Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Numerical approach for a system of second kind Volterra integral equations in magneto-electro-elastic dynamic problems*

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Received Feb. 5, 2005; revision accepted June 3, 2005

Abstract: The elastodynamic problems of magneto-electro-elastic hollow cylinders in the state of axisymmetric plane strain case can be transformed into two Volterra integral equations of the second kind about two functions with respect to time. Interpolation functions were introduced to approximate two unknown functions in each time subinterval and two new recursive formulae are derived. By using the recursive formulae, numerical results were obtained step by step. Under the same time step, the accuracy of the numerical results by the present method is much higher than that by the traditional quadrature method.

Key words: Magneto-electro-elastic, Elastodynamic problem, Volterra integral equation, Numerical solution, Recursive formula
doi:10.1631/jzus.2005.A0928 **Document code:** A **CLC number:** O347.1; O241.8

INTRODUCTION

In scientific and engineering problems, Volterra integral equations are always encountered and have attracted much attention (Christopher and Baker, 1977; Delves and Mohamed, 1985; Brunner and van der Houwen, 1986; Kress, 1989; Oja and Saveljeva, 2002; Maleknejad and Shahrezaee, 2004; Maleknejad and Aghazadeh, 2005; Zerarka and Soukeur, 2005). Generally, exact solutions are very difficult to find and solutions have been obtained for only some few special cases (Yan and Cui, 1993; Li *et al.*, 1995). While, for practical cases, approximate solutions are also useful if exact solutions cannot be obtained. It is noted that the numerical methods in many textbooks are usually described for the Volterra integral equations with general kernels and the calculating efficiency is always imperfect for practical problems. Recently, Ding *et al.*(2004) found that the elastodynamic problems for piezoelectric and pyroelectric hollow cylinders under radial deformation can be successfully transformed into a second kind Volterra integral equation with respect to a function of time. The numerical method for such problem had been discussed detailedly. In very recent study, the elastodynamic problems for magneto-electro-elastic hollow cylinders in the state of axisymmetric plane strain case can be transformed into two second kind Volterra integral equations with respect to two functions of time in the form below (Hou and Leung, 2004)

$$\chi(\tau)=Z_1\eta(\tau)+Z_2 f(\tau)+\sum_{i=1}^{m}Z_{3i}\int_0^{\tau}\eta(p)\sin\omega_i(\tau-p)\mathrm{d}p+\sum_{i=1}^{m}Z_{4i}\int_0^{\tau}f(p)\sin\omega_i(\tau-p)\mathrm{d}p$$
$$\psi(\tau)=E_1\eta(\tau)+E_2 f(\tau)+\sum_{i=1}^{m}E_{3i}\int_0^{\tau}\eta(p)\sin\omega_i(\tau-p)\mathrm{d}p+\sum_{i=1}^{m}E_{4i}\int_0^{\tau}f(p)\sin\omega_i(\tau-p)\mathrm{d}p, \tag{1}$$

*Project supported by the National Natural Science Foundation of China (No. 10472102) and Postdoctoral Foundation of China (No. 20040350712)

where $\eta(\tau)$ and $f(\tau)$ are two undetermined functions of time. $\chi(\tau)$ and $\psi(\tau)$ are functions about time τ, related to the mechanical, electric and magnetic boundary conditions on the internal and external surfaces. ω_i (i=1,2,…,m), arranged in an increasing order, are related to the positive roots of an eigenequation. Usually, the number of ω_i is infinite, and m represents the number of the terms for calculation. Z_1, Z_2, E_1, E_2 and Z_{3i}, Z_{4i}, E_{3i}, E_{4i} (i=1,2,…,m) are constants which can be determined beforehand.

Noticing that the kernel of Eq.(1) is a triangular function, so the integration can be obtained explicitly if $\eta(\tau)$ and $f(\tau)$ are polynomial functions of time τ. By using this property, we can derive the recursive formulae by using of an interpolation polynomial to approximate the unknown functions at each of the time subintervals. Then the numerical results can be obtained by the recursive formulae step by step.

