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General solutions for special orthotropic piezoelectric media*

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Abstract: This paper presents the forms of the general solution for general anisotropic piezoelectric media starting from the basic equations of piezoelasticity by using the operator method introduced by Lur'e (1964), and gives the analytical form of the general solution for special orthotropic piezoelectric media. This paper uses the non-uniqueness of the general solution to obtain the generalized LHN solution and the generalized E-L solution for special orthotropic piezoelectric media. When the special orthotropic piezoelectric media degenerate to transversely piezoelectric media, the solution given by this paper degenerates to the solution for transversely isotropic piezoelectric media accordingly, so that this paper generalized the results in transversely isotropic piezoelectric media.

Key words: Special orthotropic piezoelectric media, LHN solution, E-L solution
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INTRODUCTION

General solution is a convenient, effective method for solving space problems such as problems of infinite, semi-infinite and two-phase infinite media. For linear theory of piezoelasticity, Ding *et al*.(1996; 1997a) gave the general solution for coupled equations for transversely piezoelectric media using potential theory. Ding *et al*.(1997b; 1997c) gave the general solution of plane problem. Based on the general solution, Ding *et al*.(1997a) presented the closed form fundamental solution by using the trial-and-error method. This paper was aimed to obtain the general solution for special orthotropic piezoelectric media. The operator method introduced by Lur'e (1964) can be used to express the displacements field by the adjoint matrix of the differential operator matrix and a vector. Considering the special orthotropic piezoelectric media, the general solution can be obtained by using the operator method. The generalized LHN solution and generalized E-L solution can be obtained by using the completeness and non-uniqueness of the general solution.

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EQUIVALENT EQUATIONS EXPRESSED BY DISPLACEMENTS

For generally orthotropic piezoelectric media, the equivalent equations expressed by displacements and electric potential can be written as:

$$\left(c_{11}\frac{\partial^2}{\partial x^2}+c_{66}\frac{\partial^2}{\partial y^2}+c_{55}\frac{\partial^2}{\partial z^2}\right)u+(c_{12}+c_{66})\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2}{\partial x\partial y}[(c_{13}+c_{55})w+(e_{15}+e_{31})\phi]=0,$$

$$(c_{12}+c_{66})\frac{\partial^2 u}{\partial x\partial y}+\left(c_{66}\frac{\partial^2}{\partial x^2}+c_{22}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}\right)v+\frac{\partial^2}{\partial y\partial z}[(c_{23}+c_{44})w+(e_{24}+e_{32})\phi]=0,$$

$$(c_{13}+c_{55})\frac{\partial^2 u}{\partial x\partial z}+(c_{23}+c_{44})\frac{\partial^2 v}{\partial x\partial z}+\left(c_{55}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial y^2}+c_{33}\frac{\partial^2}{\partial z^2}\right)w+\left(e_{15}\frac{\partial^2}{\partial x^2}+e_{24}\frac{\partial^2}{\partial y^2}+e_{33}\frac{\partial^2}{\partial z^2}\right)\phi=0,$$

$$(e_{15}+e_{31})\frac{\partial^2 u}{\partial x\partial z}+(e_{24}+e_{32})\frac{\partial^2 v}{\partial y\partial z}+\left(e_{15}\frac{\partial^2}{\partial x^2}+e_{24}\frac{\partial^2}{\partial y^2}\right.$$

$$+e_{33}\frac{\partial^2}{\partial z^2}\bigg)w-\left(\varepsilon_{11}\frac{\partial^2}{\partial x^2}+\varepsilon_{22}\frac{\partial^2}{\partial y^2}+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)\phi=0, \quad (1)$$

where u, v, w and ϕ are the elastic displacements and the electric potential, respectively, while c_{ij}, ε_{ij} and e_{kij} are the elastic moduli, the dielectric constants and the piezoelectric coefficients, respectively.

Assume a particular case that all material constants satisfy the following six relations,

$$\frac{c_{44}}{c_{55}}=\frac{c_{23}}{c_{13}}=\frac{\sqrt{c_{22}}}{\sqrt{c_{11}}}=\frac{2c_{66}+c_{12}}{c_{11}}=\frac{e_{32}}{e_{31}}=\frac{e_{24}}{e_{15}}=\frac{\varepsilon_{22}}{\varepsilon_{11}}=\lambda, \quad (2)$$

where λ is constant but not equal to zero.

