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A class of quasi Bézier curves based on hyperbolic polynomials*

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Abstract: This paper presents a basis for the space of hyperbolic polynomials Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt} on the interval [0,α] from an extended Tchebyshev system, which is analogous to the Bernstein basis for the space of polynomial used as a kind of well-known tool for free-form curves and surfaces in Computer Aided Geometry Design. Then from this basis, we construct quasi Bézier curves and discuss some of their properties. At last, we give an example and extend the range of the parameter variable t to arbitrary close interval [r, s] (r<s).

Key words: Bernstein basis, Bézier curve, Hyperbolic polynomials, Extended Tchebyshev system, B-base
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INTRODUCTION

As we all know, the Bézier model is widely used in constructing free-form curves and surfaces because of its good properties. But since the Bernstein basis is a basis for the space of degree-n algebraic polynomials T_n=span{1, t, t^2, …, t^n}, it has many shortcomings, especially in representing transcendental curves, such as the helix and the hyperbola. Many articles report finding of new Bernstein-like bases in new space.

Zhang (1996) investigated curves in the space span{1, t, cos$t$, sin$t$}. Peña (1997) gave a basis for span{1, cos$t$, cos2t, …, cosmt}. Sânchez-Reyes (1998) found a basis for the space of trigonometric polynomials span{1, sin$t$, cos$t$, sin2t, cos2t, …, sinnt, cosnt}. Chen and Wang (2003) constructed a basis in the space span{1, t, t^2, …, t^n, sin$t$, cos$t$}. These bases are similar to the Bernstein basis in that they have many of the properties of Bernstein basis and are all in the space of trigonometric polynomials and the space of hyperbolic polynomials must have analogous basis. Lü *et al*.(2002) introduced hyperbolic polynomial B-splines, generated over the space span{1, t, t^2, …, t^{n-2}, sh$t$, ch$t$} which is mixed with the polynomials {1, t, t^2, …, t^{n-2}} and the hyperbola {sh$t$, ch$t$}, and we will show the hyperbolic polynomials space.

In this paper, we give a new basis on the interval [0, α], for the space of hyperbolic polynomials Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt} instead of the space of trigonometric polynomials span{1, sin$t$, cos$t$, sin2t, cos2t, …, sinnt, cosnt}.

DEFINITION OF THE BASIS

As we all know, the Bernstein basis is the term of binomial expansion of the identity 1=(1–t)+t, and we can use the analogous method to construct the quasi Bernstein basis in the space of hyperbolic polynomials. Supposing α>0 and $t\in$[0, α], we firstly take cognizance of an identical equation:

$$1\equiv\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)} \quad (1)$$

Then, by raising Eq.(1) to the mth power, we get

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$$1\equiv 1^m\equiv\left[\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\right]^m$$

$$=\sum_{\substack{s+l+k=m\\ s,l,k\geq 0}}\left\{\binom{m}{i,j,k}\left(\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}\right)^s\times\left(\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}\right)^l\left(\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\right)^k\right\}$$

$$=\frac{1}{\mathrm{sh}^{2m}(\alpha/2)}\sum_{i=0}^{2m}\left\{\sum_{\substack{0\leq l\leq i\\ (i+l)\text{ is even}}}\left\{\binom{m}{m-(i+l)/2,l,(i-l)/2}\times\left(2\mathrm{ch}(\alpha/2)\right)^l\right\}\ \mathrm{sh}^{2m-i}[(\alpha-t)/2]\mathrm{sh}^i(t/2)\right\}$$

$$=\sum_{i=0}^{2m}\left\{\frac{1}{\mathrm{sh}^{2m}(\alpha/2)}\sum_{j=0}^{\lfloor i/2\rfloor}\left\{\binom{m}{m-i+j,i-2j,j}\times\left(2\mathrm{ch}(\alpha/2)\right)^{i-2j}\right\}\mathrm{sh}^{2m-i}[(\alpha-t)/2]\mathrm{sh}^i(\ /2)\right\}$$

where $\binom{n}{i,j,k}=\frac{n!}{i!j!k!}$. Here $i, j, k\geq 0$ and $i+j+k=n$. If one of i, j, k is negative, set $\binom{n}{i,j,k}=0$ and it always holds in the whole paper. Thus, the definition of the new basis for the space Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt} is as follows.

Definition 1 Let $\alpha>0$, $t\in[0, \alpha]$ and m be a non-negative integer. Note that

$$U_{i,2m}(t)=a_{i,2m}\left(\mathrm{sh}[(\alpha-t)/2]\right)^{2m-i}\left(\mathrm{sh}(t/2)\right)^i,\quad i=0,1,2,\cdots,2m, \tag{2}$$

where the coefficient

$$a_{i,2m}=\frac{1}{\mathrm{sh}^{2m}(\alpha/2)}\sum_{j=0}^{\lfloor i/2\rfloor}\binom{m}{m-i+j,i-2j,j}\left(2\mathrm{ch}(\alpha/2)\right)^{i-2j} \tag{3}$$

It is an extended Tchebyshev system (Schumaker, 1981) and we only need to prove that $\left\{U_{i,2m}(t)\right\}_{i=0}^{2m}$ are all in the space Γ_m. Firstly, a lemma is given.

Lemma 1 $\prod_{i=0}^{m}(\lambda_i+\eta_i\mathrm{sh}t+\delta_i\mathrm{ch}t)\in$span{1, sh$t$, ch$t$, sh2$t$, ch2$t$, …, sh$mt$, ch$mt$} for any real number λ_i, η_i, δ_i, m=0,1,2,…

Theorem 1 For any $i\in[0, 2m]$, $U_{i,2m}(t)$ is in the space Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt}.

Proof We only need to show that $\mathrm{sh}^{2m-i}[(\alpha-t)/2]\times\mathrm{sh}^i(t/2)$ is in the space Γ_m for any $i\in[0, 2m]$. The problem is discussed respectively for the parity of i.

(1) When i is even,

$$\mathrm{sh}^{2m-i}\frac{\alpha-t}{2}\mathrm{sh}^i\frac{t}{2}=\left(\frac{\mathrm{ch}\alpha\mathrm{ch}t-\mathrm{sh}\alpha\mathrm{sh}t}{2}\right)^{m-i/2}\left(\frac{\mathrm{ch}t}{2}\right)^{i/2}.$$

(2) When i is odd,

$$\begin{aligned}&\mathrm{sh}^{2m-i}((\alpha-t)/2)\mathrm{sh}^i(t/2)\\&=\left[\frac{\mathrm{ch}(\alpha-t)}{2}\right]^{m-(i+1)/2}\left(\frac{\mathrm{ch}t}{2}\right)^{(i-1)/2}\left(\mathrm{sh}\frac{\alpha-t}{2}\mathrm{sh}\frac{t}{2}\right)\\&=\left(\frac{\mathrm{ch}\alpha\mathrm{ch}t-\mathrm{sh}\alpha\mathrm{sh}t}{2}\right)^{m-(i+1)/2}\left(\frac{\mathrm{ch}t}{2}\right)^{(i-1)/2}\\&\quad\times\left[\frac{\mathrm{ch}(\alpha/2)-\mathrm{ch}(\alpha/2)\mathrm{ch}t+\mathrm{sh}(\alpha/2)\mathrm{sh}t}{2}\right]\end{aligned}$$

Using Lemma 1, we conclude that $U_{i,2m}(t)$ can always be linear represented by {1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt}, and the theorem is proved.

Fig.1 gives the images of $\left\{U_{i,2m}(t)\right\}_{i=0}^{2m}$ when m=2 and m=3 respectively.

PROPERTIES OF THE BASIS

The basis for the space Γ_m defined in Section 2 has many Bernstein-like properties.

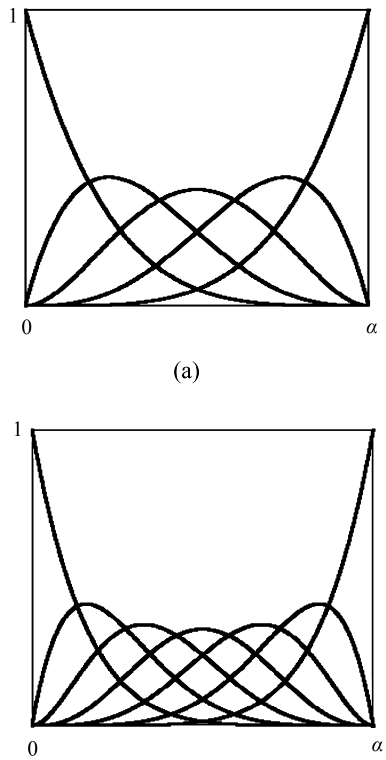
Some of characters of B-base

Proposition 1 $\left\{U_{i,2m}(t)\right\}_{i=0}^{2m}$ has following properties of B-base (Carnicer and Peña, 1994):

(1) linear independence;

(2) $U_{0,2m}(0)=U_{2m,2m}(\alpha)=1$,

$U_{i,2m}^{(j)}(0)=U_{i,2m}^{(k)}(\alpha)=0$,



(a)

(b)

Fig.1 The images of $\{U_{i,4}(t)\}_{i=0}^{4}$ (a) and $\{U_{i,6}(t)\}_{i=0}^{6}$ (b)

j=0, 1, …, i–1, k=0, 1, …, $2m$–i–1;

(3) $U_{i,2m}(t)\geq 0$, $0\leq t\leq \alpha$, $\sum_{i=0}^{2m} U_{i,2m}(t)=1$, $\forall t\in[0, \alpha]$;

(4) $U_{i,2m}(t)=U_{2m-i,2m}(\alpha-t)$ for $\forall t\in[0, \alpha]$, i=0, 1, 2, …, $2m$.

Proof The anterior three characters are obvious from Definition 1 and we only need to prove the last one. From the symmetry of $\left\{\mathrm{sh}^{2m-i}\frac{\alpha-t}{2}\mathrm{sh}^{i}\frac{t}{2}\right\}_{i=0}^{2m}$ and the normalization of $\{U_{i,2m}(t)\}_{i=0}^{2m}$, we have

$$\sum_{i=0}^{2m} a_{i,2m}\mathrm{sh}^{2m-i}[(\alpha-t)/2]\mathrm{sh}^{i}(t/2)=\sum_{i=0}^{2m} U_{i,2m}(t)=1$$

$$=\sum_{i=0}^{2m} a_{i,2m}\mathrm{sh}^{2m-i}\frac{t}{2}\mathrm{sh}^{i}\frac{\alpha-t}{2}=\sum_{i=0}^{2m} a_{2m-i,2m}\mathrm{sh}^{2m-i}\frac{\alpha-t}{2}\mathrm{sh}^{i}\frac{t}{2}.$$

Because $\{\mathrm{sh}^{2m-i}[(\alpha-t)/2]\mathrm{sh}^{i}(t/2)\}_{i=0}^{2m}$ is a basis, $a_{i,2m}=a_{2m-i,2m}$, and the proposition is proved.

Recurrence formula

Proposition 2

$$U_{i,2m}(t)=\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}U_{i-2,2(m-1)}(t)+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}U_{i-1,2(m-1)}(t)+\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}U_{i,2(m-1)}(t)$$

and
$$a_{i,2m}=\frac{a_{i-2,2(m-1)}+2\mathrm{ch}(\alpha/2)a_{i-1,2(m-1)}+a_{i,2(m-1)}}{\mathrm{sh}^2(\alpha/2)} \quad (4)$$

It is easy to prove from the definition of the basis by induction on m.

Degree-elevation

Using Definition 1, we can easily get the degree-elevation formula.

Proposition 3

$$U_{i,2m}(t)=U_{i,2m}(t)\times\left(\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\right)$$
$$=\frac{a_{i,2m}}{\mathrm{sh}^2(\alpha/2)a_{i,2(m+1)}}U_{i,2(m+1)}+\frac{2\mathrm{ch}(\alpha/2)a_{i,2m}}{\mathrm{sh}^2(\alpha/2)a_{i+1,2(m+1)}}U_{i+1,2(m+1)}+\frac{a_{i,2m}}{\mathrm{sh}^2(\alpha/2)a_{i+2,2(m+1)}}U_{i+2,2(m+1)}.$$

Differentiation

The basis $\{U_{i,2m}(t)\}_{i=0}^{2m}$ is for the space Γ_m=span{1, sh$t$, ch$t$, sh$2t$, ch$2t$, …, shmt, chmt}, and each $U_{i,2m}(t)$ can be represented as follows:

$$U_{i,2m}(t)=c_{i0}+c_{i1}\mathrm{sh}t+c_{i2}\mathrm{ch}t+\cdots+c_{2m-1}\mathrm{sh}mt+c_{2m}\mathrm{ch}mt$$

Thus, as $k\geq 1$, $U_{i,2m}^{(k)}(t)\in$span{sh$t$, ch$t$, sh$2t$, ch$2t$, …, shmt, chmt} and $\boldsymbol{p}^{(k)}(t)$ cannot be expressed by a lower-degree curve with the basis $\{U_{i,2n}(t)\}_{i=0}^{2n}$ ($n<m$), but can be represented by the basis $\{U_{i,2m}(t)\}_{i=0}^{2m}$. We only show the degree-one derivative, the higher de-

gree derivative can be analogously produced recursively.

Proposition 4 For arbitrary integer $i\in[0, 2m]$, we have the differential formula:

$$U'_{i,2m}(t)=\frac{ia_{i,2m}}{2\mathrm{sh}(\alpha/2)a_{i-1,2m}}U_{i-1,2m}(t)+\frac{(i-m)\mathrm{ch}(\alpha/2)}{\mathrm{sh}(\alpha/2)}U_{i,2m}(t)+\frac{(i-2m)a_{i,2m}}{2\mathrm{sh}(\alpha/2)a_{i+1,2m}}U_{i+1,2m}(t) \quad (5)$$

Proof The process of deriving Eq.(5) is as follows:

$$U'_{i,2m}(t)=a_{i,2m}\left[-\frac{2m-i}{2}\left(\mathrm{sh}\frac{\alpha-t}{2}\right)^{2m-1-i}\mathrm{ch}\frac{\alpha-t}{2}\left(\mathrm{sh}\frac{t}{2}\right)^{i}+\frac{i}{2}\left(\mathrm{sh}\frac{\alpha-t}{2}\right)^{2m-i}\left(\mathrm{sh}\frac{t}{2}\right)^{i-1}\mathrm{ch}\frac{t}{2}\right]$$

$$=a_{i,2m}\left(\mathrm{sh}\frac{\alpha-t}{2}\right)^{2m-1-i}\left(\mathrm{sh}\frac{t}{2}\right)^{i-1}\times\left[-m\,\mathrm{ch}\frac{\alpha-t}{2}\left(\mathrm{sh}\frac{t}{2}\right)+\frac{i}{2}\mathrm{sh}\frac{\alpha}{2}\right]$$

$$=a_{i,2m}\left(\mathrm{sh}\frac{\alpha-t}{2}\right)^{2m-1-i}\left(\mathrm{sh}\frac{t}{2}\right)^{i-1}\left[\frac{m}{2}\mathrm{sh}\left(\frac{\alpha}{2}-t\right)+\frac{i-m}{2}\mathrm{sh}\frac{\alpha}{2}\right]$$

