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Vanishing torsion of parametric curves*

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Abstract: We consider the class of parametric curves that can be represented by combination of control points and basis functions. A control point is let vary while the rest is held fixed. It's shown that the locus of the moving control point that yields points of zero torsion is the osculating plane of the corresponding discriminant curve at its point of the same parameter value. The special case is studied when the basis functions sum to one.

Key words: Parametric curve, Torsion, Discriminant curve
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INTRODUCTION

In parametric curve design and interrogation, intrinsic curves properties that are independent of the parametrization play important role. Such properties are tangent line, curvature and torsion. The detection of singularities (zero curvature, cusp, lop) of curves is essential for designers, thus there are several papers published on this topic, e.g., (Wang, 1981; Stone and DeRose, 1989; