j}$ ($j$=1,2) can be determined from Eqs.(12)–(15). To satisfy the boundary conditions Eq.(11c), the solution above should be superposed on the rigid body displacements solutions as follows

$$u_1=u_0+\omega_0z,\qquad w_1=w_0-\omega_0x \tag{16}$$

where

$$u_0=-2L\sum_{j=1}^{2}A_{2j},\ \omega_0=2L\sum_{j=1}^{2}s_jk_j(B_{3j}+2L^2B_{5j}) \tag{17a}$$

$$w_0=L^2\sum_{j=1}^{2}s_jk_j(B_{3j}+3L^2B_{5j}) \tag{17b}$$

**Cantilever beam with axial force $N$ and bending moment $M$ at the free end**

We constitute the displacement function as follows

$$\psi_j=(x^2-z_j^2)A_{2j}+(x^2z_j-\frac{1}{3}z_j^3)B_{3j},\ (j=1,2) \tag{18}$$

Substituting Eq.(18) into Eq.(5) leads to

$$u=\sum_{j=1}^{2}(2xA_{2j}+2xz_jB_{3j}),$$
$$w=\sum_{j=1}^{2}s_jk_j[-2z_jA_{2j}+(x^2-z_j^2)B_{3j}] \tag{19a}$$

$$\sigma_z=\sum_{j=1}^{2}\omega_{1j}(-2A_{2j}-2z_jB_{3j}),\quad \tau_{xz}=2x\sum_{j=1}^{2}s_j\omega_{1j}B_{3j} \tag{19b}$$

$$\sigma_x=\sum_{j=1}^{2}\omega_{2j}(-2A_{2j}-2z_jB_{3j}) \tag{19c}$$

The boundary conditions are

$$z=\pm h/2:\sigma_z=0,\quad \tau_{xz}=0 \tag{20a}$$

$$x=0:\int_{-h/2}^{h/2}\sigma_x\mathrm{d}z=N,\ \int_{-h/2}^{h/2}\sigma_xz\mathrm{d}z=M,\ \int_{-h/2}^{h/2}\tau_{xz}\mathrm{d}z=0 \tag{20b}$$

$$(x=L,z=0):u=0,\ \ w=0,\ \ \partial w/\partial x=0 \tag{20c}$$

Substituting Eqs.(19b) and (19c) into Eqs.(20a) and (20b), we have

$$\sum_{j=1}^{2}\omega_{1j}A_{2j}=0,\qquad \sum_{j=1}^{2}s_j\omega_{1j}B_{3j}=0 \tag{21}$$

$$-\frac{h^3}{6}\sum_{j=1}^{2}s_j\omega_{2j}B_{3j}=M,\quad -2h\sum_{j=1}^{2}\omega_{2j}A_{2j}=N \tag{22}$$

Then, the constants $A_{2j}$ and $B_{3j}$ can be determined from Eqs.(21) and (22). To satisfy the boundary conditions Eq.(20c), the solution above should be superposed on the rigid body displacement solutions as follows

$$u_1 = u_0 + \omega_0 z, \qquad w_1 = w_0 - \omega_0 x \tag{23}$$

where

$$u_0 = -2L\sum_{j=1}^{2} A_{2j}, \omega_0 = 2L\sum_{j=1}^{2} s_j k_j B_{3j}, w_0 = L^2\sum_{j=1}^{2} s_j k_j B_{3j} \tag{24}$$

**Cantilever beam with the *n*th power of *x* tangential loads on the upper and bottom surfaces**

The boundary conditions are taken as

$$z = \pm h/2 : \sigma_z = 0, \quad \tau_{xz} = T_n x^n \tag{25a}$$

$$x = 0 : \int_{-h/2}^{h/2} \sigma_x \mathrm{d}z = 0, \ \int_{-h/2}^{h/2} \sigma_x z \mathrm{d}z = 0, \ \int_{-h/2}^{h/2} \tau_{xz} \mathrm{d}z = 0 \tag{25b}$$

$$(x = L, z = 0) : u = 0, \ \ w = 0, \ \ \partial w / \partial x = 0 \tag{25c}$$

We introduce the displacement function as follows

$$\psi_j = B_{2j}\varphi_2^1(x,z_j) + B_{4j}\varphi_4^1(x,z_j) + \cdots + B_{n+4,j}\varphi_{n+4}^1(x,z_j) \quad (j = 1,2; n = 2,4,6,\cdots) \tag{26a}$$

$$\psi_j = B_{3j}\varphi_3^1(x,z_j) + B_{5j}\varphi_5^1(x,z_j) + \cdots + B_{n+4,j}\varphi_{n+4}^1(x,z_j) \quad (j = 1,2; n = 1,3,5,\cdots) \tag{26b}$$

where $B_{mj}$ are undetermined constants, and $\varphi_m^1(x,z_j)$ are taken from Appendix A.

Substituting Eq.(26) into Eq.(5) leads to the expressions of displacements and stresses. When $n$ is an even number, we have

$$u = \sum_{j=1}^{2}[z_j B_{2j} + (3x^2 z_j - z_j^3)B_{4j} + (5x^4 z_j - 10x^2 z_j^3 + z_j^5)B_{6j} + (7x^6 z_j - 35x^4 z_j^3 + 21x^2 z_j^5 - z_j^7)B_{8j} + \cdots] \tag{27a}$$

$$w = \sum_{j=1}^{2} s_j k_j [xB_{2j} + (x^3 - 3xz_j^2)B_{4j} + (x^5 - 10x^3 z_j^2 + 5xz_j^4)B_{6j} + (x^7 - 21x^5 z_j^2 + 35x^3 z_j^4 - 7xz_j^6)B_{8j} + \cdots] \tag{27b}$$

$$\sigma_z = \sum_{j=1}^{2} \omega_{1j}[-6xz_j B_{4j} + 20xz_j(z_j^2 - x^2)B_{6j} + (-42x^5 z_j + 140x^3 z_j^3 - 42xz_j^5)B_{8j} + \cdots] \tag{27c}$$

$$\tau_{xz} = \sum_{j=1}^{2} s_j \omega_{1j}[B_{2j} + (3x^2 - 3z_j^2)B_{4j} + (5x^4 - 30x^2 z_j^2 + 5z_j^4)B_{6j} + (7x^6 - 105x^4 z_j^2 + 105x^2 z_j^4 - 7z_j^6)B_{8j} + \cdots] \tag{27d}$$

$$\sigma_x = \sum_{j=1}^{2} \omega_{2j}[-6xz_j B_{4j} + 20xz_j(z_j^2 - x^2)B_{6j} + (-42x^5 z_j + 140x^3 z_j^3 - 42xz_j^5)B_{8j} + \cdots] \tag{27e}$$

When $n$ is an odd number, we have

$$u = \sum_{j=1}^{2}[2xz_j B_{3j} + (4x^3 z_j - 4xz_j^3)B_{5j} + (6x^5 z_j - 20x^3 z_j^3 + 6xz_j^5)B_{7j} + \cdots] \tag{28a}$$

$$w = \sum_{j=1}^{2} s_j k_j [(x^2 - z_j^2)B_{3j} + (x^4 - 6x^2 z_j^2 + z_j^4)B_{5j} + (x^6 - 15x^4 z_j^2 + 15x^2 z_j^4 - z_j^6)B_{7j} + \cdots] \tag{28b}$$

$$\sigma_z = \sum_{j=1}^{2} \omega_{1j}[-2z_j B_{3j} + (-12x^2 z_j + 4z_j^3)B_{5j} + (-30x^4 z_j + 60x^2 z_j^3 - 6z_j^5)B_{7j} + \cdots] \tag{28c}$$

$$\sigma_x = \sum_{j=1}^{2} \omega_{2j}[-2z_j B_{3j} + (-12x^2 z_j + 4z_j^3)B_{5j} + (-30x^4 z_j + 60x^2 z_j^3 - 6z_j^5)B_{7j} + \cdots] \tag{28d}$$

$$\tau_{xz} = \sum_{j=1}^{2} s_j \omega_{1j}[2xB_{3j} + 4(x^3 - 3xz_j^2)B_{5j} + 6(x^5 - 10x^3 z_j^2 + 5xz_j^4)B_{7j} + \cdots] \tag{28e}$$

When $n$=1, we substitute Eqs.(28c), (28d) and (28e) into Eqs.(25a) and (25b) and have

$$\sum_{j=1}^{2} \omega_{1j}(-hs_j B_{3j} + \frac{1}{2}h^3 s_j^3 B_{5j}) = 0, \sum_{j=1}^{2} s_j \omega_{1j} B_{5j} = 0 \tag{29}$$

$$\sum_{j=1}^{2} s_j \omega_{1j}(2B_{3j} - 3h^2 s_j^2 B_{5j}) = T_1, \quad \sum_{j=1}^{2} s_j \omega_{2j}(-10B_{3j} + 3h^2 s_j^2 B_{5j}) = 0 \tag{30}$$