If $\chi(\tau)$ and $\psi(\tau)$ in Eq.(1) are differentiable functions, then we have

$$\begin{aligned}\dot{\chi}(\tau) &= Z_1\dot{\eta}(\tau) + Z_2\dot{f}(\tau) \\ &+ \sum_{i=1}^{m} Z_{3i}\omega_i\int_0^{\tau}\eta(p)\cos\omega_i(\tau-p)\mathrm{d}p \\ &+ \sum_{i=1}^{m} Z_{4i}\omega_i\int_0^{\tau}f(p)\cos\omega_i(\tau-p)\mathrm{d}p \\ \dot{\psi}(\tau) &= E_1\dot{\eta}(\tau) + E_2\dot{f}(\tau) \\ &+ \sum_{i=1}^{m} E_{3i}\omega_i\int_0^{\tau}\eta(p)\cos\omega_i(\tau-p)\mathrm{d}p \\ &+ \sum_{i=1}^{m} E_{4i}\omega_i\int_0^{\tau}f(p)\cos\omega_i(\tau-p)\mathrm{d}p,\end{aligned} \tag{2}$$

where a dot over the quantity denotes its derivative with respect to time. Setting τ=0 in Eqs.(1) and (2), we can determine $\eta(0)$, $f(0)$, $\dot{\eta}(0)$ and $\dot{f}(0)$ as

$$\eta(0) = \frac{E_2\chi(0) - Z_2\psi(0)}{E_2Z_1 - E_1Z_2},\ f(0) = \frac{Z_1\psi(0) - E_1\chi(0)}{E_2Z_1 - E_1Z_2} \tag{3}$$

$$\dot{\eta}(0) = \frac{E_2\dot{\chi}(0) - Z_2\dot{\psi}(0)}{E_2Z_1 - E_1Z_2},\ \dot{f}(0) = \frac{Z_1\dot{\psi}(0) - E_1\dot{\chi}(0)}{E_2Z_1 - E_1Z_2} \tag{4}$$

In the following, two interpolation functions, linear interpolation function and cubic Hermite polynomial, are employed to construct the recursive formulae.

NUMERICAL APPROACH

Recursive formula: the first kind

In each time subinterval $[\tau_{j-1}, \tau_j]$ (j=1,2,…,n), the linear interpolation function is introduced to approximate the unknown functions $\eta(\tau)$ and $f(\tau)$ as

$$\begin{aligned}\eta(\tau) &= F_{0j}(\tau)\eta(\tau_{j-1}) + F_{1j}(\tau)\eta(\tau_j), \\ f(\tau) &= F_{0j}(\tau)f(\tau_{j-1}) + F_{1j}(\tau)f(\tau_j) \\ &(j = 1,2,\cdots,n)\end{aligned} \tag{5}$$

in which

$$F_{0j}(\tau) = \frac{\tau - \tau_j}{\tau_{j-1} - \tau_j},\quad F_{1j}(\tau) = \frac{\tau - \tau_{j-1}}{\tau_j - \tau_{j-1}} \quad (j = 1,2,\cdots,n) \tag{6}$$

The substitution of Eq.(5) into Eq.(1) yields

$$\begin{aligned}\chi(\tau_j) &= Z_1\eta(\tau_j) + Z_2 f(\tau_j) \\ &+ \sum_{i=1}^{m} Z_{3i}\sum_{k=1}^{j}[A_{ijk}\eta(\tau_{k-1}) + B_{ijk}\eta(\tau_k)] \\ &+ \sum_{i=1}^{m} Z_{4i}\sum_{k=1}^{j}[A_{ijk}f(\tau_{k-1}) + B_{ijk}f(\tau_k)] \\ \psi(\tau_j) &= E_1\eta(\tau_j) + E_2 f(\tau_j) \\ &+ \sum_{i=1}^{m} E_{3i}\sum_{k=1}^{j}[A_{ijk}\eta(\tau_{k-1}) + B_{ijk}\eta(\tau_k)] \\ &+ \sum_{i=1}^{m} E_{4i}\sum_{k=1}^{j}[A_{ijk}f(\tau_{k-1}) + B_{ijk}f(\tau_k)]\end{aligned} \tag{7}$$