Now transform the variables and set

$$\sqrt{\lambda}x=x',\ \ y=y',\ \ \sqrt{\lambda}z=z' \quad (3)$$

to obtain

$$\frac{\partial}{\partial x}=\sqrt{\lambda}\frac{\partial}{\partial x'},\ \frac{\partial}{\partial y}=\frac{\partial}{\partial y'}\ \frac{\partial}{\partial z}=\sqrt{\lambda}\frac{\partial}{\partial z'}. \quad (4)$$

Then Eq.(1) can be expressed by matrix as

$$\boldsymbol{AU}=\begin{bmatrix}A_{11}&A_{12}&A_{13}&A_{14}\\A_{12}&A_{22}&A_{23}&A_{24}\\A_{13}&A_{23}&A_{33}&A_{34}\\A_{14}&A_{24}&A_{34}&A_{44}\end{bmatrix}\begin{bmatrix}u\\v\\w\\\phi\end{bmatrix}=0, \quad (5)$$

where the elements of matrix $\boldsymbol{A}$ are

$$A_{11}=\frac{1}{\lambda}\left(\nabla^2+\alpha_1\frac{\partial^2}{\partial x'^2}+\alpha_2\frac{\partial^2}{\partial z'^2}\right),$$
$$A_{12}=\frac{\alpha_1}{\sqrt{\lambda}}\frac{\partial^2}{\partial x'\partial y'},\ \ A_{13}=\alpha_3\frac{\partial^2}{\partial x'\partial z'},$$
$$A_{14}=\alpha_3\frac{\partial^2}{\partial x'\partial z'},\ \ A_{22}=\nabla^2+\alpha_1\frac{\partial^2}{\partial y'^2}+\alpha_2\frac{\partial^2}{\partial z'^2},$$
$$A_{23}=\sqrt{\lambda}\alpha_3\frac{\partial^2}{\partial y'\partial z'},\ \ A_{24}=\sqrt{\lambda}\beta_1\frac{\partial^2}{\partial y'\partial z'},$$
$$A_{33}=\frac{1}{\lambda}\alpha_2\nabla^2+\alpha_4\frac{\partial^2}{\partial z'^2},\ A_{34}=\beta_2\nabla^2+\beta_3\frac{\partial^2}{\partial z'^2},$$
$$A_{44}=\gamma_1\nabla^2+\gamma_2\frac{\partial^2}{\partial z'^2}, \quad (6)$$

and

$$\nabla^2=\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2},\ \ \alpha_1=\frac{c_{12}+c_{66}}{c_{66}},\ \ \alpha_2=\frac{c_{44}}{c_{66}},$$
$$\alpha_3=\frac{c_{13}+c_{55}}{c_{66}},\ \ \alpha_4=\frac{c_{33}}{c_{66}},\ \ \beta_1=\frac{c_{15}+e_{31}}{c_{66}},\ \ \beta_2=\frac{e_{15}}{c_{66}},$$
$$\beta_3=\frac{e_{33}}{c_{66}},\ \ \gamma_1=-\frac{\varepsilon_{11}}{c_{66}},\ \ \gamma_2=-\frac{\varepsilon_{33}}{c_{66}}. \quad (7)$$

GENERAL SOLUTION OBTAINED BY OPERATOR METHOD

In this section, we will use operator method to obtain the general solution for the special orthotropic piezoelectric media assumed in the last section. Some discussions about the operator method can be referred to Xu (2005). We can calculate the determinant of the matrix $\boldsymbol{A}$,

$$|\boldsymbol{A}|=\left(\nabla^2+\lambda\tilde{\alpha}_2\frac{\partial^2}{\partial z'^2}\right)\left(a\frac{\partial^6}{\partial z'^6}+b\frac{\partial^4}{\partial z'^4}+c\frac{\partial^2}{\partial z'^2}+d\nabla^6\right), \quad (8)$$

where the coefficients are

$$\begin{aligned}a&=\tilde{\alpha}_2(\alpha_4\gamma_2-\beta_3{}^2),\\b&=\tilde{\alpha}_2\alpha_4\gamma_1+[\tilde{\alpha}_2{}^2-\alpha_3{}^2+\alpha_4(\tilde{\alpha}_1+1)]\gamma_2\\&\quad-(\tilde{\alpha}_1+1)\beta_3{}^2-2\tilde{\alpha}_2\beta_2\beta_3+2\alpha_3\beta_1\beta_3-\alpha_4\beta_1{}^2,\\c&=[\tilde{\alpha}_2{}^2-\alpha_3{}^2+\alpha_4(\tilde{\alpha}_1+1)]\gamma_1+(\tilde{\alpha}_1+1)\\&\quad\times(\tilde{\alpha}_2\gamma_2-2\beta_2\beta_3)-\tilde{\alpha}_2(\beta_1{}^2+\beta_2{}^2)+2\alpha_3\beta_1\beta_3,\\d&=(\tilde{\alpha}_1+1)(\tilde{\alpha}_2\gamma_1-\beta_2{}^2),\end{aligned} \quad (9)$$

while $\tilde{\alpha}_2=\frac{\alpha_2}{\lambda},\tilde{\alpha}_1=\frac{1+\alpha_1}{\lambda}-1.$

Assume $s_i{}^2$ are the roots of algebraic equation $as^6-bs^4+cs^2-d=0$, then the determinant of matrix $\boldsymbol{A}$ can be expressed as

$$|\boldsymbol{A}|=a\nabla_0{}^2\nabla_1{}^2\nabla_2{}^2\nabla_3{}^2, \quad (10)$$

where

$$\nabla_0{}^2=\nabla^2+\frac{1}{s_0{}^2}\frac{\partial^2}{\partial z'^2},\quad s_0{}^2=\frac{1}{\lambda\tilde{\alpha}_2}, \quad (11)$$

$$\nabla_i{}^2=\nabla^2+\frac{1}{s_i{}^2}\frac{\partial^2}{\partial z'^2},\quad i=1,2,3. \quad (12)$$

Assume the adjoint matrix of matrix ***A*** is ***B***, therefore we can calculate the elements of ***B***, which can be written as

$$
\begin{aligned}
&B_{11}=\left(a'\nabla_1^2\nabla_2^2\nabla_3^2-e\nabla_{a1}^2\nabla_{a2}^2\frac{\partial^2}{\partial x'^2}\right)/\lambda,\\
&B_{12}=B_{21}=-\frac{1}{\lambda^{3/2}}e\nabla_{a1}^2\nabla_{a2}^2\frac{\partial^2}{\partial x'\partial y'},\\
&B_{22}=\left(a'\nabla_1^2\nabla_2^2\nabla_3^2-e\nabla_{a1}^2\nabla_{a2}^2\frac{\partial^2}{\partial y'^2}\right)/\lambda^2,\\
&B_{13}=B_{31}=(\beta_1\beta_3-\alpha_3\gamma_1)\nabla_0^2\nabla_b^2\frac{\partial^2}{\partial x'\partial z'},\\
&B_{14}=B_{41}=(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\nabla_0^2\nabla_c^2\frac{\partial^2}{\partial x'\partial z'},\\
&B_{23}=B_{32}=\frac{1}{\sqrt{\lambda}}(\beta_1\beta_2-\alpha_3\gamma_1)\nabla_0^2\nabla_b^2\frac{\partial^2}{\partial y'\partial z'},\\
&B_{33}=(1+\tilde{\alpha}_1)\nabla_0^2\left[\nabla_d^2\nabla_e^2-\frac{\beta_1^2}{(1+\tilde{\alpha}_1)\gamma_1}\nabla^2\frac{\partial^2}{\partial z'^2}\right],\\
&B_{24}=B_{42}=\frac{1}{\sqrt{\lambda}}(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\nabla_0^2\nabla_c^2\frac{\partial^2}{\partial y'\partial z'},\\
&B_{34}=B_{43}=-(1+\tilde{\alpha}_1)\beta_2\nabla_0^2\left[\nabla_d^2\nabla_f^2-\frac{\alpha_3\beta_1}{(1+\tilde{\alpha}_1)\beta_2}\nabla^2\frac{\partial^2}{\partial z'^2}\right],\\
&B_{44}=(1+\tilde{\alpha}_1)\tilde{\alpha}_2\left[\nabla_d^2\nabla_g^2-\frac{\alpha_3^2}{(1+\tilde{\alpha}_1)\tilde{\alpha}_2}\nabla^2\frac{\partial^2}{\partial z'^2}\right],
\end{aligned}
\tag{13}
$$

where $\alpha'=\lambda^2 a$, and

$$\nabla_{ai}^2=\nabla^2+\frac{1}{s_{ai}^2}\frac{\partial^2}{\partial z'^2},\quad i=1,2. \tag{14}$$

In Eq.(14), s_{ai}^2 are the roots of the algebraic equation

$$es^4-fs^2+g=0, \tag{15}$$

where e, f, g are defined by

$$
\begin{aligned}
&e=(\lambda\alpha_1\alpha_4-\lambda^2\alpha_3^2)\gamma_2-\lambda\alpha_1\beta_3^2+2\lambda^2\alpha_3\beta_1\beta_3-\lambda^2\beta_1\alpha_4,\\
&f=-(\lambda\alpha_1\alpha_4-\lambda^2\alpha_3^2)\gamma_1-\alpha_1(\alpha_2\gamma_2-2\lambda\beta_2\beta_3)\\
&\quad+\lambda\alpha_2\beta_1^2-2\lambda\alpha_3\beta_1\beta_2,\\
&g=\alpha_1(\alpha_2\gamma_1-\lambda\beta_2^2).
\end{aligned}
\tag{16}
$$

Additionally, we have,

$$
\begin{aligned}
&\nabla_b^2=\nabla^2+\frac{\beta_1\beta_3-\alpha_3\gamma_2}{\beta_1\beta_2-\alpha_3\gamma_1}\frac{\partial^2}{\partial z'^2},\\
&\nabla_c^2=\nabla^2+\frac{\alpha_3\beta_3-\alpha_4\beta_1}{\alpha_3\beta_2-\tilde{\alpha}_2\beta_1}\frac{\partial^2}{\partial z'^2},\\
&\nabla_d^2=\nabla^2+\frac{\alpha_2}{1+\alpha_1}\frac{\partial^2}{\partial z'^2},\quad \nabla_e^2=\nabla^2+\frac{\gamma_2}{\gamma_1}\frac{\partial^2}{\partial z'^2},\\
&\nabla_f^2=\nabla^2+\frac{\beta_3}{\beta_2}\frac{\partial^2}{\partial z'^2},\quad \nabla_g^2=\nabla^2+\frac{\alpha_4}{\tilde{\alpha}_2}\frac{\partial^2}{\partial z'^2}.
\end{aligned}
\tag{17}
$$

Use of Theorem 1 in Appendix A yields the general solution for special orthotropic media, which are given by

$$
\begin{aligned}
u_1&=\frac{1}{\lambda}a'\nabla_1^2\nabla_2^2\nabla_3^2\varphi_1-\frac{1}{\lambda}e\nabla_{a1}^2\nabla_{a2}^2\frac{\partial}{\partial x'}\left(\frac{\partial\varphi_1}{\partial x'}+\frac{1}{\sqrt{\lambda}}\frac{\partial\varphi_2}{\partial y'}\right)\\
&\quad+\frac{\partial^2}{\partial x'\partial z'}\nabla_0^2[(\beta_1\beta_2-\alpha_3\gamma_1)\nabla_b^2\varphi_3+(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\nabla_c^2\varphi_4],\\
u_2&=\frac{1}{\lambda^2}a'\nabla_1^2\nabla_2^2\nabla_3^2\varphi_1-\frac{1}{\lambda^{3/2}}e\nabla_{a1}^2\nabla_{a2}^2\\
&\quad\times\frac{\partial}{\partial y'}\left(\frac{\partial\varphi_1}{\partial x'}+\frac{1}{\sqrt{\lambda}}\frac{\partial\varphi_2}{\partial y'}\right)+\frac{1}{\sqrt{\lambda}}\frac{\partial^2}{\partial y'\partial z'}\\
&\quad\times\nabla_0^2[(\beta_1\beta_2-\alpha_3\gamma_1)\nabla_b^2\varphi_3+(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\nabla_c^2\varphi_4],\\
u_3&=(1+\tilde{\alpha}_1)\nabla_0^2\nabla_d^2(\gamma_1\nabla_e^2\varphi_3-\beta_2\nabla_f^2\varphi_4)\\
&\quad+(\beta_1\beta_2-\alpha_3\gamma_1)\frac{\partial}{\partial z'}\nabla_0^2\nabla_b^2\left(\frac{\partial\varphi_1}{\partial x'}+\frac{1}{\sqrt{\lambda}}\frac{\partial\varphi_2}{\partial y'}\right)\\
&\quad-\beta_1\frac{\partial^2}{\partial z'^2}\nabla_0^2\nabla^2(\beta_1\varphi_3-\alpha_3\varphi_4),\\
u_4&=(1+\tilde{\alpha}_1)\nabla_0^2\nabla_d^2(\tilde{\alpha}_2\nabla_g^2\varphi_4-\beta_2\nabla_f^2\varphi_3)\\
&\quad+(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\frac{\partial}{\partial z'}\nabla_0^2\nabla_c^2\left(\frac{\partial\varphi_1}{\partial x'}+\frac{1}{\sqrt{\lambda}}\frac{\partial\varphi_2}{\partial y'}\right)\\
&\quad+\alpha_3\frac{\partial^2}{\partial z'^2}\nabla_0^2\nabla^2(\beta_1\varphi_3-\alpha_3\varphi_4).
\end{aligned}
\tag{18}
$$

GENERALIZED LHN GENERAL SOLUTION AND E-L GENERAL SOLUTION FOR SPECIAL ORTHOTROPIC MEDIA

If there are no more than two points of intersection for any line parallel to z-axis though the elastic field, we can use the non-uniqueness of the general solution and Eq.(18) to obtain the generalized solution for special orthotropic media. The non-uniqueness Theorem 2 is given in Appendix A. Omitting the deducing details, the LHN solution are given by

$$u_1=-\frac{1}{\lambda}\frac{\partial\psi_0}{\partial y'}+\frac{\partial^2}{\partial x'\partial y'}[(\beta_1\beta_2-\alpha_3\gamma_1)\nabla_b^{\,2}F_1+(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\nabla_c^{\,2}F_2],$$
$$u_2=\frac{1}{\lambda^{5/2}}\frac{\partial\psi_0}{\partial x'}+\frac{1}{\sqrt{\lambda}}\frac{\partial^2}{\partial y'\partial z'}[(\beta_1\beta_2-\alpha_3\gamma_1)\nabla_b^{\,2}F_1+(\alpha_3\beta_2-\tilde{\alpha}_2\beta_1)\nabla_c^{\,2}F_2],$$
$$u_3=(1+\tilde{\alpha}_1)\nabla_d^{\,2}(\gamma_1\nabla_e^{\,2}F_1-\beta_2\nabla_f^{\,2}F_2)-\beta_1\frac{\partial^2}{\partial z'^2}\nabla^2(\beta_1F_1-\alpha_3F_2),$$
$$\phi=(1+\tilde{\alpha}_1)\nabla_d^{\,2}(\tilde{\alpha}_2\nabla_e^{\,2}F_2-\beta_2\nabla_f^{\,2}F_1)+\alpha_3\frac{\partial^2}{\partial z'^2}(\beta_1F_1-\alpha_3F_2), \tag{19}$$

where ψ_0, F_i (i=1,2) satisfy

$$\nabla_0^{\,2}\psi_0=0,\quad \nabla_1^{\,2}\nabla_2^{\,2}\nabla_3^{\,2}F_i=0\,. \tag{20}$$

If there are no more than two points of intersection for any line parallel to z-axis though the elastic field, and the eigenvalues s_i^2 (i=0,1,2,3) are not equal to each other, we can derive the E-L general solution by using Almansi theorem and the LHN solution Eq.(19), which are expressed by

$$u_1=-\frac{1}{\lambda}\frac{\partial\psi_0}{\partial y'}+\sum_{j=1}^{3}\frac{\partial\psi_j}{\partial x'},\quad u_2=\frac{1}{\lambda^{3/2}}\frac{\partial\psi_0}{\partial x'}+\frac{1}{\sqrt{\lambda}}\sum_{j=1}^{3}\frac{\partial\psi_j}{\partial y'},\quad u_3=\sum_{j=1}^{3}k_{1j}\frac{\partial\psi_j}{\partial z'},\quad \phi=\sum_{j=1}^{3}k_{2j}\frac{\partial\psi_j}{\partial z'}, \tag{21}$$

where the functions ψ_i (i=0,1,2,3) satisfy

$$\nabla_j^{\,2}\psi_j=0, \tag{22}$$

and the coefficients k_{1j} and k_{2j} are

$$k_{1j}=\frac{(1+\tilde{\alpha}_1)\gamma_1-[(1+\tilde{\alpha}_1)\gamma_2+\alpha_2\gamma_1-\beta_1^2]s_j^{\,2}+\tilde{\alpha}_2\gamma_2s_j^{\,4}}{(\alpha_3\gamma_1-\beta_1\beta_2)s_j^{\,2}+(\beta_1\beta_3-\alpha_3\gamma_2)s_j^{\,4}},$$
$$k_{2j}=-\frac{(1+\tilde{\alpha}_1)\beta_2-[(1+\tilde{\alpha}_1)\beta_3+\tilde{\alpha}_2\gamma_1-\alpha_3\beta_1]s_j^{\,2}+\tilde{\alpha}_2\beta_3s_j^{\,4}}{(\alpha_3\gamma_1-\beta_1\beta_2)s_j^{\,2}+(\beta_1\beta_3-\alpha_3\gamma_2)s_j^{\,4}}. \tag{23}$$

In Eq.(22), j are not summed.

In view of the hypothesis on special orthotropic piezoelectric media, we can see that the media degenerate to transversely piezoelectric media if λ=1. It is obvious that the LHN general solution Eq.(19) and the E-L general solution Eq.(21) degenerate to the case of transversely piezoelectric media if λ=1. When λ=1, it can be seen that Eq.(21) is the same as the results given by Ding *et al*.(1996). So this paper has generalized the results in transversely isotropic piezoelectric media.

APPENDIX A

In this appendix, we present two theorems used in this paper, the proof for the theorems can be referred to Xu (2005).

Theorem 1 The equivalent equations for anisotropic piezoelectric media can be expressed by operator matrix

$$\boldsymbol{AU}=0, \tag{24}$$

where $\boldsymbol{A}$ is a 4×4 matrix,

$$\boldsymbol{A}=\begin{bmatrix} c_{ijkl}\partial_k\partial_l & e_{jil}\partial_j\partial_l \\ e_{jkl}\partial_j\partial_l & -\varepsilon_{jl}\partial_j\partial_l \end{bmatrix}, \tag{25}$$

and $\partial_j=\dfrac{\partial}{\partial x_j}$, additionally, $\boldsymbol{U}$ is a vector

$$\boldsymbol{U}=[u_1\ \ u_2\ \ u_3\ \ \phi]^{\mathrm{T}}\,. \tag{26}$$

Assume the vector $\boldsymbol{\varphi}$ satisfy

$$|\boldsymbol{A}|\boldsymbol{\varphi}=0, \tag{27}$$

where $|\boldsymbol{A}|$ is the determinant of the matrix $\boldsymbol{A}$, then

$$\boldsymbol{U}=\boldsymbol{B\varphi} \tag{28}$$

is the solution of Eq.(24), and the solution Eq.(28) is complete.

Theorem 2 The solution of Eq.(24) is non-unique. If $\boldsymbol{\varphi}$ in Eq.(28) is replaced by

$$\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \boldsymbol{A}\boldsymbol{h}, \quad (29)$$

where $\boldsymbol{h}$ satisfies

$$|\boldsymbol{A}|\boldsymbol{h}=0, \quad (30)$$

then we have

$$|\boldsymbol{A}|\tilde{\boldsymbol{\varphi}} = 0, \quad \boldsymbol{U} = \boldsymbol{B}\tilde{\boldsymbol{\varphi}}. \quad (31)$$

If there are two kinds of expressions for $\boldsymbol{U}$,

$$\boldsymbol{U}=\boldsymbol{B}\boldsymbol{\varphi}^{(i)}, \quad i=1,2, \quad (32)$$

then there exist vector $\boldsymbol{h}$, which satisfies

$$\boldsymbol{A}\boldsymbol{h}=\boldsymbol{\varphi}^{(1)}-\boldsymbol{\varphi}^{(2)}, \quad |\boldsymbol{A}|\boldsymbol{h}=0. \quad (33)$$

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