Then, the unknown constants $B_{3j}$ and $B_{5j}$ can be determined from Eqs.(29) and (30). To satisfy the boundary conditions in Eq.(25c), the solution above should be superposed on the rigid body displacements solutions as follows

$$u_1 = \omega_0 z, \qquad w_1 = w_0 - \omega_0 x \tag{31}$$

where

$$\omega_0 = 2L\sum_{j=1}^{2} s_j k_j (B_{3j} + 2L^2 B_{5j}),$$

$$w_0 = L^2 \sum_{j=1}^{2} s_j k_j (B_{3j} + 3L^2 B_{5j}) \quad (32)$$

When $n$=3, we substitute Eqs.(28c), (28d) and (28e) into Eqs.(25a) and (25b) and have

$$\sum_{j=1}^{2} \omega_{1j}\left(-6hs_j B_{5j} + \frac{15}{2}h^3 s_j^3 B_{7j}\right) = 0, \quad \sum_{j=1}^{2} s_j \omega_{1j} B_{7j} = 0 \quad (33)$$

$$\sum_{j=1}^{2} \omega_{1j}\left(-s_j h B_{3j} + \frac{1}{2}s_j^3 h^3 B_{5j} - \frac{3}{16}h^5 s_j^5 B_{7j}\right) = 0 \quad (34)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}(4B_{5j} - 15 s_j^2 h^2 B_{7j}) = T_3 \quad (35)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(2B_{3j} - 3h^2 s_j^2 B_{5j} + \frac{15}{8}s_j^4 h^4 B_{7j}\right) = 0 \quad (36)$$

$$\sum_{j=1}^{2} s_j \omega_{2j}\left(-\frac{1}{3}B_{3j} + \frac{1}{10}s_j^2 h^2 B_{5j} - \frac{3}{112}s_j^4 h^4 B_{7j}\right) = 0 \quad (37)$$

Then, $B_{3j}$, $B_{5j}$ and $B_{7j}$ can be determined from Eqs.(33)–(37). To satisfy the left boundary conditions in Eq.(25c), the solution above should be superposed on the rigid body displacement solutions as follows

$$u_1 = \omega_0 z, \quad w_1 = w_0 - \omega_0 x \quad (38)$$

where

$$\omega_0 = 2L\sum_{j=1}^{2} s_j k_j (B_{3j} + 2L^2 B_{5j} + 3L^4 B_{7j}) \quad (39a)$$

$$w_0 = L^2 \sum_{j=1}^{2} s_j k_j (B_{3j} + 3L^2 B_{5j} + 5L^4 B_{7j}) \quad (39b)$$

When $n$=2, we substitute Eqs.(27c), (27d) and (27e) into Eqs.(25a) and (25b) and have

$$\sum_{j=1}^{2} \omega_{1j}\left(-3hs_j B_{4j} + \frac{5}{2}h^3 s_j^3 B_{6j}\right) = 0, \quad \sum_{j=1}^{2} s_j \omega_{1j} B_{6j} = 0 \quad (40)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(3B_{4j} - \frac{15}{2}h^2 s_j^2 B_{6j}\right) = T_2 \quad (41)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(B_{2j} - \frac{3}{4}h^2 s_j^2 B_{4j} + \frac{5}{16}s_j^4 h^4 B_{6j}\right) = 0 \quad (42)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(B_{2j} - \frac{1}{4}s_j^2 h^2 B_{4j} + \frac{1}{16}s_j^4 h^4 B_{6j}\right) = 0 \quad (43)$$

Substituting Eq.(27b) into the third of Eq.(25c), we have

$$\sum_{j=1}^{2} s_j k_j (B_{2j} + 3L^2 B_{4j} + 5L^4 B_{6j}) = 0 \quad (44)$$

Then, the constants $B_{2j}$, $B_{4j}$ and $B_{6j}$ can be determined from Eqs.(40)–(44). To satisfy the left boundary conditions of Eq.(25c), the solution above should be superposed on the rigid body displacements solutions as follows

$$w_1 = w_0 = -L\sum_{j=1}^{2} s_j k_j (B_{2j} + L^2 B_{4j} + L^4 B_{6j}) \quad (45)$$

