Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

More general results on mixed extreme value distributions*

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Received May 8, 2004; revision accepted Sept. 8, 2004

Abstract: The sequences $\{Z_{i,n}, 1\le i\le n\}$, $n\ge 1$ are multi-nomial distribution among i.i.d. random variables $\{X_{1,i}, i\ge 1\}$, $\{X_{2,i}, i\ge 1\}$, …, $\{X_{m,i}, i\ge 1\}$. The extreme value distribution $G_Z(x)$ of this particular triangular array of i.i.d. random variables $Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n}$ is discussed. A new type of not max-stable extreme value distributions which are Fréchet mixture, Gumbel mixture and Weibull mixture has been found if $F_j,\ldots,F_m$ belong to the same MDA. Whether mixtures of different types of extreme value distributions exist or not and the more general case are discussed in this paper. We found that $G_Z(x)$ does not exist as mixture forms of the different types of extreme value distributions after we investigated all cases.

Key words: Extreme value distribution, Maximum domain of attraction (MDA), Mixed distribution functions
doi:10.1631/jzus.2005.A0769 **Document code:** A **CLC number:** O211.4

INTRODUCTION

Let $\{X_{1,i}, i\ge 1\}$, $\{X_{2,i}, i\ge 1\}$, …, $\{X_{m,i}, i\ge 1\}$ be m sequences of independent and identically distributed random variables with distribution functions $F_j(x)$ for j, $1\le j\le m$. The sequences $\{Z_{i,n}, 1\le i\le n\}$, $n\ge 1$, are defined in Jiang (2004a; 2005) where we found $G_Z(x)$ exists as the forms of Fréchet mixture, Gumbel mixture and Weibull mixture if $F_j,\ldots,F_m$ have the same right endpoint and belong to the same MDA. In this paper we continue to deal with the other cases.

The case in which $F_j(x)$ $(1\le j\le m)$ have the same right endpoint and belong to different MDA's with $p_{j,n}\to 0$ is discussed in Section 2; the case with different right endpoints is dealt with in Section 3, while the more general case with $p_j\ge 0$ $(1\le j\le m)$ is discussed in Section 4. Finally, an example is analyzed in Section 5.

RESULTS FOR DIFFERENT MDA'S

Now we deal with the case that all $F_j(x)$ $(1\le j\le m)$ do not belong to the same MDA.

Lemma 1 Assume that $F_j(x)\in MDA(\Phi_{\alpha_j})$ for $1\le j\le m$ and $F_j(x)\in MDA(\Lambda)$ for $k+1\le j\le m$ with $x_F=\infty$ satisfy Eqs.(2), (3), and (4) in Jiang (2005). If for all j, s with $1\le j,s\le m$

$$np_{j,n}\overline{F}_j(\alpha'_{s,n})\to A_{j,s}\in[0,\infty]$$

and for all s with $k+1\le s\le m$ and all j with $1\le j\le m$

$$np_{j,n}\overline{F}_j(\beta'_{s,n})\to A_{j,s}\in[0,\infty],$$

then there exist

i) an index $r_0=\max\{k+1\le s\le m: A_{j,s}<\infty$ if $\lambda_{j/s}>0$ or $A_{j,s}=0$ if $\lambda_{j/s}=0$ for every j with $k+1\le j\le s\}$ and

ii) an index $r=\max\{1\le s\le k: A_{j,s}<\infty$ for every j with $1\le j\le s\}$.

Proof r_0 and r exist by Lemma 3 and Lemma 4 in Jiang (2005), respectively.

Theorem 1 Under the assumptions of Lemma 1.

i) If $A_{j,r_0}=0$ for j with $1\le j\le k$, then the limit distribution of $M_n(Z)$ with the normalizing sequences

*Project partially supported by the National Natural Science Foundation of Switzerland

$\alpha_n=\alpha'_{r_0,n}$ and $\beta_n=\beta'_{r_0,n}$ is

$$G_Z(x)=\prod_{j=k+1}^{r_0-1}\Lambda^{A_{j,r_0}}(\lambda_{j/r_0}x)\times\Lambda(x),$$

where r_0 is given in Lemma 1.

ii) Otherwise, with the normalizing sequences $\alpha_n=\alpha'_{r,n}$ and $\beta_n=\beta'_{r,n}$ for $x\geq A_{r,r_0}^{-1/\alpha_r}$

$$G_Z(x)=\prod_{j=1}^{r-1}\Phi_{\alpha_j}^{A_{j,r}}(x)\times\Phi_{\alpha_r}(x),$$

where r is given in Lemma 1.

