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Global dimension of weak smash product*

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Abstract: In Artin algebra representation theory there is an important result which states that when the order of G is invertible in Λ then $gl.dim(\Lambda G)=gl.dim(\Lambda)$. With the development of Hopf algebra theory, this result is generalized to smash product algebra. As known, weak Hopf algebra is an important generalization of Hopf algebra. In this paper we give the more general result, that is the relation of homological dimension between an algebra A and weak smash product algebra $A\#H$, where H is a finite dimensional weak Hopf algebra over a field k and A is an H-module algebra.

Key words: Weak Hopf algebra, Weak smash product, Gobal dimension
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INTRODUCTION

In (Auslander, 1995), the author established the relationship of homological dimension between skew group algebra ΛG and Λ. Recall from Corollary 4.7 and Corollary 4.10 in (Auslander, 1995) that $gl.dim(\Lambda G)=gl.dim(\Lambda)$ where ΛG is a skew group algebra with the order of G invertible in Λ.

Recently, more and more mathematicians have studied homological dimension for Hopf algebra. Yang (2002) gave the relationship of homological dimension between smash product algebra $A\#H$ and A. Recall from Theorem 2.2 in (Auslander, 1995) that if there is a coaction ρ on A such that if A is H-commutative, $\rho(1)=1\otimes 1$ and $\rho(t\circ a)=\sum t_1 a_{(-1)}S(t_3)\otimes t_2\circ a_{(0)}$, then $gl.dim(A\#H)=n$ if and only if $gl.dim(A)=n$ and $t\circ c=1$. Zhu and Zhang (2003) and Zhu (2005) gave the homological dimension for Hopf actions and smash coproduct respectively.

The purpose of our paper is to consider the same relationship in the setting of weak Hopf algebras because for a weak Hopf algebra we do not require Δ preserving the unit and ε preserving the multiplication, we need some technical deal rather than trivial generalization.

PRELIMINARY RESULT

We use the Swdeeler-Heyneman notations for comultiplication and coaction: if H is a coalgebra, with comultiplication Δ, and M is a left H-comodule, then we write:

$$\Delta(h)=\sum h_1\otimes h_2 \text{ and } \rho(m)=\sum m_{(-1)}\otimes m_{(0)}.$$

Definition 1 A weak bialgebra (WBA) is a quintuple $(H, m, \mu, \Delta, \varepsilon)$ satisfying Axioms 1~3 below. If $(H, m, \mu, \Delta, \varepsilon)$ satisfies Axioms 1~4 below it is called a weak Hopf algebra (WHA).

Axiom 1 H is a finite dimensional associative algebra over a field k with multiplication m and unit μ;

Axiom 2 H is a coalgebra over the field k with comultiplication Δ and counit ε;

Axiom 3 For compatibility of the algebra and coalgebra structures we assume:

(1) For all $h, g\in H$,

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$$\Delta(hg)=\Delta(h)\Delta(g); \quad (1)$$

(2) For all $h, g, l\in H$,

$$\varepsilon(hgl)=\sum\varepsilon(hg_1)\varepsilon(g_2l)=\sum\varepsilon(hg_2)\varepsilon(g_1l); \quad (2)$$

(3) $\Delta^2(1)=[1\otimes\Delta(1)][\Delta(1)\otimes1]$

$$=[\Delta(1)\otimes1][1\otimes\Delta(1)]. \quad (3)$$

Axiom 4 There exists a k-linear map S: $H\to H$, called the antipode, satisfying the following:

For all $h\in H$,

(1) $\sum h_1S(h_2)=\sum\varepsilon(1_1h)1_2$; (4)

(2) $\sum S(h_1)h_2=\sum\varepsilon(h1_2)1_1$; (5)

(3) $\sum S(h_1)h_2S(h_3)=S(h)$. (6)

Throughout this paper, H is a finite dimensional weak Hopf algebra over the field k. Recall from Theorem 2.10 in (Böhm *et al*., 1999) that the antipode of a finite dimensional weak Hopf algebra is always bijective. One can find more details of weak Hopf algebra in (Böhm *et al*., 1999; Nikshych and Vainerman, 2002).