where

$$\begin{aligned}A_{ijk} &= \int_{\tau_{k-1}}^{\tau_k} F_{0k}(p)\sin\omega_i(\tau_j - p)\mathrm{d}p \\ B_{ijk} &= \int_{\tau_{k-1}}^{\tau_k} F_{1k}(p)\sin\omega_i(\tau_j - p)\mathrm{d}p \\ &(k = 1,2,\cdots,j;\quad j = 1,2,\cdots,n)\end{aligned} \tag{8}$$

Then a recursive formula can be obtained from Eq.(7) as

$$\begin{aligned}\eta(\tau_j) &= (b_{1j}a_{22j} - b_{2j}a_{12j})/W_j \\ f(\tau_j) &= (b_{2j}a_{11j} - b_{1j}a_{21j})/W_j\end{aligned} \tag{9}$$

where

$$a_{11j} = Z_1 + \sum_{i=1}^{m} Z_{3i}B_{ijj},\quad a_{12j} = Z_2 + \sum_{i=1}^{m} Z_{4i}B_{ijj}$$

$$a_{21j}=E_1+\sum_{i=1}^{m}E_{3i}B_{ijj},\quad a_{22j}=E_2+\sum_{i=1}^{m}E_{4i}B_{ijj}$$
$$W_j=a_{11j}a_{22j}-a_{12j}a_{21j}$$
$$\begin{aligned}b_{1j}=\chi(\tau_j)&-\sum_{i=1}^{m}Z_{3i}A_{ijj}\eta(\tau_{j-1})-\sum_{i=1}^{m}Z_{4i}A_{ijj}f(\tau_{j-1})\\&-\sum_{i=1}^{m}Z_{3i}\sum_{k=1}^{j-1}[A_{ijk}\eta(\tau_{k-1})+B_{ijk}\eta(\tau_k)]\\&-\sum_{i=1}^{m}Z_{4i}\sum_{k=1}^{j-1}[A_{ijk}f(\tau_{k-1})+B_{ijk}f(\tau_k)]\end{aligned}$$
$$\begin{aligned}b_{2j}=\psi(\tau_j)&-\sum_{i=1}^{m}E_{3i}A_{ijj}\eta(\tau_{j-1})-\sum_{i=1}^{m}E_{4i}A_{ijj}f(\tau_{j-1})\\&-\sum_{i=1}^{m}E_{3i}\sum_{k=1}^{j-1}[A_{ijk}\eta(\tau_{k-1})+B_{ijk}\eta(\tau_k)]\\&-\sum_{i=1}^{m}E_{4i}\sum_{k=1}^{j-1}[A_{ijk}f(\tau_{k-1})+B_{ijk}f(\tau_k)]\end{aligned}\quad(10)$$

As $\eta(0)$ and $f(0)$ have been obtained in Eq.(3), we can determine $\eta(\tau_j)$ and $f(\tau_j)$ (j=1,2,…,n) step by step by virtue of Eq.(9).

Recursive formula: the second kind

Cubic Hermite polynomials are adopted here to approximate the unknown functions $\eta(\tau)$ and $f(\tau)$ as

$$\begin{aligned}\eta(\tau)&=H_{0j}(\tau)\eta(\tau_{j-1})+H_{1j}(\tau)\eta(\tau_j)\\&\quad+H_{2j}(\tau)\dot\eta(\tau_{j-1})+H_{3j}(\tau)\dot\eta(\tau_j)\\f(\tau)&=H_{0j}(\tau)f(\tau_{j-1})+H_{1j}(\tau)f(\tau_j)\\&\quad+H_{2j}(\tau)\dot f(\tau_{j-1})+H_{3j}(\tau)\dot f(\tau_j)\\&\quad(j=1,2,\cdots,n)\end{aligned}\quad(11)$$