Proof i) By Theorem 3 in Jiang (2005) and using $A_{j,r_0}=0$ for j with $1\leq j\leq k$, the claim follows.

ii) By the proof of Theorem 1 in Jiang (2005) for any j with $r<j\leq k$

$$np_{j,n}\bar{F}_j(\alpha'_{r,n}x)\to 0$$

As in the proof of Theorem 3 in Jiang (2005) we have for any j with $m\geq j>r_0$

$$np_{j,n}\bar{F}_j(\beta'_{r_0,n})n\to 0=A_{j,r_0}$$

By Lemma 2 in Jiang (2004b) for every j with $k+1\leq j\leq m$ and $x\geq A_{r,r_0}^{-1/\alpha_r}$

$$\lim_{x\to\infty}\frac{\bar{F}_j(\alpha'_{r,n}x)}{\bar{F}_j(\beta'_{r_0,n})}=0$$

Hence, for every j with $k+1\leq j\leq m$ and $x\geq A_{r,r_0}^{-1/\alpha_r}$

$$np_{j,n}\bar{F}_j(\alpha'_{r,n}x)=(np_{j,n}\bar{F}_j(\beta'_{r_0,n}))\frac{\bar{F}_j(\alpha'_{r,n}x)}{\bar{F}_j(\beta'_{r_0,n})}\to 0,$$

which completes the proof.

Similarly, if $x_F<\infty$, we have the following theorem.

Theorem 2 Assume that $F_j(x)\in MDA(\Psi_{\alpha_j})$ for $1\leq j\leq k$ and $F_j(x)\in MDA(\Lambda)$ for $k+1\leq j\leq m$ with $x_F<\infty$ satisfying Eqs.(1), (2), and (3) in Jiang (2005). If for all j, s with $1\leq j,s\leq k$

$$np_{j,n}\bar{F}_j(\gamma'_{s,n})\to A_{j,s}\in[0,\infty]$$

and for all s with $k+1\leq s\leq m$ and all j with $1\leq j\leq m$

$$np_{j,n}\bar{F}_j(\beta'_{s,n})\to A_{j,s}\in[0,\infty],$$

then there exist

i) an index $r_0=\max\{k+1\leq s\leq m: A_{j,s}<\infty$ if $\lambda_{j/s}>0$ or $A_{j,s}=0$ if $\lambda_{j/s}=0$ for every j with $k+1\leq j\leq s\}$ and

ii) an index $r=\max\{1\leq s\leq k: A_{j,s}<\infty$ for every j with $1\leq j\leq s\}$.

iii) If $A_{j,r_0}=0$ for j with $1\leq j\leq k$, then the limit distribution of $M_n(Z)$ with the normalizing sequences $\alpha_n=\alpha'_{r_0,n}$ and $\beta_n=\beta'_{r_0,n}$ is

$$G_Z(x)=\prod_{j=k+1}^{r_0-1}\Lambda^{A_{j,r_0}}(\lambda_{j/r_0}x)\times\Lambda(x),$$

for any $x\leq x_F$.

Otherwise, with the normalizing sequences $\alpha_n=\alpha'_{r,n}$ and $\beta_n=\beta'_{r,n}$ for $-A_{r,r_0}^{-1/\alpha_r}\leq x<0$

$$G_Z(x)=\prod_{j=1}^{r-1}\Psi_{\alpha_j}^{A_{j,r_0}}(x)\times\Psi_{\alpha_r}(x).$$

Remark 1 By Theorem 1 and Theorem 2, the extreme value distribution $G_Z(x)$ does not exist as mixture forms of the different types of extreme value distributions.