In a WBA we define the maps $\cap^l$, $\cap^r$: $H\to H$ by

$$\cap^l(h)=h^l=\sum\varepsilon(1_1h)1_2, \quad (7)$$

$$\cap^r(h)=h^r=\sum\varepsilon(h1_2)1_1. \quad (8)$$

Definition 2 A left (right) integral in a weak Hopf algebra H is an element $t\in H$ ($q\in H$) verifying

$$\text{for all } h\in H, \quad ht=h^lt \ (qh=qh^r).$$

In addition, if $t^l=1$ ($q^r=1$) then it is called a normalized integral.

Definition 3 [Definition 1.4 in (Zhang and Zhu, 2004)] Let H be a weak Hopf algebra. An algebra A is called a left H-module algebra if it is a left H-module such that for all $a, b\in A$, $h\in H$,

(1) $h\circ(ab)=\sum(h_1\circ a)(h_2\circ b)$; (9)

(2) $h\circ1=h^l\circ1$. (10)

Lemma 1 Let H be a weak Hopf algebra and A a left H-module algebra. For all $h\in H$ and $a\in A$, we have:

$$h^l\circ a=(h^l\circ1)a, \quad (11)$$

$$h^r\circ a=a(h^r\circ1). \quad (12)$$

Proof We only prove Eq.(11), and Eq.(12) can be obtained similarly.

$$h^l\circ a=\sum(1_1h^l\circ1)(1_2\circ a)=(h^l\circ1)a.$$

Definition 4 [Definition 2.1 in (Böhm *et al*., 1999)] An algebra A with the left coaction ρ: $A\to H\otimes A$ is called a left H-comodule algebra if the following equations are satisfied:

(1) For all $a, b\in A$, $\rho(ab)=\rho(a)\rho(b)$; (13)

(2) For all $a\in A$, $\sum1_{(-1)}\otimes a1_{(0)}=\sum a^r_{(-1)}\otimes a_{(0)}$. (14)

WEAK SMASH PRODUCTS

Definition 5 Let H be a weak Hopf algebra and A a left H-module algebra. Then the smash product algebra $A\#H$ is defined on a k-vector space $A\otimes_{H^l}H$, where H is a left H^l-module via the multiplication of H and A is a right H^l-module via for all $a\in A$, $z\in H^l$,

$$a\circ z=a(z\circ1)=S^{-1}(z)\circ a. \quad (15)$$

Let $[a\#h]$ be the class of $a\otimes h$ in $A\otimes_{H^l}H$, its multiplication and unit are given respectively by $[a\#h][b\#g]=\sum[a(h_1\circ b)\#h_2g]$ and unit is $[1\#1]$.

Remark 1 (1) We have known that A and H are both subalgebras of $A\#H$. For this reason, we frequently abbreviate the element $a\#h$ by ah. In this notation, we may write

$$ha=\sum(h_1\circ a)h_2.$$

(2) $A\#H=(1\#H)(A\#1)$. In fact, for all $a\in A$, $h\in H$, we have

$$\sum(1\#h_2)(S^{-1}(h_1)\circ a\#1)=\sum S^{-1}(h_1^l)\circ a\#h_2=a\#h.$$

We claim that A is an $A\#H$-module under the action given by $(a\#h)\circ b=a(h\circ b)$ since A is a left H-module algebra. We only need to show for all $a, b\in A$, $h, g\in H$,

$$\begin{aligned}(a\circ h^l\#g-a\#h^lg)\circ b&=(a\circ h^l)(g\circ b)-a(h^lg\circ b)\\&=(a\circ h^l)(g\circ b)-a(h^l\circ1)(g\circ b)\\&=(a\circ h^l)(g\circ b)-(a\circ h^l)(g\circ b)=0.\end{aligned}$$

