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On the Ruled surfaces in Minkowski 3-space R_1^3

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Abstract: Izumiya and Takeuchi (2003) obtained some characterizations for Ruled surfaces. Turgut and Hacısalihoğlu (1998) defined timelike Ruled surfaces and obtained some characterizations in timelike Ruled surfaces. Choi (1995) and Jung and Pak (1996) studied Ruled surfaces. This study uses the method in (Izumiya and Takeuchi, 2003) to investigate cylindrical helices and Bertrand curves as curves on timelike Ruled surfaces in Minkowski 3-space $\boldsymbol{R}_1^3$. We have studied singularities of the rectifying developable (surface) of a timelike curve. We observed that the rectifying developable along a timelike curve α is non-singular if and only if α is a cylindrical helice. In this case the rectifying developable is a cylindrical surface.

Key words: Timelike Ruled surfaces, Bertrand curve, Cylindrical helices
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INTRODUCTION

Let $\mathbb{R}^3=\{(x_1,x_2,x_3)|x_1,x_2,x_3\in\mathbb{R}\}$ be a 3D vector space, $\boldsymbol{x}=(x_1,x_2,x_3)$ and $\boldsymbol{y}=(y_1,y_2,y_3)$ be two vectors in $\mathbb{R}^3$. The pseudo scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$<\boldsymbol{x},\boldsymbol{y}>=-x_1y_1+x_2y_2+x_3y_3.$$

We call $(\mathbb{R}^3,<,>)$ a 3D pseudo Euclidean space or Minkowski 3-space. We use $\boldsymbol{R}_1^3$ for $(\mathbb{R}^3,<,>)$.

We say that a vector $\boldsymbol{x}$ in $\boldsymbol{R}_1^3$ is spacelike, lightlike or timelike if $<\boldsymbol{x},\boldsymbol{x}>>0$, $<\boldsymbol{x},\boldsymbol{x}>=0$ or $<\boldsymbol{x},\boldsymbol{x}><0$, respectively. The norm of the vector $\boldsymbol{x}\in\boldsymbol{R}_1^3$ is defined by

$$\|\boldsymbol{x}\| = \sqrt{|<\boldsymbol{x},\boldsymbol{x}>|}\ .$$

Let $\alpha:I\rightarrow\boldsymbol{R}_1^3$, $\alpha(t)=(\alpha_1(t),\alpha_2(t),\alpha_3(t))$ be a smooth regular curve in $\boldsymbol{R}_1^3$ (i.e. $\alpha'(t)\neq0$ for any $t\in I$), where I is an open interval. The curve α is called spacelike if $<\alpha',\alpha'>>0$ and timelike if $<\alpha',\alpha'><0$ and lightlike if $<\alpha',\alpha'>=0$.

The arc-length of a spacelike curve α, measured from $\alpha(t_0)$, $t_0\in I$ is

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \mathrm{d}t\ .$$

Then the parameter s is determined such that $\|\alpha'(s)\|=1$, where $\alpha'(s)=\mathrm{d}\alpha/\mathrm{d}s$. So we say that a spacelike curve α is parametrized by arc-length if it satisfies $\|\alpha'(s)\|=1$. Let us denote $\boldsymbol{t}(s)=\alpha'(s)$ and call $\boldsymbol{t}(s)$ a unit tangent vector of α at s. We define the curvature by

$$\kappa(s) = \sqrt{|<\alpha''(s),\alpha''(s)>|}\ .$$

If $\kappa(s)\neq0$ then the unit principal normal vector $\boldsymbol{n}$ of a timelike curve α at s is given by $\alpha''(s)=\kappa(s)\boldsymbol{n}(s)$.

For any $\boldsymbol{x}=(x_1,x_2,x_3)$, $\boldsymbol{y}=(y_1,y_2,y_3)\in\boldsymbol{R}_1^3$, the pseudo-vector product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x}\wedge\boldsymbol{y} = (-(x_2y_3-x_3y_2),\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1).$$

The unit vector $\boldsymbol{b}(s)=\boldsymbol{t}(s)\wedge\boldsymbol{n}(s)$ is called a unit binormal vector of a timelike α at s (Beem and Ehrlich, 1981; O'Neill, 1983).