RESULTS FOR DIFFERENT RIGHT ENDPOINTS

In this section we discuss the limit distribution $G_Z(x)$ assuming that $p_{j,n}\to 0$ for $1\leq j\leq m-1$ where the right endpoints x_{F_j} are not the same. Suppose

$$x_F^*=\max(x_{F_1},\ldots,x_{F_{m-1}}).$$

Theorem 3 Suppose $\{X_{i,i}, j\geq 1\}$, $1\leq i\leq m$, are m sequences of independent and identically distributed random variables with distribution functions $F_j(x)\in MDA(G_j)$ and right endpoint x_{F_j}, $1\leq i\leq m$, respectively.

If $x_F^*<x_{F_m}$, then with the normalizing sequences $\alpha_n=\alpha_{m,n}$ and $\beta_n=\beta_{m,n}$,

$$G_Z(x)=G_m(x)$$

Proof If we set $\alpha_n=\alpha_{m,n}$ and $\beta_n=\beta_{m,n}$, then $\alpha_n x+\beta_n\to x_{F_m}>x_{F_j}$ for $1\le j\le m-1$. Hence, for large n

$$n\overline{F}_{Z,n}(\alpha_n x+\beta_n)=\sum_{j=1}^{m-1}np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)-\log G_m(x)+o(1)=-\log G_m(x)+o(1)$$

The statement follows.

Under the conditions of Theorem 3, let $x_F^*=x_{F_m}$. If there exist some $r\in R=\{1\le r\le m-1: x_{F_r}<x_F^*\}$, then for large n

$$n\overline{F}_{r,n}(\alpha_n x+\beta_n)=0 \quad (1)$$

since $\alpha_n x+\beta_n\to x_F^*=x_{F_m}<x_{F_r}$. Hence, the term $np_{r,n}\overline{F}_r(\alpha_n x+\beta_n)$ has no influence on the limit distribution $G_Z(x)$. Hence, w.l.o.g. we can suppose that every $F_j(x)$ ($1\le j\le m-1$) in our model satisfies $x_{F_j}=x_{F_m}$, which was derived in Section 4.2. Therefore, we need only discuss the situation $x_F^*>x_{F_m}$ and w.l.o.g. we suppose

$$x_{F_1}\ge x_{F_2}\cdots\ge x_{F_{m-1}}\ge x_{F_m} \quad (2)$$

where if $x_{F_i}=x_{F_j}$ and $i<j$, then assume $\lim_{x\to x_{F_i}}\frac{\overline{F}_i(x)}{\overline{F}_j(x)}\ge 1$ and if $\lim_{x\to x_F}\frac{\overline{F}_i(x)}{\overline{F}_j(x)}=1$ we can order them as we want.

Mixture distribution for different right endpoints

A nondegenerate extreme value distribution function $G(x)$ is a mixture distribution for different right endpoints, if there exists an integer $r\ge 2$, $A_i\in[0,\infty)$ for i, $1\le j\le r$, such that

$$T_r(x)=\prod_{i=1}^{r}\exp\{-A_i(1-F_i(x))\}I(x\ge x_{F_r}) \quad (3)$$

where $F_i(x)$ for i with $1\le i\le r$ are distribution functions with $x_{F_i}>x_{F_r}$. Obviously, if $A_i=0$, the factor $\exp\{-A_i(1-F_i(x))\}=1$ is not needed in Eq.(3).

Theorem 4 Under the conditions of Theorem 3, assume that $F_j(x)$ ($1\le j\le m$) satisfy Eq.(2). Let $np_{j,n}\to A_j\in[0,\infty]$ for $1\le j\le m$.

i) If $A_j=0$ for all $1\le j<m$, then with the normalizing sequences $\alpha_n=\alpha_{m,n}$ and $\beta_n=\beta_{m,n}$,

$$G_Z(x)=G_m(x)$$

ii) If $A_j\in(0,\infty)$ for all $1\le j\le m-1$ and at least one $A_r\ne 0$, $1\le r\le m-1$, then with the normalizing sequences $\alpha_n=1$ and $\beta_n=0$,

$$G_Z(x)=T_{m-1}(x)$$

iii) Let $s=\min\{j\le m-1: A_j=\infty\}$.