Let $(A\#H)^H=\{x\in A\#H|hx=h^lx\}$. If $f\in Hom_{A\#H}(A,$

$A\#H$), then $f(1)\in(A\#H)^H$. In fact, for all $h\in H$,

$$hf(1)=f(h\circ 1)=f(h^l\circ 1)=h^l f(1).$$

Let $x\in(A\#H)^H$, if we define f_x: $A\to A\#H$ by $f_x(a)=ax$ for all $a\in A$, then $f_x\in Hom_{A\#H}(A,A\#H)$. Indeed for all $h\in H$ and a, $b\in A$,

$$f_x((b\#h)\circ a)=b(h\circ a)x=\sum b(h_1\circ a)h_2x=(b\#h)f_x(a).$$

So we have the following lemma:

Lemma 2 Let H be a weak Hopf algebra and A a left H-module algebra. Then as a k-space

$$Hom_{A\#H}(A,A\#H)\cong(A\#H)^H.$$

It is easy to prove $(1\#t)(A\#1)\subseteq(A\#H)^H$. As known, the invariants of a module M over a finite dimensional weak Hopf algebra H are the elements of the kernel of the map:

$$\delta_M: M\to Hom_k(H,k)\otimes_{H^l}M,$$
$$\delta_M(m)=\sum\beta^n\otimes_{H^l}b_n\circ m-\varepsilon\otimes_{H^l}m,$$

where $\{b_n, \beta^n\}$ are a pair of dual basis of H and $Hom_k(H,k)$. The left H^l-module structure on $Hom_k(H, k)$ is defined by for all h, $h'\in H$,

$$\varphi\in Hom_k(H, k),\ (\varphi\circ h^l)(h')=\varphi(S^2(h^l)h').$$

Furthermore, $A\#H$ is isomorphic to $H\otimes_{H^l}A$ as a left H-module, where the left H-module structure on $H\otimes_{H^l}A$ is given by the multiplication in the first factor, via the isomorphism described before. Finally, $\delta_{H\otimes_{H^l}A}=\delta_H\otimes_{H^l}A$. These observations imply the chain of isomorphisms:

$$(A\#H)^H\cong(H\otimes_{H^l}A)^H\cong Ker(\delta_{H\otimes_{H^l}A}\cong Ker(\delta_H\otimes_{H^l}A).$$

Since H^l is a separable algebra over a field (Proposition 2.11 in (Böhm *et al*., 1999)), any H^l-module is flat. Hence $Ker(\delta_{H\otimes_{H^l}A})=Ker(\delta_H\otimes_{H^l}A)$. Since the left integrals in H are the elements of $Ker(\delta_H)$, these imply the following lemma:

Lemma 3 Let H be a weak Hopf algebra and A a left H-module algebra. Then the space of $(A\#H)^H$ is spanned by elements of the form $(1\#t)(a\#1)$, where t is a left integral in H and a is an element of A.

Let us define E: $A\#H\to A$ by $E(a\#h)=S^{-1}(h^l)\circ a$ for all $a\in A$, $h\in H$. We claim that E is an $A\#H$-module epimorphism.

First we show it is well-defined. Indeed, for all $a\in A$ and h, $g\in H$,

$$E(a\circ h^l\#g)=S^{-1}(h^lg^l)\circ a=S^{-1}((h^lg)^l)\circ a=E(a\#h^lg),$$

where we obtained the second step by using Eq.(2.13a) in (Böhm *et al*., 1999). Next we claim that it is an $A\#H$-module. In fact, for all a, $b\in A$ and h, $g\in H$,

$$E[(a\#h)(b\#g)]=\sum S^{-1}((h_2g^l)^l)\circ[a(h_1\circ b)]$$
$$=\sum[a(h_1\circ b)][((h_2g^l)^l)\circ 1]=a[hS^{-1}(g^l)\circ b]=(a\#h)E(b\#g),$$

where we used Lemma 2.9 in (Böhm *et al*., 1999) and Lemma 1 to get the second one.