where

$$\begin{aligned}H_{0j}(\tau)&=\left(1+2\frac{\tau-\tau_{j-1}}{\tau_j-\tau_{j-1}}\right)\left(\frac{\tau-\tau_j}{\tau_j-\tau_{j-1}}\right)^2\\H_{1j}(\tau)&=\left(1+2\frac{\tau_j-\tau}{\tau_j-\tau_{j-1}}\right)\left(\frac{\tau-\tau_{j-1}}{\tau_j-\tau_{j-1}}\right)^2\\H_{2j}(\tau)&=(\tau-\tau_{j-1})\left(\frac{\tau-\tau_j}{\tau_j-\tau_{j-1}}\right)^2\\H_{3j}(\tau)&=(\tau-\tau_j)\left(\frac{\tau-\tau_{j-1}}{\tau_j-\tau_{j-1}}\right)^2\quad(j=1,2,\cdots,n)\end{aligned}\quad(12)$$

Substitution of Eq.(11) into Eqs.(1) and (2) yields

$$\begin{aligned}\chi(\tau_j)&=Z_1\eta(\tau_j)+\sum_{i=1}^{m}Z_{3i}\sum_{k=1}^{j}[L_{0ijk}\eta(\tau_{k-1})\\&\quad+L_{1ijk}\eta(\tau_k)+L_{2ijk}\dot\eta(\tau_{k-1})+L_{3ijk}\dot\eta(\tau_k)]\\&\quad+Z_2f(\tau_j)+\sum_{i=1}^{m}Z_{4i}\sum_{k=1}^{j}[L_{0ijk}f(\tau_{k-1})\\&\quad+L_{1ijk}f(\tau_k)+L_{2ijk}\dot f(\tau_{k-1})+L_{3ijk}\dot f(\tau_k)]\\\psi(\tau_j)&=E_1\eta(\tau_j)+\sum_{i=1}^{m}E_{3i}\sum_{k=1}^{j}[L_{0ijk}\eta(\tau_{k-1})\\&\quad+L_{1ijk}\eta(\tau_k)+L_{2ijk}\dot\eta(\tau_{k-1})+L_{3ijk}\dot\eta(\tau_k)]\\&\quad+E_2f(\tau_j)+\sum_{i=1}^{m}E_{4i}\sum_{k=1}^{j}[L_{0ijk}f(\tau_{k-1})\\&\quad+L_{1ijk}f(\tau_k)+L_{2ijk}\dot f(\tau_{k-1})+L_{3ijk}\dot f(\tau_k)]\end{aligned}\quad(13)$$

$$\begin{aligned}\dot\chi(\tau_j)&=Z_1\dot\eta(\tau_j)+\sum_{i=1}^{m}Z_{3i}\omega_i\sum_{k=1}^{j}[K_{0ijk}\eta(\tau_{k-1})\\&\quad+K_{1ijk}\eta(\tau_k)+K_{2ijk}\dot\eta(\tau_{k-1})+K_{3ijk}\dot\eta(\tau_k)]\\&\quad+Z_2\dot f(\tau_j)+\sum_{i=1}^{m}Z_{4i}\omega_i\sum_{k=1}^{j}[K_{0ijk}f(\tau_{k-1})\\&\quad+K_{1ijk}f(\tau_k)+K_{2ijk}\dot f(\tau_{k-1})+K_{3ijk}\dot f(\tau_k)]\\\dot\psi(\tau_j)&=E_1\dot\eta(\tau_j)+\sum_{i=1}^{m}E_{3i}\omega_i\sum_{k=1}^{j}[K_{0ijk}\eta(\tau_{k-1})\\&\quad+K_{1ijk}\eta(\tau_k)+K_{2ijk}\dot\eta(\tau_{k-1})+K_{3ijk}\dot\eta(\tau_k)]\\&\quad+E_2\dot f(\tau_j)+\sum_{i=1}^{m}E_{4i}\omega_i\sum_{k=1}^{j}[K_{0ijk}f(\tau_{k-1})\\&\quad+K_{1ijk}f(\tau_k)+K_{2ijk}\dot f(\tau_{k-1})+K_{3ijk}\dot f(\tau_k)]\end{aligned}\quad(14)$$

where

$$\begin{aligned}L_{lijk}&=\int_{\tau_{k-1}}^{\tau_k}H_{lk}(p)\sin\omega_i(\tau_j-p)\mathrm{d}p\\K_{lijk}&=\int_{\tau_{k-1}}^{\tau_k}H_{lk}(p)\cos\omega_i(\tau_j-p)\mathrm{d}p\\&(l=0,1,2,3;\quad k=1,2,\cdots,j;\quad j=1,2,\cdots,n)\end{aligned}\quad(15)$$