a) If $s=1$ and $x_{F_j}<x_{F_1}$ for all j with $2\le j\le m-1$, then with the normalizing sequences $\alpha_n=\alpha'_{1,n}$ and $\beta_n=\beta'_{1,n}$,

$$G_Z(x)=G_1(x).$$

b) If $s\ge 2$, then with the normalizing sequences $\alpha_n=1$ and $\beta_n=0$,

$$G_Z(x)=T_{s-1}(x).$$

Remark 2 The case $s=1$ and $x_{F_j}=x_{F_1}$ for some j with $2\le j\le m-1$ will be dealt with later on.

Proof i) Setting $\alpha_n=\alpha_{m,n}$ and $\beta_n=\beta_{m,n}$, $\alpha_n x+\beta_n\to x_{F_m}<x_{F_j}$ for $1\le j\le m-1$, we have

$$n\overline{F}_{Z,n}(\alpha_n x+\beta_n)\sim\sum_{j=1}^{m-1}np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)-\log G_m(x)+o(1)\to-\log G_m(x)$$

and the statement follows.

ii) If $x<x_{F_m}$, then $F_j(x)<1$ for all $1\le j\le m$. We have

$$\Pr\{M_n(Z)\le x\}=\left\{\sum_{j=1}^{m}p_{j,n}F_j(x)\right\}^n\le\{\max(F_1(x),...,F_m(x))\}^n\to 0$$

as $n\to\infty$.

If $x_{F_m} \le x \le x_F^*$, then $F_m(x)=1$ and $F_j(x)\le 1$ for all $1\le j\le m-1$.

We have

$$\Pr\{M_n(Z)\le x\} = \left\{\sum_{j=1}^{m} p_{j,n}F_j(x)\right\}^n$$
$$= \left\{1-\sum_{j=1}^{m-1} p_{j,n}(1-F_j(x))\right\}^n$$
$$\sim \exp\left\{-n\sum_{j=1}^{m-1} p_{j,n}(1-F_j(x))\right\}$$
$$\to \exp\left\{-\sum_{j=1}^{m-1} A_j(1-F_j(x))\right\}$$
$$= \prod_{j=1}^{m-1}\exp\left\{-A_j(1-F_j(x))\right\}$$

as $n\to\infty$, which implies the claim.

iii) Let $s=1$ and set $\alpha_n=\alpha'_{1,n}$ and $\beta_n=\beta'_{1,n}$, then $\alpha_n x+\beta_n \to x_{F_1}=x_{F_s}>x_{F_j}$ for all $2\le j\le m$. Thus $\overline{F}_j(\alpha_n x+\beta_n)=0$ for all $2\le j\le m$ and large n, which shows a).

Let $s\ge 2$ and $x<x_{F_s}$, then $F_j(x)<1$ for $j\le s$. Hence,

$$\Pr\{M_n(Z)\le x\} = \left\{\sum_{j=1, j\ne s}^{m} p_{j,n}F_j(x)+p_{s,n}F_s(x)\right\}^n$$
$$\le \left\{\sum_{j=1, j\ne s}^{m} p_{j,n}+p_{s,n}F_s(x)\right\}^n = \left\{1-p_{s,n}(1-F_s(x))\right\}^n$$
$$\sim \exp\{-np_{s,n}(1-F_s(x))\}\to 0,$$

since $A_s=\infty$.

If $x\ge x_{F_s}$, then $F_j(x)=1$ for all j with $s\le j\le m$. Hence we have

$$\Pr\{M_n(Z)\le x\} = \left\{\sum_{j=1}^{s-1} p_{j,n}F_j(x)+\sum_{j=s}^{m} p_{j,n}\right\}^n$$
$$= \left\{1-\sum_{j=1}^{s-1} p_{j,n}(1-F_j(x))\right\}^n$$
$$\sim \exp\left\{-n\sum_{j=1}^{s-1} p_{j,n}(1-F_j(x))\right\}$$
$$\to \exp\left\{-\sum_{j=1}^{s-1} A_j(1-F_j(x))\right\}$$
$$= \prod_{j=1}^{s-1}\exp\{-A_j(1-F_j(x))\},$$

as $n\to\infty$ which implies the claim.