$$=\frac{a_{i,2m}}{2\mathrm{sh}\frac{\alpha}{2}}\left(\mathrm{sh}\frac{\alpha-t}{2}\right)^{2m-1-i}\left(\mathrm{sh}\frac{t}{2}\right)^{i-1}\times\left[i\,\mathrm{sh}^2\frac{\alpha-t}{2}+2\mathrm{ch}\frac{\alpha}{2}(i-m)\mathrm{sh}\frac{\alpha-t}{2}\mathrm{sh}\frac{t}{2}+(i-2m)\mathrm{sh}^2\frac{t}{2}\right]$$

$$=\frac{ia_{i,2m}}{2\mathrm{sh}\frac{\alpha}{2}a_{i-1,2m}}U_{i-1,2m}(t)+\frac{(i-m)\mathrm{ch}\frac{\alpha}{2}}{\mathrm{sh}\frac{\alpha}{2}}U_{i,2m}(t)+\frac{(i-2m)a_{i,2m}}{2\mathrm{sh}\frac{\alpha}{2}a_{i+1,2m}}U_{i+1,2m}(t)\quad(1\le i\le 2m-1)$$

When i=0 or i=2m, the equation above also holds.

Limit of the basis

Proposition 5 As $\alpha\to0$, the limit of the basis $\{U_{i,2m}(t)\}_{i=0}^{2m}$ in the space Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt} approaches the Bernstein basis in the space T_{2m}=span{1, t, t^2, …, t^{2m}}.

Proof By mathematical induction.

(1) When m=0, the proposition is true obviously.

(2) Suppose that it holds in the space Γ_{m-1}. After reparameterizing by $\tau=t/\alpha\in[0, 1]$ and setting the Bernstein basis $B_{i,n}(\tau)=\binom{n}{i}(1-t)^{n-i}t^i$, by the inductive hypothesis,

$$\lim_{\alpha\to0}U_{i,2(m-1)}(t)=\lim_{\alpha\to0}U_{i,2(m-1)}(\alpha\tau)=B_{i,2(m-1)}(\tau),$$

$$i=0, 1, \dots, 2(m-1)$$

(3) From Proposition 2, we have

$$\lim_{\alpha\to0}U_{i,2m}(\alpha\tau)=\lim_{\alpha\to0}U_{i,2m}(t)$$

$$=\lim_{\alpha\to0}\left[\frac{\mathrm{sh}^2(\alpha\tau/2)}{\mathrm{sh}^2(\alpha/2)}U_{i-2,2(m-1)}(\alpha\tau)+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-\alpha\tau)/2]\mathrm{sh}(\alpha\tau/2)}{\mathrm{sh}^2(\alpha/2)}U_{i-1,2(m-1)}(\alpha\tau)+\frac{\mathrm{sh}^2[(\alpha-\alpha\tau)/2]}{\mathrm{sh}^2(\alpha/2)}U_{i,2(m-1)}(\alpha\tau)\right]$$

$$=\lim_{\alpha\to0}\left[\frac{(\alpha\tau/2)^2}{(\alpha/2)^2}U_{i-2,2(m-1)}(\alpha\tau)+\frac{2\mathrm{ch}(\alpha/2)[(\alpha-\alpha\tau)/2](\alpha\tau/2)}{(\alpha/2)^2}U_{i-1,2(m-1)}(\alpha\tau)+\frac{[(\alpha-\alpha\tau)/2]^2}{(\alpha/2)^2}U_{i,2(m-1)}(\alpha\tau)\right]$$

$$=\tau^2B_{i-2,2(m-1)}(\tau)+2\tau(1-\tau)B_{i-1,2(m-1)}(\tau)+(1-\tau)^2B_{i,2(m-1)}(\tau)$$

$$=\tau B_{i-1,2m-1}(\tau)+(1-\tau)B_{i,2m-1}(\tau)=B_{i,2m}(\tau).$$

By induction on m, the proposition is right.

From the above characters, $\{U_{i,2m}(t)\}_{i=0}^{2m}$ is a quasi Bernstein basis and is also a blending system.

GEOMETRIC PROPERTIES OF THE QUASI BEZIER CURVE

The corresponding curve $\boldsymbol{p}(t)$ with control points $\{\boldsymbol{p}_i\}_{i=0}^{2m}$ is defined by

$$\boldsymbol{p}(t)=\sum_{i=0}^{2m}U_{i,2m}(t)\boldsymbol{p}_i,\quad t\in[0,\alpha]. \quad (6)$$

Marking some linear operators: $\boldsymbol{E}\boldsymbol{p}_i=\boldsymbol{p}_{i+1}$, $\boldsymbol{I}\boldsymbol{p}_i=\boldsymbol{p}_i$,

$\Delta \boldsymbol{p}_i=(\boldsymbol{E}-\boldsymbol{I})\boldsymbol{p}_i=\boldsymbol{p}_{i+1}-\boldsymbol{p}_i$, we also have another representation of the new curve Eq.(6) by simple calculations.

$$\boldsymbol{p}(t)=\left[\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{I}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}^2\right]^m\boldsymbol{p}_0$$
$$=\left[(\boldsymbol{I}+f(t)\Delta)(\boldsymbol{I}+g(t)\Delta)\right]^m\boldsymbol{p}_0, \quad (7)$$

where $f(t)=\dfrac{\mathrm{sh}(t/2)}{\mathrm{sh}(\alpha/2)\left(\mathrm{ch}[(\alpha-t)/2]-\mathrm{sh}[(\alpha-t)/2]\right)}$, $g(t)=\dfrac{\mathrm{sh}(t/2)\left(\mathrm{ch}[(\alpha-t)/2]-\mathrm{sh}[(\alpha-t)/2]\right)}{\mathrm{sh}(\alpha/2)}$ satisfied that $0\le f(t), g(t)\le 1$. Because the quasi Bernstein basis $\{U_{i,2m}(t)\}_{i=0}^{2m}$, $\boldsymbol{p}(t)$ is a quasi Bézier curve, it has many Bézier-like characters.