Let α be a timelike curve in $\boldsymbol{R}_1^3$ and let us denote $\boldsymbol{t}(s)=\alpha'(s)$. Then we have the Frenet-Serret formulae

$$\begin{aligned}&\boldsymbol{t}'(s)=\kappa(s)\boldsymbol{n}(s),\\&\boldsymbol{n}'(s)=\kappa(s)\boldsymbol{t}(s)+\tau(s)\boldsymbol{b}(s),\\&\boldsymbol{b}'(s)=-\tau(s)\boldsymbol{n}(s),\end{aligned} \tag{1}$$

where $\tau(s)$ is the torsion of a timelike curve α at s. For any unit speed timelike curve $\alpha:I\rightarrow\boldsymbol{R}_1^3$, we can define the Darboux vector field by

$$\boldsymbol{D}(s)=-\tau(s)\boldsymbol{t}(s)-\kappa(s)\boldsymbol{b}(s). \tag{2}$$

We define a vector field

$$\tilde{\boldsymbol{D}}(s)=-(\tau/\kappa)(s)\boldsymbol{t}(s)-\boldsymbol{b}(s) \tag{3}$$

along a timelike curve α under the condition that $\kappa(s)\neq0$ and we call it modified Darboux vector field of a timelike curve α.

Definition 1 A timelike curve $\alpha:I\rightarrow\boldsymbol{R}_1^3$ with $\kappa(s)\neq0$ is called a cylindrical helice if the tangent lines of α make a constant angle with a fixed direction.

It has been known that a timelike curve $\alpha(s)$ is a cylindrical helice if and only if $(\tau/\kappa)(s)$ is constant. We call a timelike curve circular helice if both $\kappa(s)\neq0$ and $\tau(s)$ are constant (Ikawa, 1985).

Definition 2 Let α and $\bar{\alpha}$ be two regular curves with $\kappa(s)\neq0$, $\bar{\kappa}(s)\neq0, s\in I$. Let $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ and $\{\bar{\boldsymbol{t}},\bar{\boldsymbol{n}},\bar{\boldsymbol{b}}\}$ be the frenet frame on $\boldsymbol{R}_1^3$ along α and $\bar{\alpha}$, respectively. If $\{\boldsymbol{n},\bar{\boldsymbol{n}}\}$ is linearly dependent, in other words, if the principal normal lines of α and $\bar{\alpha}$ at $s\in I$ are equal, then the curve α is called a Bertrand curve. In this case $\bar{\alpha}$ is called Bertrand mate of α and we can write

$$\bar{\alpha}(s)=\alpha(s)+r\boldsymbol{n}(s),\ \forall s\in I. \tag{4}$$

The mate of Bertrand curve is denoted by $(\alpha,\bar{\alpha})$ (Balgetir *et al*., 2004).

Theorem 1 Let $(\alpha,\bar{\alpha})$ be a mate of Bertrand curve in Minkowski 3-space $\boldsymbol{R}_1^3$. Then r is constant and defined by Eq.(4) (Balgetir *et al*., 2004).

Theorem 2 Let α and $\bar{\alpha}$ be two regular curves of Minkowski 3-space $\boldsymbol{R}_1^3$. Then $(\alpha,\bar{\alpha})$ is a mate of Bertrand curve if and only if there exists a linear relation

$$A\kappa+B\tau=1, \tag{5}$$

where A,B are nonzero constants and κ and τ are curvature and torsion of α, respectively (Balgetir *et al*., 2004).

Corollary 1 Let α be a regular curve of in Minkowski 3-space $\boldsymbol{R}_1^3$. Then α has more than one Bertrand mate if and only if α is a circular helice.

Any plane curve α is a Bertrand curve whose Bertrand mates are parallel curves of α (Balgetir *et al*., 2004).

Theorem 3 (Schell's Theorem) Let $(\alpha,\bar{\alpha})$ be a mate of Bertrand curve in Minkowski 3-space $\boldsymbol{R}_1^3$. Then the product of torsions τ and $\bar{\tau}$ at the corresponding points of the Bertrand curve is constant, where τ and $\bar{\tau}$ are the torsions of the curves α and $\bar{\alpha}$, respectively (Balgetir *et al*., 2004).

We have the following corollary.