Now we suppose that $x_{F_1}=x_{F_2}=\cdots=x_{F_k}>x_{F_{k+1}}$ and $np_{1,n}\to A_1=\infty$. M_Z has the following limit distribution.

Theorem 5 Under the conditions of Theorem 3, assume that $F_j(x)$ satisfy Eq.(2) with $x_{F_1}=x_{F_2}=\cdots=x_{F_k}$, $1\le j\le m$. Let $np_{1,n}\to A_1=\infty$ and $r=\min\{1\le j\le m: A_j=\infty\}$.

i) If $r>k$, then with the normalizing sequences $\alpha_n=\alpha'_{1,n}$ and $\beta_n=\beta'_{1,n}$,

$$G_Z(x)=G_1(x).$$

ii) If $r\le k$, let $s=\max\{j\le k: A_j=\infty\}$, then $G_Z(x)$ is a mixture extreme value distribution.

Proof i) Since $r>k$, $A_j\in(0,\infty)$ for all j with $2\le j\le k$. If we set $\alpha_n=\alpha'_{1,n}$ and $\beta_n=\beta'_{1,n}$, then $\alpha_n x+\beta_n\to x_{F_1}$ and $\overline{F}_j(\alpha_n x+\beta_n)=0$ for all j with $k+1\le j\le m$ and large n. Hence, as $n\to\infty$

$$n\overline{F}_{Z,n}(\alpha_n x+\beta_n)=\sum_{j=1}^{k} np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)$$
$$=-\log G_1(x)+o(1)+\sum_{j=2}^{k} np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)$$
$$\to -\log G_1(x),$$

the statement follows.

ii) If we set $\alpha_n=\alpha'_{s,n}$ and $\beta_n=\beta'_{s,n}$, then $\alpha_n x+\beta_n\to x_{F_1}$ and $\overline{F}_j(\alpha_n x+\beta_n)=0$ for all j with $k+1\le j\le m$ and large n. Since $A_j\le\infty$ for j with $s<j\le k$, as $n\to\infty$ we get

$$n\overline{F}_{Z,n}(\alpha_n x+\beta_n)$$
$$=\sum_{j=r}^{s-1} np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)-\log G_s(x)+o(1)$$
$$\sim \sum_{j=r}^{s-1} np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)I\{A_j=\infty\}$$

$$-\log G_s(x)+o(1).$$

We note that $x_{F_1}=x_{F_2}=\cdots=x_{F_s}$, by using the same approach as in the proofs of Theorems 1 and 2 in Jiang (2005), the statement follows.

RESULTS FOR GENERAL CASE

In this section we discuss the limit distribution $G_Z(x)$ for the case in which $p_{j,n}\to p_j\geq 0$ for $1\leq j\leq m$.

W.l.o.g. we suppose that $p_j=0$ ($1\leq j\leq r$) and $p_j>0$ ($r\leq j\leq m$). Let $x'_F=\max(x_{F_1},\cdots,x_{F_m})$. For any normalizing sequences α_n and β_n with $n\overline{F}_j(\alpha_n x+\beta_n)<\infty$ for each j ($m\geq j\geq r$) and $\alpha_n x+\beta_n\to x'_F$. Then we have

$$\begin{aligned}n\overline{F}_{Z,n}(\alpha_n x+\beta_n)&=\left(\sum_{j=1}^{r}+\sum_{j=r+1}^{m}\right)np_{j,n}\overline{F}_j(\alpha_n x+\beta_n)\\&=n\sum_{j=1}^{r}p_{j,n}\overline{F}_j(\alpha_n x+\beta_n)+n\sum_{j=r+1}^{m}p_j\overline{F}_j(\alpha_n x+\beta_n)+o(1)\\&=n\sum_{j=1}^{r}p_{j,n}\overline{F}_j(\alpha_n x+\beta_n)+n\overline{F}''_j(\alpha_n x+\beta_n)+o(1),\end{aligned}$$

where $\overline{F}''_{r+1}(x)=\sum_{j=r+1}^{m}p_j\overline{F}_j(x)$.