Geometric properties at the endpoints

The geometric properties at the endpoints of the quasi Bézier curves are obvious from those of the basis $\{U_{i,2m}(t)\}_{i=0}^{2m}$.

(1) $\boldsymbol{p}(0)=\boldsymbol{p}_0$, $\boldsymbol{p}(\alpha)=\boldsymbol{p}_{2m}$;

(2) $\boldsymbol{p}^{(k)}(0)=\sum_{i=0}^{k}U_{i,2m}^{(k)}(0)\boldsymbol{p}_i$, $\boldsymbol{p}^{(k)}(\alpha)=\sum_{i=2m-k}^{2m}U_{i,2m}^{(k)}(\alpha)\boldsymbol{p}_i$.

Especially, by the differentiation formula Eq.(5), we also have

(3) $\boldsymbol{p}'(0)=\dfrac{m\mathrm{ch}(\alpha/2)}{\mathrm{sh}(\alpha/2)}(\boldsymbol{p}_1-\boldsymbol{p}_0)$,

$\boldsymbol{p}'(\alpha)=\dfrac{m\mathrm{ch}(\alpha/2)}{\mathrm{sh}(\alpha/2)}(\boldsymbol{p}_{2m-1}-\boldsymbol{p}_{2m})$

Convex hull property

The quasi Bézier curve Eq.(6) must lie inside its control polygon spanned by $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, …, $\boldsymbol{p}_{2m}$. This property is a consequence of the third subproposition of Proposition 1 and we show it in Fig.2.

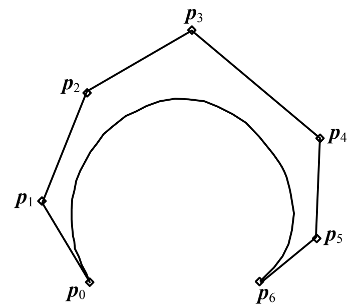


Fig.2 Convex hull property

Degree-elevation

The problem of elevating $2k$ degrees is stated as follows: Given $2m+1$ points $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, …, $\boldsymbol{p}_{2m}$, and a natural number k, we should find $2(m+k)+1$ points $\boldsymbol{p}_0^{[k]}, \boldsymbol{p}_1^{[k]}, \cdots, \boldsymbol{p}_{2(m+k)}^{[k]}$, which satisfy

$$\boldsymbol{p}(t)=\sum_{i=0}^{2m}U_{i,2m}(t)\boldsymbol{p}_i=\sum_{i=0}^{2(m+k)}U_{i,2(m+k)}(t)\boldsymbol{p}_i^{[k]}.$$

Using the normalization of the quasi Bernstein basis, we have

$$\boldsymbol{p}(t)=\sum_{i=0}^{2m}U_{i,2m}(t)\boldsymbol{p}_i=\left(\sum_{i=0}^{2m}U_{i,2m}(t)\boldsymbol{p}_i\right)\left(\sum_{i=0}^{2k}U_{i,2k}(t)\right)$$
$$=\left(\sum_{i=0}^{2m}a_{i,2m}\left(\mathrm{sh}[(\alpha-t)/2]\right)^{2m-i}\left(\mathrm{sh}(t/2)\right)^i\boldsymbol{p}_i\right)\times\left(\sum_{i=0}^{2k}a_{i,2k}\left(\mathrm{sh}[(\alpha-t)/2]\right)^{2k-i}\left(\mathrm{sh}(t/2)\right)^i\right)$$
$$=\left(\sum_{i=0}^{2(m+k)}a_{i,2(m+k)}\left(\mathrm{sh}[(\alpha-t)/2]\right)^{2(m+k)-i}\left(\mathrm{sh}(t/2)\right)^i\boldsymbol{p}_i^{[k]}\right)$$

And the formulas of $\boldsymbol{p}_0^{[k]}, \boldsymbol{p}_1^{[k]}, \cdots, \boldsymbol{p}_{2(m+k)}^{[k]}$ are obtained by comparing the coefficients of $\left\{\left(\mathrm{sh}[(\alpha-t)/2]\right)^{2(m+k)-i}\left(\mathrm{sh}(t/2)\right)^i\right\}_{i=0}^{2(m+k)}$.

$$\boldsymbol{p}_i^{[k]}=\frac{\sum_{j=0}^{i}a_{j,2m}a_{i-j,2k}\boldsymbol{p}_j}{a_{i,2(m+k)}}=\frac{\sum_{j=\max\{0,i-2k\}}^{i}a_{j,2m}a_{i-j,2k}\boldsymbol{p}_j}{a_{i,2(m+k)}}.$$

