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A variational energy approach for electromechanical analysis of thick piezoelectric beam*

LAU C.W.H.[1,2], LIM C.W.[†1], LEUNG A.Y.T.[1]
([1]Department of Building and Construction, City University of Hong Kong, Hong Kong, China)
([2]Department of Construction, Institute of Vocational Education (Tsing Yi), Hong Kong, China)
[†]E-mail: bccwlim@cityu.edu.hk
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Abstract: A new two dimensional coupled electromechanical model for a thick, laminated beam with piezoelectric and isotropic lamina subjected to static external electric loading is developed. The model combined the first order shear deformation theory for the relatively thick elastic core and linear piezoelectric theory for the piezoelectric lamina. The actuation response is induced through the application of external electric voltage. Rayleigh-Ritz method is adopted to model the displacement and potential fields of the beam and governing equations were finally derived from the variational energy principle. The model allows the piezoelectric lamina to be formulated via a two-dimensional model because of the strong electro-mechanical coupling and the presence of a two-dimensional electric field. Numerical examples of piezoelectric laminated beam are presented. It is shown in this paper that a one-dimensional model for the piezoelectric beam-like layer is inadequate.

Key words: Energy, Electromechanical, Linear piezoelectricity, Ritz method, Thick beam
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INTRODUCTION

Piezoelectricity is an electromechanical phenomenon which couples elasticity and electricity through the existence of pressure induced electrical field or electric induced stress field. The piezoelectric materials are usually surface bonded in patches or fully embedded in the host structure, so that the structure becomes a laminated piezoelectric beam.

Tzou and Gadre (1989), Lee (1990), Crawly and Lazarus (1991) developed laminated plate models incorporating the piezoelectric property of materials by using classical laminated theory approximation. Later, Tzou and Zhong (1993) revised the model based on first order shear deformation theory. Tzou (1993)'s paper is very good reference for piezoelectric plate and shell. Lin *et al*.(2000) derived an analytic solution of a laminated piezoelectric beam based on the two-dimensional constitutive relationships. Huang and Sun (2001) developed an approximate analytical solution based on linear piezoelectric and Mindlin lamination theory.

This paper presents an efficient two-dimensional model for analysis of the electromechanical response of thick laminated piezoelectric beams. Timoshenko beam theory including shear deformation is used to model the thick elastic core. Although the piezoelectric lamina is a beam-like layer, it is formulated via a two-dimensional model because of the strong electro-mechanical coupling, and the presence of a two-dimensional electric field. It is shown in this paper that a one-dimensional model for the piezoelectric beam-like layer is inadequate. Variational energy principle is employed in the modelling of bending strain energy and the work of the electric potential. Numerical examples illustrate the effects of various parameters on the various actuation responses.

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FORMULATION OF THICK LAMINATED BEAM

The thick laminated, cantilever beam considered has length L and width b and consists of a thick, elastic core with thickness of h_e and covered at the top with a piezoelectric lager of thickness h_p as shown in Fig.1. The top and bottom surfaces of the laminate are free of shear traction and an electric potential difference V is applied across the layer.

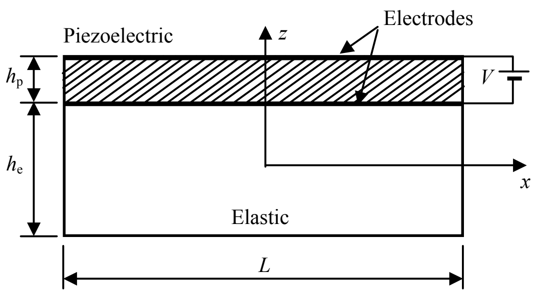

Fig.1 Laminated thick piezoelectric beam

For simplicity, the piezoelectric element in the laminate is assumed to be poled along the positive direction of x_3 and linear piezoelectricity theory is adopted. The stress and strain tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the displacement, electric field and electric displacement vectors, $\boldsymbol{u}$, $\boldsymbol{E}$ and $\boldsymbol{D}$, are related as follows

$$\boldsymbol{\sigma}=\boldsymbol{C\varepsilon}-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{E}, \qquad \boldsymbol{D}=\boldsymbol{e\varepsilon}+\boldsymbol{kE}$$

with $\boldsymbol{C}$, $\boldsymbol{e}$ and $\boldsymbol{k}$ being 6×6, 3×6 and 3×3 matrices representing the elasticity, piezoelectricity and dielectricity tensors.