Corollary 2 Let $\alpha:I\rightarrow\boldsymbol{R}_1^3$ be a timelike curve with $\kappa(s)\neq0$ and $\tau(s)\neq0$. Then a timelike curve α is a Bertrand curve if and only if there exists a real number $A\neq0$ such that

$$A(\tau'(s)\kappa(s)-\kappa'(s)\tau(s))-\tau'(s)=0. \tag{6}$$

In this case the Bertrand mate of α is given by

$$\bar{\alpha}(s)=\alpha(s)+A\boldsymbol{n}(s). \tag{7}$$

Proof By Theorem 2 and Theorem 3, a timelike curve α is a Bertrand curve if and only if there exists a real number $A\neq0$ and B such that $A\kappa(s)+B\tau(s)=1$. This is equivalent to the condition that there exists a real number $A\neq0$ such that $(1-A\kappa(s))/\tau(s)$ is constant. Differentiating both sides of the last equality, we have

$$A(\tau'(s)\kappa(s)-\kappa'(s)\tau(s))=\tau'(s). \tag{8}$$

The converse assertion is also true.

TIMELIKE RULED SURFACES IN MINKOWSKI 3-SPACE $\boldsymbol{R}_1^3$

Let $\alpha:I\rightarrow\boldsymbol{R}_1^3$ be a differentiable timelike curve in

Minkowski 3-Space $\boldsymbol{R}_1^3$ parametrized by arc-length. The tangent vector field of a timelike curve α will be denoted by t.

A spacelike straight line,

$$l:\mathbb{R}\to\boldsymbol{R}_1^3,$$
$$v\to l(v)=(\alpha_1(t)+va_1(t),\ \ldots,\ \alpha_3(t)+va_3(t)),\qquad(9)$$

where the scalars $a_i(t)\in\mathbb{R}$ for all $1<i<3$, are components of the director vector at the point $\alpha(t)$, and can be chosen so that the director vector of l and the tangent vector of a timelike curve α are linearly independent at every point of a timelike curve α.

As l moves along a timelike curve α it generates a Ruled surface given by the parametrization $(I\times\mathbb{R},\ \Psi)$, where

$$\Psi_{(\alpha,l)}:I\times\mathbb{R}\to\boldsymbol{R}_1^3,$$
$$(t,v)\to\Psi_{(\alpha,l)}(t,v)$$
$$=(\alpha_1(t)+va_1(t),\ \ldots,\ \alpha_3(t)+va_3(t)),\qquad(10)$$

which can be obtained in Minkowski 3-Space $\boldsymbol{R}_1^3$. We call a timelike curve α the base curve and l the director curve. The straight lines $v\to\alpha(t)+vl(t)$ are called rulings (Turgut and Hacısalihoğlu, 1998).

In particular, if l is constant, then it is said to be cylindrical, and if it is not so, then the surface is said to be non-cylindrical (Choi, 1995).

We can calculate that

$$\frac{\partial\Psi_{(\alpha,l)}}{\partial t}(t,v)\wedge\frac{\partial\Psi_{(\alpha,l)}}{\partial v}(t,v)=\alpha'(t)\wedge l(t)+vl'(t)\wedge l(t).\qquad(11)$$

Therefore (t_0,v_0) is a singular point of $\Psi_{(\alpha,l)}$ if and only if

$$\alpha'(t_0)\wedge l(t_0)+vl'(t_0)\wedge l(t_0)=0.\qquad(12)$$

We say that the timelike Ruled surface $\Psi_{(\alpha,l)}$ is a cylindrical surface if $l'(t_0)\wedge l(t_0)=0$. Thus we say that the timelike Ruled surface $\Psi_{(\alpha,l)}$ is non-cylindrical if $l'(t_0)\wedge l(t_0)\neq 0$.

We now consider a curve $\beta(t)$ on the timelike Ruled surface $\Psi_{(\alpha,l)}$ with the property that $<\beta'(t), l'(t)>=0$. We call such a curve a line of striction. If $\Psi_{(\alpha,l)}$ is non-cylindrical, it has been known that there exists the line of striction uniquely.

In this study, we consider the following two special timelike Ruled surfaces associated with a timelike curve α with $\kappa(s)\neq 0$ which are respectively related to cylindrical helices and Bertrand curves.

Definition 3 A timelike Ruled surface $\Psi_{(\alpha,\tilde{\boldsymbol{D}})}(s,v)=\alpha(s)+v\tilde{\boldsymbol{D}}(s)$ is called a rectifying developable of timelike curve α.

Definition 4 A timelike Ruled surface $\Psi_{(\alpha,\boldsymbol{n})}(s,v)=\alpha(s)+v\boldsymbol{n}(s)$ is called the principal normal surface of a timelike curve α.

We now consider the rectifying developable of unit speed timelike curve $\alpha(s)$ with $\kappa(s)\neq 0$. We can calculate that

$$\tilde{\boldsymbol{D}}'(s)=(-\tau/\kappa)'(s)\boldsymbol{t}(s).\qquad(13)$$

Therefore (s_0,v_0) is a singular point of $\Psi_{(\alpha,\tilde{\boldsymbol{D}})}$ if and only if

$$(-\tau/\kappa)'(s_0)\neq 0\ \text{ and }\ v_0=-[(-\tau/\kappa)'(s_0)]^{-1}.\qquad(14)$$

We have the following proposition.

Proposition 1 For a unit speed timelike curve $\alpha:I\to\boldsymbol{R}_1^3$ with $\kappa(s)\neq 0$, the following are equivalent.

(1) The rectifying developable $\Psi_{(\alpha,\tilde{\boldsymbol{D}})}:I\times\mathbb{R}\to\boldsymbol{R}_1^3$ of a timelike curve α is a non-singular surface.

(2) A timelike curve α is a cylindrical helice.

(3) The rectifying developable $\Psi_{(\alpha,\tilde{\boldsymbol{D}})}$ of a timelike curve α is a cylindrical surface.

Proof By the previous calculation, $\Psi_{(\alpha,\tilde{\boldsymbol{D}})}$ is non-singular at any point in $I\times\mathbb{R}$ if and only if $(-\tau/\kappa)'(s_0)=0$. This means that a timelike curve α is a cylindrical helice.

On the other hand, we have calculated that

$$\tilde{\boldsymbol{D}}'(s)=(-\tau/\kappa)'(s)\boldsymbol{t}(s).\qquad(15)$$

The rectifying developable $\Psi_{(\alpha,\tilde{\boldsymbol{D}})}$ is cylindrical if and only if $\tilde{\boldsymbol{D}}'(s)\equiv 0$, so that Condition (2) is equivalent to Condition (3). Thus the proof is completed.

Now we consider the principal normal surface

$\Psi_{(\alpha,\boldsymbol{n})}(s,v)$ of a unit speed timelike curve $\alpha(s)$ with $\kappa(s)\neq 0$. Let us consider the singular point of $\Psi_{(\alpha,\boldsymbol{n})}(s,v)$. By the Frenet-Serret formulae, we get

$$\alpha'(s)\wedge \boldsymbol{n}(s)+v\boldsymbol{n}'(s)\wedge \boldsymbol{n}(s)=(1+v\kappa(s))\boldsymbol{b}(s)+\tau(s)v\boldsymbol{t}(s). \tag{16}$$

Therefore (s_0,v_0) is a singular point of $\Psi_{(\alpha,\boldsymbol{n})}$ if and only if

$$\tau(s_0)=0 \quad \text{and} \quad v_0=-[\kappa(s_0)]^{-1}. \tag{17}$$

The principal normal surface $\Psi_{(\alpha,\boldsymbol{n})}$ is non-singular under the assumption that $\tau(s)\neq 0$. For example, the principal normal surface of a circular helice is the helicoid.

For a Bertrand curve, we have the following proposition.

Proposition 2 Let a timelike curve $\alpha:I\to \boldsymbol{R}_1^3$ be a Bertrand curve. The principal normal surface $\Psi_{(\alpha,\boldsymbol{n})}$ has a singular point if and only if a timelike curve α is a plane curve. In this case the image of $\Psi_{(\alpha,\boldsymbol{n})}$ is a plane in $\boldsymbol{R}_1^3$.

Proof If there exists a point $s_0\in I$ such that $\tau(s_0)=0$, then α is a plane curve. On the other hand, the singular point of $\Psi_{(\alpha,\boldsymbol{n})}$ corresponds to the point $s_0\in I$ with $\tau(s_0)=0$. The last assertion of the proposition is clear.

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