From Eqs.(13) and (14), the recursive formula about $\eta(\tau_j)$, $f(\tau_j)$, $\dot\eta(\tau_j)$ and $\dot f(\tau_j)$ can then be obtained. The specific form of the recursive formula is omitted here to reduce the length.

NUMERICAL TESTS

In the following, numerical test is presented to verify the validity of the recursive formulae. We consider the exact solutions of $\eta(\tau)$ and $f(\tau)$ are $\eta(\tau)=100+10\cos 2\tau$ and $f(\tau)=100+20e^{-0.1\tau}$. The analytical expressions of $\chi(\tau)$ and $\psi(\tau)$ are obtained by substituting the exact solutions of $\eta(\tau)$ and $f(\tau)$ into Eq. (1). In the demonstration, we take m=30, ω_i={1.43, 7.41, 14.61, 21.29, 29.23, 36.57, 43.87, 51.18, 58.94, 65.67, 73.05, 80.36, 87.86, 94.96, 102.52, 109.45, 116.68, 124.71, 131.84, 138.27, 146.14, 153.45, 160.76, 168.70, 175.37, 182.68, 189.99, 197.02, 204.05, 211.48}, Z_1=−0.61, Z_2=0.82, Z_{3i}={0.12, 0.62, 0.0034, 0.19, 0.01, 0.12, 0.0029, 0.084, 0.0021, 0.065, 0.0017, 0.053, 0.0014, 0.04, 0.0012, 0.039, 0.001, 0.034, 0.00091, 0.03, 0.00086, 0.028, 0.00078, 0.025, 0.00072, 0.023, 0.00066, 0.021, 0.00061, 0.02}, Z_{4i}={0.44, 1.86, 0.1, 0.58, 0.049, 0.34, 0.032, 0.24, 0.024, 0.19, 0.019, 0.158, 0.016, 0.13, 0.013, 0.11, 0.012, 0.1, 0.01, 0.09, 0.0096, 0.082, 0.0087, 0.075, 0.008, 0.069, 0.0074, 0.064, 0.0069, 0.059}, E_1=−0.99, E_2=2.56, E_{3i}={0.043, 0.1, 0.0058, 0.031, 0.0026, 0.018, 0.0017, 0.013, 0.0012, 0.01, 0.001, 0.0082, 0.00084, 0.0069, 0.00072, 0.006, 0.00063, 0.0053, 0.00056, 0.0047, 0.0005, 0.0043, 0.00045, 0.0039, 0.00042, 0.0036, 0.00038, 0.0033, 0.00036, 0.0031}, E_{4i}={0.078, 0.26, 0.05, 0.086, 0.023, 0.051, 0.015, 0.036, 0.011, 0.028, 0.0091, 0.022, 0.0075, 0.019, 0.0064, 0.016, 0.0056, 0.014, 0.005, 0.013, 0.0045, 0.012, 0.0041 ,0.01, 0.0037, 0.01, 0.0034, 0.0093, 0.0032, 0.0086}. It is noted here that uniform step is adopted in the calculation. Comparisons between the quadrature method (the trapezium rule) and the present two methods for different time steps are shown in Tables 1~4.