Especially, when $k=1$,

$$\boldsymbol{p}_i^{[1]}=\frac{\sum_{j=i-2}^{i}a_{j,2m}a_{i-j,2}\boldsymbol{p}_j}{a_{i,2(m+1)}}$$

$$=\frac{a_{i-2,2m}\boldsymbol{p}_{i-2}+2\mathrm{ch}(\alpha/2)a_{i-1,2m}\boldsymbol{p}_{i-1}+a_{i,2m}\boldsymbol{p}_i}{a_{i,2(m+1)}\mathrm{sh}^2(\alpha/2)} \quad (8)$$

From Eq.(4), we have

$$a_{i,2(m+1)}=\frac{a_{i-2,2m}+2\mathrm{ch}(\alpha/2)a_{i-1,2m}+a_{i,2m}}{\mathrm{sh}^2(\alpha/2)},$$

and

$$\left(\frac{a_{i-2,2m}}{a_{i,2(m+1)}\mathrm{sh}^2(\alpha/2)},\frac{2\mathrm{ch}(\alpha/2)a_{i-1,2m}}{a_{i,2(m+1)}\mathrm{sh}^2(\alpha/2)},\frac{a_{i,2m}}{a_{i,2(m+1)}\mathrm{sh}^2(\alpha/2)}\right)$$

is the barycentric coordinate of $\boldsymbol{p}_i^{[1]}$ in $\Delta\boldsymbol{p}_{i-2}\boldsymbol{p}_{i-1}\boldsymbol{p}_i$, when $2\le i\le 2m-2$. Fig.3 shows that the curve degree-elevates from degree-5 (m=2) to degree-7 (m=3). The primary control polygon is represented by solid while the elevated control polygon is represented by dashed line.

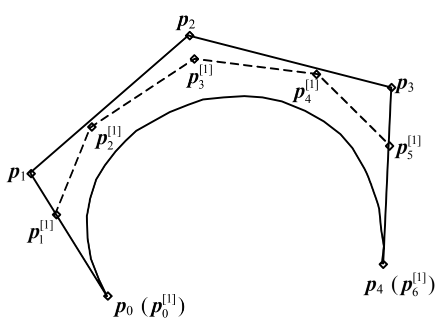


Fig.3 Degree-elevation

For geometric significance, we represent the degree-elevation formula with the form of cutting angles. Eq.(8) can be denoted:

$$\boldsymbol{p}_i^{[1]}=(1-s_i)\boldsymbol{q}_{i-1}+s_i\boldsymbol{q}_i$$
$$=(1-s_i)\left[(1-k_{i-1})\boldsymbol{p}_{i-2}+k_{i-1}\boldsymbol{p}_{i-1}\right]+s_i\left[(1-k_i)\boldsymbol{p}_{i-1}+k_i\boldsymbol{p}_i\right]$$

where

$$k_i=\frac{\mathrm{ch}\dfrac{(m+1-i)\alpha}{2}a_{i,2m}}{\mathrm{ch}\dfrac{(m-i)\alpha}{2}a_{i-1,2m}+\mathrm{ch}\dfrac{(m+1-i)\alpha}{2}a_{i,2m}}\in[0,1]$$
$$i=0,1,\cdots,2m+1;$$

$$s_i=\frac{\left[\mathrm{ch}\dfrac{(m-i)\alpha}{2}\Big/\mathrm{ch}\dfrac{(m+1-i)\alpha}{2}\right]a_{i-1,2m}+a_{i,2m}}{a_{i,2(m+1)}\mathrm{sh}^2(\alpha/2)}\in[0,1]$$
$$i=0,1,\cdots,2m+2.$$

Fig.4 shows the process of cutting angles.

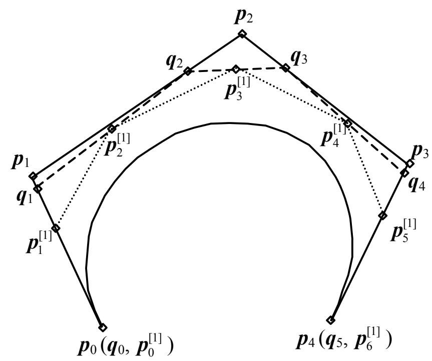


Fig.4 Degree-elevation of cutting angles

Recursive evaluation

First, a recursive definition Eq.(9) is given.

$$\boldsymbol{p}_i^{2k}(t)\triangleq\begin{cases}\dfrac{\mathrm{sh}^2\dfrac{\alpha-t}{2}}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{p}_{i-2}^{2(k-1)}(t)+\dfrac{2\mathrm{ch}(\alpha/2)\mathrm{sh}\dfrac{\alpha-t}{2}\mathrm{sh}\dfrac{t}{2}}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{p}_{i-1}^{2(k-1)}(t) \\ \quad +\dfrac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{p}_i^{2(k-1)}(t) \quad k=1,2,...,m,\ \ i=2k,...,2m, \\ \boldsymbol{p}_i \qquad\qquad k=0,\ \ i=0,1,...,2m.\end{cases} \quad (9)$$

Then, from the definition Eq.(9), we have

$$\boldsymbol{p}_i^{2k}(t)=\left[\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{I}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}^2\right]\boldsymbol{p}_{i-2}^{2(k-1)}$$
$$=\cdots=\left[\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{I}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2)]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}^2\right]^k\boldsymbol{p}_{i-2k}^0$$

Especially from Eq.(7), the following formula is hold.

$$\boldsymbol{p}_{2m}^{2m}(t)=\left[\frac{\mathrm{sh}^2[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{I}+\frac{2\mathrm{ch}(\alpha/2)\mathrm{sh}[(\alpha-t)/2]\mathrm{sh}(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}+\frac{\mathrm{sh}^2(t/2)}{\mathrm{sh}^2(\alpha/2)}\boldsymbol{E}^2\right]^m\boldsymbol{p}_0^0=\boldsymbol{p}(t)$$

Fig.5 gives the graphic representation of the recursive evaluation. By setting m=2, t=0.4α, and giving the five points $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, $\boldsymbol{p}_4$, we first get $\boldsymbol{p}_2^2(t), \boldsymbol{p}_3^2(t), \boldsymbol{p}_4^2(t)$ in $\Delta\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$, $\Delta\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3$, $\Delta\boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_4$ respectively. Then, we get $\boldsymbol{p}_4^4(t)=\boldsymbol{p}(t)$ with the barycentric coordinate in $\Delta\boldsymbol{p}_2^2(t)\boldsymbol{p}_3^2(t)\boldsymbol{p}_4^2(t)$.

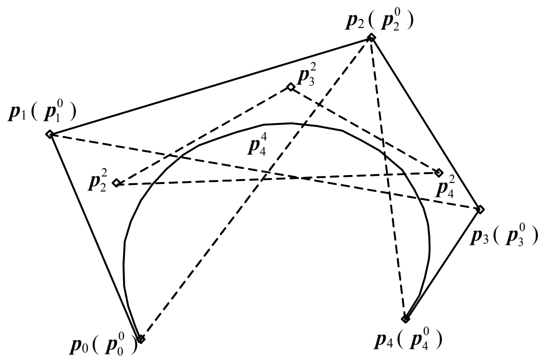


Fig.5 Recursive evaluation

For geometric significance, we want to produce the recursive evaluation points with the form of cutting angles, so the points $\boldsymbol{p}_i^{2k-1}(t)$ are added to Eq.(7). Eq.(9) is changed to

$$\boldsymbol{p}_i^j(t)=\begin{cases}\boldsymbol{p}_i, & j=0,\ i=j,j+1,...,2m\\ \left(1-\frac{\mathrm{sh}(t/2)\left(\mathrm{ch}[(\alpha-t)/2]-\mathrm{sh}[(\alpha-t)/2]\right)}{\mathrm{sh}(\alpha/2)}\right)\boldsymbol{p}_{i-1}^{j-1}(t)+\frac{\mathrm{sh}(t/2)\left(\mathrm{ch}[(\alpha-t)/2]-\mathrm{sh}[(\alpha-t)/2]\right)}{\mathrm{sh}(\alpha/2)}\boldsymbol{p}_i^{j-1}(t), & j=1,3,\cdots,2m-1,\ i=j,j+1,...,2m\\ \left(1-\frac{\mathrm{sh}(t/2)}{\mathrm{sh}(\alpha/2)\left(\mathrm{ch}[(\alpha-t)/2]-\mathrm{sh}[(\alpha-t)/2]\right)}\right)\boldsymbol{p}_{i-1}^{j-1}(t)+\frac{\mathrm{sh}(t/2)}{\mathrm{sh}(\alpha/2)\left(\mathrm{ch}[(\alpha-t)/2]-\mathrm{sh}[(\alpha-t)/2]\right)}\boldsymbol{p}_i^{j-1}(t), & j=2,4,\cdots,2m,\ i=j,j+1,...,2m\end{cases}$$

Fig.6 shows the process of cutting angles.

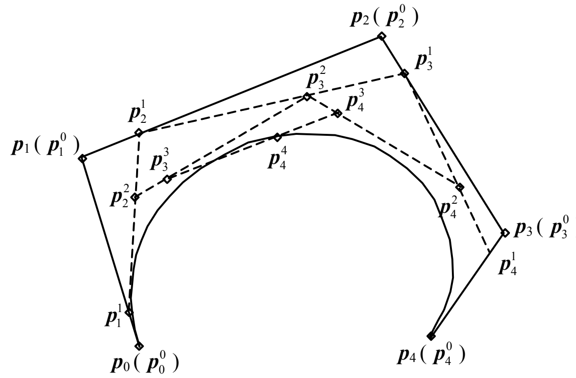


Fig.6 Recursive evaluation with the form of cutting angles

Differentiation

Let $\boldsymbol{p}^{(k)}(t)$ be the degree-k derivative of $\boldsymbol{p}(t)$ in Eq.(6). From Eq.(5), we get

$$\boldsymbol{p}'(t)=\sum_{i=0}^{2m}U_{i,2m}(t)\left[\frac{(i-1-2m)a_{i-1,2m}}{2\mathrm{sh}(\alpha/2)a_{i,2m}}\boldsymbol{p}_{i-1}+\frac{(i-m)\mathrm{ch}(\alpha/2)}{\mathrm{sh}(\alpha/2)}\boldsymbol{p}_i+\frac{(i+1)a_{i+1,2m}}{2\mathrm{sh}(\alpha/2)a_{i,2m}}\boldsymbol{p}_{i+1}\right]$$

where $a_{-1,2m}=a_{2m+1,2m}=0$.