Defining the distribution

$$\overline{F}'_{r+1}(x)=1-\overline{F}'_{r+1}(x)\Big/\left(1-\sum_{j=1}^{r}p_j\right)$$

the case reduces to that we dealt with in Section 2 and Section 3. Therefore, we discuss the case $p_j>0$ for all j with $1\leq j\leq m$.

Theorem 6 Suppose $\{X_{i,j},\ i\geq 1\}$, $1\leq j\leq m$, are m sequences of independent and identically distributed random variables with distribution functions $F_j(x)\in MDA(G_j)$ and right endpoints x_{F_j}, $1\leq j\leq m$, respectively. Let $G_V(x)$ be the extreme value distribution of $V=\max(X_{1,n}, X_{2,n}, \ldots, X_{m,n})$.

If $p_j>0$ and

$$\lim_{x\to x'_F}\frac{\overline{F}_j(x)}{\max\left(\overline{F}_1(x),\ldots,\overline{F}_m(x)\right)}=k_j$$

for each j with $1\leq j\leq m$, then $G_Z(x)$ and $G_V(x)$ are of the same type.

Proof For any $x_n\to x'_F$, we have

$$\overline{F}_{Z,n}(x_n)=\sum_{j=1}^{m}p_{j,n}\overline{F}_j(x_n)\sim\sum_{j=1}^{m}p_j\overline{F}_j(x_n)$$

which implies $F_{Z,n}(x)$ and $F_Z(x)=\sum_{j=1}^{m}p_jF_j(x)$ have the same extreme value distribution (Jiang, 2002). As $x\to x'_F$

$$\begin{aligned}\overline{F}_V(x)&=1-\prod_{j=1}^{m}F_j(x)=1-\prod_{j=1}^{m}(1-\overline{F}_j(x))\\&=1-\exp\left\{\sum_{j=1}^{m}\log(1-\overline{F}_j(x))\right\}\\&\sim 1-\exp\left\{-\sum_{j=1}^{m}\overline{F}_j(x)\right\}\sim\sum_{j=1}^{m}\overline{F}_j(x)\end{aligned}$$

Now we check

$$\begin{aligned}\lim_{x\to x'_F}\frac{1-F_Z(x)}{1-F_V(x)}&=\lim_{x\to x'_F}\frac{\sum_{j=1}^{m}p_j\overline{F}_j(x)}{\sum_{j=1}^{m}\overline{F}_j(x)}\\&=\lim_{x\to x'_F}\frac{\sum_{j=1}^{m}p_j\dfrac{\overline{F}_j(x)}{\max\left(\overline{F}_1(x),\ldots,\overline{F}_m(x)\right)}}{\sum_{j=1}^{m}\dfrac{\overline{F}_j(x)}{\max\left(\overline{F}_1(x),\ldots,\overline{F}_m(x)\right)}}=\frac{\sum_{j=1}^{m}p_jk_j}{\sum_{j=1}^{m}k_j}.\end{aligned}$$

Since

$$0<\min(p_1,\cdots,p_m)\leq\sum_{j=1}^{m}p_jk_j\Big/\sum_{j=1}^{m}k_j\leq 1.$$

Hence, according to Theorem 2.12 in Resnick (1987), $G_V(x)$ and $G_Z(x)$ are of the same type.

CONCLUSION AND EXAMPLE

We observed only mixtures of the same type, which means that a mixture of two different types, e.g. Fréchet and Gumbel distributions, is not possible. We have shown that this is also not possible even in the more general mixture case (m>2). With the theorems we have established, it is easy to deal with the mixed extreme value distribution as the following example shows.