Table 1 Numerical results of $\eta(\tau)$ for time step length $\Delta\tau$=0.01

Time	Theoretical results	The quadrature method (The trapezium rule)		Present, the first kind		Present, the second kind	
		Numerical results	Relative error	Numerical results	Relative error	Numerical results	Relative error
0.0	110.0000	110.0000	0.000	110.0000	0.000	110.0000	0.000
0.5	105.4030	100.9010	−4.271E-2	105.4024	−5.309E-6	105.4030	−2.596E-11
1.0	95.8385	87.9440	−8.237E-2	95.8379	−6.210E-6	95.8385	−5.724E-11
1.5	90.1001	74.1798	−1.767E-1	90.0993	−8.821E-6	90.1001	−1.272E-10
2.0	93.4636	56.0428	−4.004E-1	93.4612	−2.539E-5	93.4636	−4.612E-9

Table 2 Numerical results $f(\tau)$ for time step length $\Delta\tau$=0.01

Time	Theoretical results	The quadrature method (The trapezium rule)		Present, the first kind		Present, the second kind	
		Numerical results	Relative error	Numerical results	Relative error	Numerical results	Relative error
0.0	120.0000	120.0000	0.000	120.0000	0.000	120.0000	0.000
0.5	119.0246	117.3739	−1.387E-2	119.0244	−1.721E-6	119.0246	−8.447E-12
1.0	118.0967	115.2187	−2.437E-2	118.0965	−1.822E-6	118.0967	−1.676E-11
1.5	117.2142	111.4259	−4.938E-2	117.2139	−2.436E-6	117.2142	−3.549E-11
2.0	116.3746	102.7563	−1.170E-1	116.3738	−7.420E-6	116.3746	−1.360E-9

Table 3 Numerical results of $\eta(\tau)$ for time step length $\Delta\tau$=0.02

Time	Theoretical results	The quadrature method (The trapezium rule)		Present, the first kind		Present, the second kind	
		Numerical results	Relative error	Numerical results	Relative error	Numerical results	Relative error
0.0	110.0000	110.0000	0.000	110.0000	0.000	110.0000	0.000
0.5	105.4030	85.9447	−1.846E-1	105.4008	−2.132E-5	105.4030	−5.790E-10
1.0	95.8385	62.9179	−3.435E-1	95.8361	−2.490E-5	95.8385	−6.605E-10
1.5	90.1001	24.4640	−7.285E-1	90.0969	−3.541E-5	90.1001	−6.028E-10
2.0	93.4636	−57.8244	−1.619	93.4540	−1.019E-4	93.4636	−3.313E-9

Table 4 Numerical results $f(\tau)$ for time step length $\Delta\tau$=0.02

Time	Theoretical results	The quadrature method (The trapezium rule)		Present, the first kind		Present, the second kind	
		Numerical results	Relative error	Numerical results	Relative error	Numerical results	Relative error
0.0	120.0000	120.0000	0.000	120.0000	0.000	120.0000	0.000
0.5	119.0246	111.8893	−5.995E-2	119.0238	−6.910E-6	119.0246	−1.877E-10
1.0	118.0967	106.0977	−1.016E-1	118.0959	−7.306E-6	118.0967	−1.939E-10
1.5	117.2142	93.3562	−2.035E-1	117.2130	−9.781E-6	117.2142	−1.637E-10
2.0	116.3746	61.3298	−4.730E-1	116.3711	−2.979E-5	116.3746	−9.708E-10

CONCLUSION

From Tables 1~4, we find that both of the two recursive formulae have very high accuracy for numerical computation. And for the same time step, the numerical results for the second kind recursive formula are more accurate than those for the first kind. For the quadrature method (the trapezium rule), the relative errors are about $-10^{-1}\sim-10^{-2}$ for $\Delta\tau$=0.01, which can be accepted constrainedly in engineering. While the relative errors increase significantly for $\Delta\tau$=0.02. Thus, to obtain satisfactory results, small time step must be employed for the quadrature method. We also notice that for the two presented recursive formulae, the numerical results still keep very high accuracy after 100 times computation for $\Delta\tau$=0.02. That is to say, the computation by using the present recursive formulae is very stable, and a relatively large time step can further be adopted in the calculation. It is noted that the present methods provide a powerful way for solving some practical engineering cases such as the transient responses of magneto-electro-elastic media.

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