In accordance with the variational energy principle (Reddy, 1984; Tzou, 1993), an energy functional Π is defined as the difference of a structure's total strain energy, $U=U_e+U_p$, and the work done by an external electric potential, W_p, as follows

$$\begin{aligned}\Pi &= \iiint_{V_e}\left(\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\varepsilon}\right)_e \mathrm{d}V_e \\ &+\iiint_{V_p}\left(\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{E}-\frac{1}{2}\boldsymbol{E}^{\mathrm{T}}k\boldsymbol{E}\right)_p \mathrm{d}V_p \\ &-\iint_{S_p}\left(\frac{1}{2}\bar{Q}\varphi\right)_p \mathrm{d}S_p\end{aligned}$$

where subscripts e and p stand for the elastic core and piezoelectric layer, respectively.

Because the elastic core is a thick structure, a Timoshenko beam model is adopted here. For the piezoelectric layer, the displacement components $u_p(x,z)$ and $w_p(x,z)$ are functions of x and z while the electric potential $\varphi_p(z)$ is only dependent on z due to symmetry of the applied potential and electric boundary conditions.

Introducing dimensionless coordinates $\bar{x}=x/L$, $\bar{z}=z/L$ and dimensionless displacement components $\bar{u}_e=u_e/L$, $\bar{w}_e=w_e/L$, $\bar{u}_p=u_p/L$, $\bar{w}_p=w_p/L$ and dimensionless thicknesses $\bar{h}_e=h_e/L$ and $\bar{h}_p=h_p/L$, the strain energy functional becomes

$$\begin{aligned}\Pi=&\frac{bh_eLE}{2}\int_{-0.5}^{0.5}\left[\frac{h_e^2}{12L^2}\left(\frac{\mathrm{d}\theta_e}{\mathrm{d}\bar{x}}\right)^2+\frac{\kappa^2G}{E}\left(\theta_e+\frac{\mathrm{d}\bar{w}_e}{\mathrm{d}\bar{x}}\right)^2\right]\mathrm{d}\bar{x} \\ &+bL^2\iint_{\bar{A}_p}\left\{\frac{c_{11}}{2}\left(\frac{\partial\bar{u}_p}{\partial\bar{x}}\right)^2+c_{13}\frac{\partial\bar{u}_p}{\partial\bar{x}}\frac{\partial\bar{w}_p}{\partial\bar{z}}+\frac{c_{33}}{2}\left(\frac{\partial\bar{w}_p}{\partial\bar{z}}\right)^2\right. \\ &+\frac{c_{55}}{2}\left[\left(\frac{\partial\bar{u}_p}{\partial\bar{z}}\right)^2+2\left(\frac{\partial\bar{u}_p}{\partial\bar{z}}\frac{\partial\bar{w}_p}{\partial\bar{x}}\right)+\left(\frac{\partial\bar{w}_p}{\partial\bar{x}}\right)^2\right] \\ &+\frac{e_{31}V}{L}\frac{\partial\bar{u}_p}{\partial\bar{x}}\frac{\partial\bar{\varphi}_p}{\partial\bar{z}}+\frac{e_{33}V}{L}\frac{\partial\bar{w}_p}{\partial\bar{z}}\frac{\partial\bar{\varphi}_p}{\partial\bar{z}} \\ &\left.-\frac{k_{33}}{2}\frac{V^2}{L^2}\left(\frac{\partial\bar{\varphi}_p}{\partial\bar{z}}\right)^2\right\}\mathrm{d}\bar{x}\mathrm{d}\bar{z}-\frac{1}{2}QV\end{aligned}$$