Thus A is a projective $A\#H$-module if and only if there exists an element $f\in Hom_{A\#H}(A,A\#H)$ such that $Ef=id_A$, equivalently if and only if there is an element $x\in(A\#H)^H$ such that $E(x)=1$, so $E[\sum(1\#t_i)(a_i\#1)]=\sum S^{-1}(t_{i2}^l)t_{i1}\circ a=\sum t_i\circ a_i=1$. Therefore we have:

Proposition 1 Let H be a weak Hopf algebra and A a left H-module. Then A is a projective $A\#H$-module if and only if there exist finitely many left integrals t_i and corresponding elements $a_i\in A$ such that $\sum t_i\circ a_i=1$.

If we take A to be the trivial H-module algebra H^l with the action defined by $h\circ z=(hz)^l$ for all $h\in H$, $z\in H^l$, then by this proposition H^l is a projective $H^l\#H(\cong H)$-module if and only if there is some $z\in H^l$ such that $t\circ z=(tz)^l=1$. This implies the existence of a normalized left integral. Recall from Theorem 3.13 in (Böhm *et al*., 1999) that H is semisimple if and only if there exists a normalized left integral. Thus we have:

Corollary 1 Let H be a weak Hopf algebra. Then H is semisimple if and only if H^l viewed as a trivial H-module is projective.

Let A be a H-module algebra. If H is semisimple, then there exists a normalized left integral t. Thus $t\circ 1=t^l\circ 1=1$, we have:

Corollary 2 Let H be a semisimple weak Hopf algebra and A a left H-module algebra. Then A is a projective $A\#H$-module.

Lemma 4 Let H be a semisimple weak Hopf algebra and A a left H-module algebra, and let P be an $A\#H$-module. Then P is a projective $A\#H$-module if and only if it is a projective A-module.

Proof Necessity can be obtained directly since $A\#H$ is a projective A-module. It suffices to consider the sufficiency. Let f: $X\rightarrow Y$ and g: $P\rightarrow Y$ be two $A\#H$-module morphisms such that f is surjective. We only need to prove that there is an $m\in Hom_{A\#H}(P,X)$ satisfying $g=fm$. Since P is a projective A-module, there exists $m^*\in Hom_A(P,X)$ such that $g=fm^*$, where we consider an $A\#H$-module as an A-module in the natural way. Define $m(p)=\sum S(t_1)\circ m^*(t_2\circ p)$ for all $p\in P$, where t is a left integral of H, then we claim that m is both A-module morphism and H-module morphism. In fact, for all $h\in H$, $a\in A$ and $p\in P$, there is an element g of H such that $h=S(g)$ since S is bijective. We have:

$$h\circ m(p)=\sum S(t_1g)\circ m^*(t_2\circ p)$$
$$=\sum S(t_1)\circ m^*(t_2S(g)\circ p)=m(h\circ p),$$

where we obtained the first step by using Lemma 3.2 in (Böhm, 2000).

We also have

$$a\circ m(p)=\sum S(t_1)(S^{-1}(S(t_2))\circ a)\circ m^*(t_3\circ p)$$
$$=\sum S(t_1)\circ m^*(t_2a\circ p)=m(a\circ p),$$

so m is an $A\#H$-module morphism.

Lemma 5 Let H be a weak Hopf algebra and A a left H-module algebra. Then M is an $A\#H$-module if and only if M is both A-module and H-module satisfying for all $m\in M$, $h\in H$, $a\in A$,

$$h\circ(am)=\sum(h_1\circ a)(h_2\circ m).$$

Proof If M is an $A\#H$-module, then clearly M is both A-module and H-module. We also have

$$h\circ(am)=\sum(h_1\circ a\#h_2)\circ m=\sum(h_1\circ a)(h_2\circ m).$$

Conversely, if M is both A-module and H-module verifying $h\circ(am)=\sum(h_1\circ a)(h_2\circ m)$, then we can define $A\#H$-module structure by $(a\#h)\circ m=a\circ(h\circ m)$ for all $a\in A$, $h\in H$ and $m\in M$. We only show it is well-defined, the remaining proof is the same as in the setting of Hopf algebras. In fact, for all $m\in M$, $h, g\in H$, $a\in A$,

$$(a\#h^lg)\circ m=\sum a(1_1h^lg_1\circ 1)(1_2g_2\circ m)=\sum a(h^lg_1\circ 1)(g_2\circ m)$$
$$=\sum a(h^l\circ 1)(g_1\circ 1)(g_2\circ m)=(a\circ h^l)(g\circ m)=(a\circ h^l\#g)\circ m,$$

where we used Eq.(2.7) in (Böhm *et al*., 1999) to get the first step, and obtained the second and the third ones by using Lemma 1.

MAIN RESULTS

Definition 6 Let H be a weak Hopf algebra, and A a left H-module algebra and a left H-comodule algebra. We say that A is H-commutative if for all $a, b\in A$,

$$ab=\sum b_{(0)}[S^{-1}(b_{(-1)})\circ a].$$

Definition 7 Let (H,R) be a quasi-triangular weak Hopf algebra, and A a left H-module algebra. We call A is quantum commutative if for all $a, b\in A$, $ab=\sum(R^2\circ b)(R^1\circ a)$, where $R==\sum R^1\otimes R^2$ is R-matrix.

Example 1 Let H be a weak Hopf algebra. Taking $A:=H^l$ and $\rho:=\Delta$. It is not hard to verify A is H-commutative with the left H-module structure given by $h\circ g^l=(hg^l)^l$ for all $h, g\in H$.

Example 2 Let (H,R) be a quasi-triangular weak Hopf algebra and A a left H-module algebra. By Definition 5.1 and Proposition 5.6 in (Nikshych *et al*., 2003) it is not hard to verify A is a left H-comodule algebra via the coaction for all $a\in A$, $\rho(a)=\sum R^2\otimes R^1\circ a$. Then by Lemma 1 it is easy to prove that A is H-commutative if and only if it is quantum commutative.

Lemma 6 Let H be a weak Hopf algebra and A be H-commutative. If M is an $A\#H$-module, then M is an (A,A)-bimodule with the natural action and the right action defined by for all $b\in A$, $m\in M$,

$$m\leftarrow b=\sum b_{(0)}(S^{-1}(b_{(-1)})\circ m).$$

Proof It is the same as the proof in the setting of Hopf algebras to verify $(m\leftarrow b)\leftarrow a=m\leftarrow ba$ for all $a, b\in A$, $m\in M$.

We must show $m\leftarrow 1=m$. Indeed,

$$m\leftarrow 1=\sum 1_{(0)}(S^{-1}(1_{(-1)})_1\circ 1)(S^{-1}(1_{(-1)})_2\circ m)$$
$$=\sum 1_{(0)}(1_1S^{-1}(1_{(-1)})\circ 1)(1_2\circ m)$$
$$=\sum 1_{(0)}(S^{-1}(1_{(-1)})\circ 1)m=1\circ m=m,$$

where we got the first and the third steps by using

Lemma 4, and used Eq.2.7(b) in (Böhm *et al*., 1999) and Definition 2.1 in (Böhm *et al*., 1999) to obtain the second one.

Finally we claim M is an (A,A)-bimodule. Indeed, for all $a, b\in A$, $m\in M$,

$$(a\circ m)\leftarrow b=\sum b_{(0)}S^{-1}(b_{(-1),2})\circ a)(S^{-1}(b_{(-1),1})\circ m)$$
$$=\sum ab_{(0)}(S^{-1}(b_{(-1)}\circ m)=a\circ(m\leftarrow b),$$

where we got the second equation by using Lemma 5 and obtained the third one since A is H-commutative.

For any ring R, we denote the global dimension and projective dimension of R by $gl.dim(R)$ and $P.dim(R)$ respectively.

After these preparations we now can give our main result.

Proposition 2 Let H be a semisimple weak Hopf algebra and A a H-module algebra. Assume that A is also a left H-comodule algebra such that A is H-commutative, then $gl.dim(A\#H)=gl.dim(A)$.

Proof Clearly it is harmless to assume the global dimension A is finite, say n. For any $A\#H$-module M, by Lemma 4 any projective resolution of M as $A\#H$-module is also a projective resolution of M as A-module. Hence its n-syzygy is projective as an A-module, and thus is projective as an $A\#H$-module. This implies $gl.dim(A\#H)=gl.dim(A)$.

Next we have to show $gl.dim(A)\leq gl.dim(A\#H)$. We claim that the morphism E: $A\#H\rightarrow A$ defined by $E(a\#h)=S^{-1}(h^{l})\circ a$, for all $a\in A$, $h\in H$, is a split (A,A)-bimodule epimorphism.

$A\#H$ is a right A-module by the formula $(a\#h)\leftarrow b=\sum b_{(0)}(S^{-1}(b_{(-1)})\circ(a\#h))$ for all $a, b\in A$, $h\in H$. We have shown that E is well-defined and a left A-module map before.

It suffices to prove E is a right A-module map. In fact, for all $a, b\in A$, $h\in H$,

$$E[(a\#h)\leftarrow b]=E(\sum ab_{(0)}\#S^{-1}(b_{(-1)})h)$$
$$=\sum S^{-1}\{[S^{-1}(b_{(-1)})h]^{l}\}\circ(ab_{(0)})$$
$$=\sum ab_{(0)}\{S^{-1}[(S^{-1}(b_{(-1)})h)^{l}]\circ 1\}$$
$$=\sum ab_{(0)}[S^{-1}(b_{(-1)})h\circ 1]=a(h\circ 1)b$$
$$=[S^{-1}(h^{l})\circ a]b=E(a\#h)b,$$

where we used Lemma 1 to get the third step and obtained the fourth one from Eq.(2.25) in (Böhm *et al*., 1999).

Therefore A is an (A,A)-bimodule direct summand of $A\#H$. This implies every A-module M is a direct summand of the A-module $(A\#H)\otimes_{A}M$, $P.dim_{A}(M)\leq P.dim_{A}((A\#H)\otimes_{A}M)$. We consider the left $A\#H$-module $(A\#H)\otimes_{A}M$. Since any of its projective resolution as an $A\#H$-module is also a projective resolution as A-module by Lemma 4, we have

$$P.dim_{A}[(A\#H)\otimes_{A}M]\leq P.dim_{A\#H}[(A\#H)\otimes_{A}M].$$

Hence

$$gl.dim(A)\leq gl.dim(A\#H).$$

References

Auslander, M., 1995. Representation Theory of Artin Algebras. Cambridge Press.

Böhm, G., Nill, K., Szlachanyi, K., 1999. Weak Hopf algebras (I): Integral theory and C^{*}-structure. *J. Algebra*, **221**(2):385-438. [doi:10.1006/jabr.1999.7984]

Böhm, G., 2000. Doi-Hopf modules over weak Hopf algebras. *Comm. Algebra*, **28**:4687-4689.

Nikshych, D., Vainerman, L., 2002. Finite Quantum Groupoids and Their Applications. *In*: Montgomery, S., Schneider, H.J. (Eds.), New Direction in Hopf Algebras, Series: Mathematical Sciences Research Institute Publications, Cambridge Univ. Press, Cambridge, **43**:211-262.

Nikshych, D., Turaev, V., Vainerman, L., 2003. Invariants of knots and 3-manifolds from quantum groupoids. *Topology and its Applications*, **127**(1-2):91-123. [doi:10.1016/S0166-8641(02)00055-X]

Yang, S.L., 2002. Global dimension for Hopf actions. *Comm. Algebra*, **30**(8):3653-3667. [doi:10.1081/AGB-120005811]

Zhang, L.Y., Zhu, S.L., 2004. Fundamental theorems of weak doi-Hopf modules and semisimple weak smash product Hopf algebras. *Comm. Algebra*, **32**(9):3403-3416. [doi:10.1081/AGB-120039396]

Zhu, J.G., Zhang, L.Y., 2003. Semisimple extensions and homological dimension of invariants for Hopf actions. *Acta Mathematica Sinica, English Series*, **46**(1):137-142.

Zhu, J.G., 2005. Cosemisimple coextesnsion and homological dimension of smash coproduct. *Acta Mathematica Sinica, English Series*, **21**(3):563-568. [doi:10.1007/s10114-005-0527-8]