When $n$=4, we substitute Eqs.(27c), (27d) and (27e) into Eqs.(25a) and (25b) and have

$$\sum_{j=1}^{2} s_j \omega_{1j} B_{8j} = 0, \quad \sum_{j=1}^{2} \omega_{1j}\left(-10 s_j h B_{6j} + \frac{35}{2}s_j^3 h^3 B_{8j}\right) = 0 \quad (46)$$

$$\sum_{j=1}^{2} \omega_{1j}\left(-3s_j h B_{4j} + \frac{5}{2}s_j^3 h^3 B_{6j} - \frac{21}{16}s_j^5 h^5 B_{8j}\right) = 0 \quad (47)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(5B_{6j} - \frac{105}{4}s_j^2 h^2 B_{8j}\right) = T_4 \quad (48)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(3B_{4j} - \frac{15}{2}s_j^2 h^2 B_{6j} + \frac{105}{16}s_j^4 h^4 B_{8j}\right) = 0 \quad (49)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(B_{2j} - \frac{3}{4}s_j^2 h^2 B_{4j} + \frac{5}{16}s_j^4 h^4 B_{6j} - \frac{7}{64}s_j^6 h^6 B_{8j}\right) = 0 \quad (50)$$

$$\sum_{j=1}^{2} s_j \omega_{1j}\left(B_{2j} - \frac{1}{4}s_j^2 h^2 B_{4j} + \frac{1}{16}s_j^4 h^4 B_{6j} - \frac{1}{64}s_j^6 h^6 B_{8j}\right) = 0 \quad (51)$$

From Eq.(27b) and Eq.(25c), we have

$$\sum_{j=1}^{2} s_j k_j (B_{2j} + 3L^2 B_{4j} + 5L^4 B_{6j} + 7L^6 B_{8j}) = 0 \quad (52)$$

Then, $B_{2j}$, $B_{4j}$, $B_{6j}$ and $B_{8j}$ can be determined from Eqs.

(46)–(52). To satisfy the left boundary conditions in Eq.(25c), the solution above should be superposed on the rigid body displacement solutions as follows

$$w_1 = w_0 = -L\sum_{j=1}^{2} s_j k_j (B_{2j} + L^2 B_{4j} + L^4 B_{6j} + L^6 B_{8j}) \quad (53)$$

## References


Ding, H.J., Wang, G.Q., Chen, W.Q., 1997a. General solution of plane problem of piezoelectric media expressed by "harmonic functions". *Applied Mathematics and Mechanics*. **18**:757-764.

Ding, H.J., Wang, G.Q., Chen, W.Q., 1997b. Green's functions for a two-phase infinite piezoelectric plane. *Proceedings of Royal Society of London (A)*, **453**:2241-57.

Lekhnitskii, S.G., 1969. Anisotropic Plate. Gordon and Breach, London.

Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity (3rd Ed). McGraw Hill, New York.


APPENDIX A

Harmonic polynomials for the plane problems can be written in the following form:

$$\varphi_n^m(x,z) = x^{n-m}z^m + \sum_{i=1}^{[(n-m)/2]} (-1)^i \frac{(n-m)(n-m-1)\cdots(n-m-2i+1)}{(2i+m)!} x^{n-2i-m} z^{2i+m}$$
$$(m=0,1;\ n=1,2,\ldots) \quad \text{(A1)}$$

where $[(n-m)/2]$ denotes the largest integer $\le(n-m)/2$. From Eq.(A1),the first seventeen harmonic polynomials can be written as follows:

$$\varphi_0^0(x,z)=1,$$
$$\varphi_1^0(x,z)=x,\ \varphi_1^1(x,z)=z,$$
$$\varphi_2^0(x,z)=x^2-z^2,\ \varphi_2^1(x,z)=xz,$$
$$\varphi_3^0(x,z)=x^3-3xz^2,\ \varphi_3^1(x,z)=x^2z-\frac{1}{3}z^3,$$
$$\varphi_4^0(x,z)=x^4-6x^2z^2+z^4,$$
$$\varphi_4^1(x,z)=x^3z-xz^3,$$
$$\varphi_5^0(x,z)=x^5-10x^3z^2+5xz^4,$$
$$\varphi_5^1(x,z)=x^4z-2x^2z^3+\frac{1}{5}z^5,$$
$$\varphi_6^0(x,z)=x^6-15x^4z^2+15x^2z^4-z^6,$$
$$\varphi_6^1(x,z)=x^5z-\frac{10}{3}x^3z^3+xz^5,$$
$$\varphi_7^0(x,z)=x^7-21x^5z^2+35x^3z^4-7xz^6,$$
$$\varphi_7^1(x,z)=x^6z-5x^4z^3+3x^2z^5-\frac{1}{7}z^7,$$
$$\varphi_8^0(x,z)=x^8-28x^6z^2+70x^4z^4-28x^2z^6+z^8,$$
$$\varphi_8^1(x,z)=x^7z-7x^5z^3+7x^3z^5-xz^7 \quad \text{(A2)}$$