BOUDARY AND CONTINUITY CONDITIONS

For a fixed support, all displacement components at the fixed end must be zero,

$$\bar{u}_e=\bar{w}_e=\bar{u}_p=\bar{w}_p=0$$

and the gradients of $\bar{w}_e$ and $\bar{w}_p$ must also vanish at the clamped boundary,

$$\frac{\partial\bar{w}_e}{\partial\bar{x}}=\frac{\partial\bar{w}_p}{\partial\bar{x}}=0$$

For a pinned support, all displacement components at the pinned end must be zero,

$$\bar{u}_e=\bar{w}_e=\bar{u}_p=\bar{w}_p=0$$

For a laminated structure, the displacement components must be continuous across the interface. Hence, on the interface, $\bar{u}_e = \bar{u}_p$ and $\bar{w}_e = \bar{w}_p$.

For the piezoelectric layer sandwiched by two electrodes with negligible thickness and stiffness, an electric potential V is applied at the top electrode while electrode at the bottom is earthed, so the electric boundary and interface conditions are

$$\bar{\varphi}_p = 1 \text{ at } \bar{w}_p = w_p / L \text{ and } \bar{\varphi}_p = 0 \text{ at } \bar{z} = \bar{h}_e / 2.$$

ADMISSIBLE FUNCTIONS AND NONHOMOGENEOUS EQUATION

In the present study, the Ritz method with a variational energy functional is adopted (Reddy, 1984). With reference to the boundary and interface conditions (see Section 3), the admissible displacement functions of the elastic and piezoelectric layers can be represented as

$$\bar{u}_e = \bar{z}\theta_e = BF_u \times \bar{z}[C^1_{\theta_e} + C^2_{\theta_e}\bar{x} + C^3_{\theta_e}\bar{x}^2 + C^4_{\theta_e}\bar{x}^3 + ...]$$
$$\bar{w}_e = BF_w \times \left[C^1_{w_e} + C^2_{w_e}\bar{x} + C^3_{w_e}\bar{x}^2 + C^4_{w_e}\bar{x}^3 + \cdots \right]$$
$$\bar{u}_p = \bar{h}_e\theta_e / 2 + BF_u \times \left(\bar{z} - \bar{h}_e / 2\right)\left[C^1_{u_p} + C^2_{u_p}\bar{x} + C^3_{u_p}\bar{z} + C^4_{u_p}\bar{x}^2 + C^5_{u_p}\overline{xz} + C^6_{u_p}\bar{z}^2 + \cdots \right]$$
$$\bar{w}_p = \bar{w}_e + BF_w \times \left(\bar{z} - \bar{h}_e / 2\right)\left[C^1_{w_p} + C^2_{w_p}\bar{x} + C^3_{w_p}\bar{z} + C^4_{w_p}\bar{x}^2 + C^5_{w_p}\overline{xz} + C^6_{w_{pu}}\bar{z}^2 + \cdots \right]$$

BF is the basic function satisfying the geometric boundary conditions of the beam.

For a cantilever beam fixed at $\bar{x} = -0.5$ and free at $\bar{x} = 0.5$,

$$BF_u = \bar{x} + 0.5, \quad \text{and} \quad BF_w = (\bar{x} + 0.5)^2$$

For a simply supported beam with pinned ends at both supports,

$$BF_u = BF_w = (\bar{x} + 0.5)(\bar{x} - 0.5)$$

For a clamped beam with fixed ends at both supports,

$$BF_u = (\bar{x} + 0.5)(\bar{x} - 0.5)$$

$$BF_w = (\bar{x} + 0.5)^2(\bar{x} - 0.5)^2$$

Similarly, the admissible electric potential function can be approximated as

$$\bar{\varphi}_p = (\bar{z} - \bar{h}_e / 2)\left[1/\bar{h}_p + (\bar{z} - \bar{h}_e / 2 - \bar{h}_p) \times (C^1_{\varphi_p} + C^2_{\varphi_p}\bar{z} + C^3_{\varphi_p}\bar{z}^2 + C^4_{\varphi_p}\bar{z}^3 + \cdots) \right]$$

The principle of extremum energy was used to minimize the energy functional with respect to the coefficients, C^i_α ($\alpha = \theta_e, w_e, u_p, w_p, \varphi_p$), so

$$\frac{\partial \Pi}{\partial C^i_\alpha} = \frac{\partial U_e}{\partial C^i_\alpha} + \frac{\partial U_p}{\partial C^i_\alpha} - \frac{\partial W_p}{\partial C^i_\alpha} = \frac{\partial U_e}{\partial C^i_\alpha} + \frac{\partial U_p}{\partial C^i_\alpha} = 0$$

which yields the following set of nonhomogeneous equations:

$$\begin{bmatrix} \boldsymbol{k}_{\theta_e\theta_e} & \boldsymbol{k}_{\theta_e w_e} & \boldsymbol{k}_{\theta_e u_p} & \boldsymbol{k}_{\theta_e w_p} & \boldsymbol{0} \\ & \boldsymbol{k}_{w_e w_e} & \boldsymbol{k}_{w_e u_p} & \boldsymbol{k}_{w_e w_p} & \boldsymbol{0} \\ & & \boldsymbol{k}_{u_p u_p} & \boldsymbol{k}_{u_p w_p} & \boldsymbol{k}_{u_p \varphi_p} \\ & & & \boldsymbol{k}_{w_p w_p} & \boldsymbol{k}_{w_p \varphi_p} \\ \text{sym} & & & & \boldsymbol{k}_{\varphi_p \varphi_p} \end{bmatrix} \begin{Bmatrix} \boldsymbol{C}_{\theta_e} \\ \boldsymbol{C}_{w_e} \\ \boldsymbol{C}_{u_p} \\ \boldsymbol{C}_{w_p} \\ \boldsymbol{C}_{\varphi_p} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{t}_1 \\ \boldsymbol{0} \\ \boldsymbol{t}_3 \\ \boldsymbol{t}_4 \\ \boldsymbol{0} \end{Bmatrix}$$

Hence we have a system of linearly independent simultaneous equations for the unknown coefficients which can be solved numerically.

NUMERICAL EXEMPLES OF A LAMINATED PIEZOELECTRIC BEAM

The elastic core is steel and the piezoelectric layer is PZT-4, whose material properties are shown in Table 1. A potential difference V of 10 V is applied to the electrodes across the PZT-4 layer. A numerical convergence study was carried out to determine the optimal number of terms required in the displacement functions $\bar{u}_e, \bar{w}_e, \bar{u}_p, \bar{w}_p$ and the potential function $\bar{\varphi}_p$. The deflection converges to a relatively steady value when the numbers of terms for elastic, piezoelectric and potential functions are 5, 18 and 2 respectively.

Having determined numerical convergence, the numerical solutions must be verified to ensure accu-

Table 1 Material properties

	Elastic core (steel)	Piezoelectric layer (PZT-4)
Length (m)	0.3	0.3
Thickness (m)	0.02	0.005
Poisson's ratio	0.3	–
Elastic constants (GPa)	E=210	c_{11}=139 c_{12}=77.8 c_{13}=74.3 c_{33}=113 c_{44}=25.6
Piezoelectric constants (C/m^2)		e_{31}=−6.98 e_{33}=13.84 e_{15}=13.44
Dielectric constants (CV/m)		k_{11}=6×10^{-9} k_{33}=5.47×10^{-9}

racy of the approach. Here, we refer to published data of an analytical solution from Lin *et al.*(2000) and to finite element solutions using ABAQUS (2002). With reference to Lin *et al.*(2000), the beam has L=0.3 m, h_e=0.02 m, h_p=0.005 m and V=10 V. The results of comparison are presented in Figs.2 to 3. It is obvious that the results presented here agree better with FEM results than with Lin *et al.*(2000)'s results.

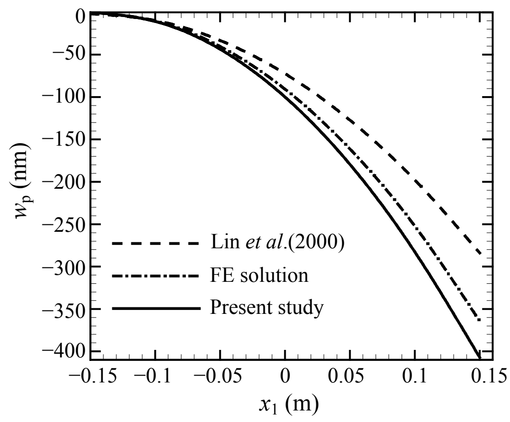

Fig.2 Displacement w_p of the piezoelectric layer at beam free-end z=h_e/2+h_p

To illustrate the application of the proposed approach, two different cases of a cantilevered beam have been investigated and the results are shown in Figs.3 and 4, respectively, on the effect of thickness ratio $\bar{h}_p/\bar{h}_e$ and length ratio L'/L on deflection $\bar{w}_p$ at $\bar{x}=0.5$ and $\bar{z}=\bar{h}_e/2+\bar{h}_p$ for varying electric potential where L' is the new length of beam under investigation. As can be observed, increasing the thickness ratio $\bar{h}_p/\bar{h}_e$ tends to increase $\bar{w}_p$ initially but the latter decreases beyond a certain maximum value which depends on the potential difference. Similar to the effect of $\bar{h}_p/\bar{h}_e$, increasing the length ratio L'/L also increases $\bar{w}_p$ initially, the latter then reaches a maximum value and decreases afterwards. Again, the maximum value depends on the potential difference. For both cases, we also observe that increasing the potential difference increases $\bar{w}_p$. This observation can be explained as follows. We may treat potential difference as similar to mechanical load to a certain extent. Therefore, increasing the potential difference

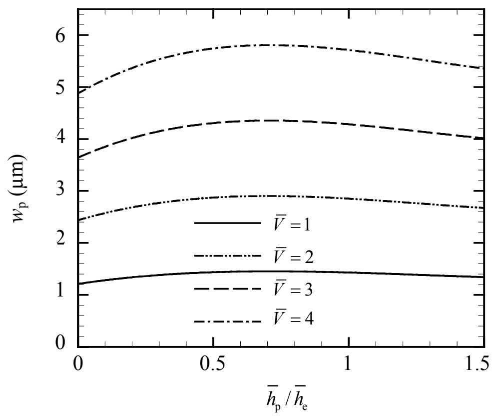

Fig.3 Effect of thickness ratio, $\bar{h}_p/\bar{h}_e$ on deflection, $\bar{w}_p$ at $\bar{x}=0.5$ and $\bar{z}=\bar{h}_e/2+\bar{h}_p$

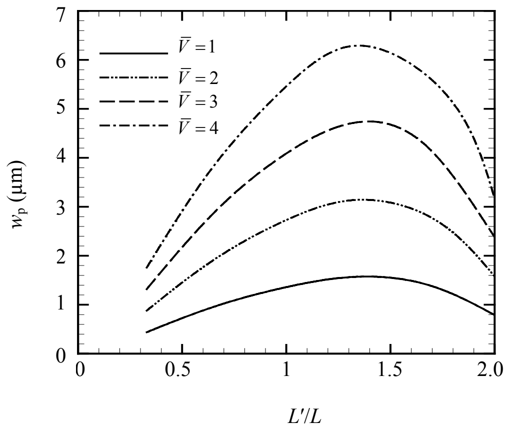

Fig.4 Effect of length ratio L'/L on deflection, $\bar{w}_p$ at $\bar{x}=0.5$ and $\bar{z}=\bar{h}_e/2+\bar{h}_p$

should have the similar effect of increasing mechanical load which results in larger deflection of a beam.

CONCLUSION

An efficient two-dimensional laminate model was developed to investigate the actuation response of a thick laminated piezoelectric beam. The model is based on the coupling of a one-dimensional thick layer and a two-dimensional linear piezoelectric layer. The problem is solved via the variational energy principle. Although the model only ensures continuity of displacement across the layer interface, the results from the model are in good agreement with the finite element solutions.

A thick laminated beam clamped at one end and subjected to an applied potential difference has been investigated. The results led to the conclusion that the electric potential developed across the piezoelectric layer is linear through the thickness. In addition, the deflection response of the beam is proportional to the applied voltage. The approach and information presented here have practical in engineering control using a beam.

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