However, from Eq.(7), we have

$$\boldsymbol{p}'(t)=m\left[\left(\boldsymbol{I}+f(t)\Delta\right)\left(\boldsymbol{I}+g(t)\Delta\right)\right]^{m-1}\times\left[\left(\boldsymbol{I}+g(t)\Delta\right)f'(t)+\left(\boldsymbol{I}+f(t)\Delta\right)g'(t)\right]\Delta\boldsymbol{p}_0$$
$$=\sum_{i=0}^{2m-1}\left[U_{i-1,2(m-1)}(t)\frac{\mathrm{sh}(t/2)\mathrm{ch}(t/2)}{\mathrm{sh}^2(\alpha/2)}+U_{i,2(m-1)}(t)\frac{\mathrm{sh}[(\alpha-t)/2]\mathrm{ch}[(\alpha-t)/2]}{\mathrm{sh}^2(\alpha/2)}\right]\Delta\boldsymbol{p}_i$$

Thus, $\boldsymbol{p}^{(k)}(t)$ ($k\geq 1$) can be linear represented by $\Delta\boldsymbol{p}_i$ (i=0, 1, …, $2m$–1).

Limit of the quasi Bézier curves

From Proposition 5 and the definition of the curve Eq.(6), we have the following proposition.

Proposition 6 As $\alpha\to 0$, the limit of the curve Eq.(6) in the space Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt} approaches a Bézier curve in the space T_{2m}=span{1, t, t^2, …, t^{2m}}.

In Fig.7, we show that when α is reduced gradually, the dashed curve Eq.(6) on [0, α] approaches to the Bézier curve drawn with solid line.

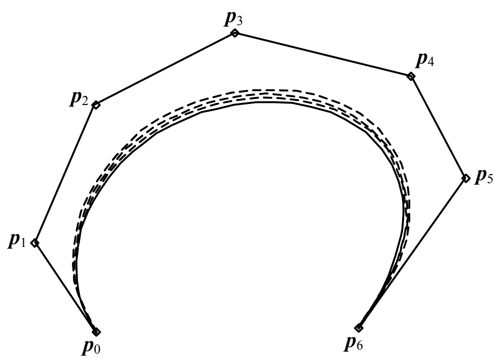


Fig.7 The limit of these curves

REPRESENTATION OF HYPERBOLIA

For the infinity of hyperbola, we can only represent a portion of the whole hyperbola. Fig.8 is a part of an equilateral hyperbola.

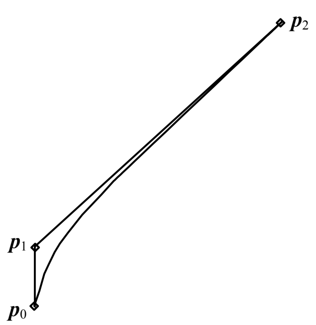


Fig.8 An equilateral hyperbola

EXTENDING TO [r, s]

The quasi Bernstein basis for the space Γ_m=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt} can also be defined on the general interval [r, s] ($r<s$) instead of [0, α] as follows:

$$U_{i,2m}=a_{i,2m}\left(\mathrm{sh}\frac{s-t}{2}\right)^{2m-i}\left(\mathrm{sh}\frac{t-r}{2}\right)^{i},\ t\in[r,s],\qquad(10.1)$$

where

$$a_{i,2m}=\frac{1}{\mathrm{sh}^{2m}\left[(s-r)/2\right]}\sum_{j=0}^{\lfloor i/2\rfloor}\binom{m}{m-i+j,i-2j,j}\left(2\mathrm{ch}\frac{s-r}{2}\right)^{i-2j}\qquad(10.2)$$

It has the same properties as those of the basis on [0, α], and when the interval [r, s] is given, the Bernstein-like basis is unique for its normalization because of the unique representation of the element $1\in\Gamma_m$=span{1, sh$t$, ch$t$, sh2t, ch2t, …, shmt, chmt}.

Reference

Carnicer, J.M., Peña, J.M., 1994. Totally positive bases for shape preserving curve design and optimality of B-splines. *Computer Aided Geometric Design*, **11**:635-656.

Chen, Q.Y., Wang, G.Z., 2003. A class of Bézier-like curves. *Computer Aided Geometric Design*, **20**:29-39.

Lü, Y.G., Wang, G.Z., Yang, X.N., 2002. Uniform hyperbolic polynomial B-spline curves. *Computer Aided Geometric Design*, **19**:379-393.

Peña, J.M., 1997. Shape preserving representations for trigonometric polynomial curves. *Computer Aided Geometric Design*, **14**:5-11.

Sânchez-Reyes, J., 1998. Harmonic rational Bézier curves, p-Bézier curves and trigonometric polynomials. *Computer Aided Geometric Design*, **15**:909-923.

Schumaker, L.L., 1981. Spline Functions: Basic Theory. Wiley-Interscience, New York, p.363-419.

Zhang, J.W., 1996. C-curves: An extension of cubic curves. *Computer Aided Geometric Design*, **13**:199-217.