Example 1 Suppose for large x

$$\overline{F}_1(x)\sim k_1x^{-a_1},\ \overline{F}_2(x)\sim k_2x^{-a_2},$$
$$\overline{F}_3(x)\sim k_3\mathrm{e}^{-c_3x},\ \overline{F}_4(x)\sim k_4\mathrm{e}^{-c_4x},$$

and

$$\overline{F}_5(x)\sim k_5\mathrm{e}^{-c_5x},$$

where, $0<C_3<C_4<C_5<\infty$, $0<a_1<a_2<\infty$ and $k_j>0$ for $1\le j\le 5$.

We discuss two cases.

i) Assume that $p_{j,n}=o(n^{-5/4})$ for j=1,2 and $n^{1-c_j^{-1}C_5}p_{j,n}\to d_j\in(0,\infty)$ for j=3,4, as $n\to\infty$. We want to derive $G_Z(x)$ and the corresponding normalizing sequences.

It is easy to check that with normalizing sequences $\alpha_{j,n}=(nk_j)^{1/a_j}$ and $\beta_{1,n}=0$ for j=1,2

$$G_j(x)=\Phi_{a_j}(x)$$

and with normalizing sequences $\alpha_{j,n}=c_j^{-1}$ and $\beta_{j,n}=c_j^{-1}\log(k_jn)$ for j=3,4,5

$$G_j(x)=\Lambda(x)$$

If we set normalizing sequences $\alpha_n=\alpha'_{5,n}=c_5^{-1}$ and $\beta_n=\beta'_{5,n}=c_5^{-1}\log(k_5np_{5,n})$, then for j=1,2

$$\begin{aligned}np_{j,n}c_5^{a_j}(\log k_5np_{5,n}))k_j&\sim c_5^{a_j^0}(n^{-1/4})(\log n)^{-a_j}k_j\\&\to 0=A_{j,5}\end{aligned}\tag{4}$$

and for j=3,4

$$\begin{aligned}np_{j,n}\mathrm{e}^{-c_jc_5^{-1}\log(k_5np_{5,n})}k_j&\sim k_jk_5^{-c_jc_5^{-1}}n^{1-c_j^{-1}c_5}p_{j,n}^{1-c_j^{-1}c_5}\\&\to dk_jk_5^{-c_jc_5^{-1}}=A_{j,5}\in(0,\infty).\end{aligned}$$

By Theorem 1, with normalizing sequences $\alpha_n=c_5^{-1}$ and $\beta_n=c_5^{-1}\log(k_5np_{5,n})$,

$$G_Z(x)=\Lambda^{A_{3,5}}(c_3c_5^{-1}x)\Lambda^{A_{4,5}}(c_4c_5^{-1}x)\Lambda(x).$$

ii) Assume that $n^{1-a_1/(4a_2)}p_{1,n}\to d_1\in(0,\infty)$ and $n^{3/4}p_{2,n}\to d_2$, as $n\to\infty$. By Eq.(4) we have

$$\begin{aligned}np_{2,n}c_5^{a_2}(\log(k_5np_{5,n}))^{-a_2}&\sim d_2c_5^{a_2}n^{0.25}(\log n)^{-a_2}\\&\to\infty=A_{2,5}.\end{aligned}$$

By Theorem 1, we set the normalizing sequences $\alpha_n=\alpha'_{2,n}=(k_2np_{2,n})^{1/a_2}$ and $\beta_n=\beta'_{2,n}=0$. Hence,

$$\begin{aligned}k_1np_{1,n}(k_2np_{2,n})^{-a_1/a_2}&=k_1k_2^{-a_1/a_2}n^{1-a_1/a_2}p_{1,n}p_{2,n}^{-a_1/a_2}\\&\sim k_1(d_2k_2)^{-a_1/a_2}n^{1-a_1/(4a_2)}p_{1,n}\\&\to d_1k_1(d_2k_2)^{-a_1/a_2}=A_{1,2}\in(0,\infty).\end{aligned}$$

Hence,

$$G_Z(x)=\Phi_{a_1}^{A_{1,2}}(x)\Phi_{a_2}(x)$$

for $x\ge 0$.

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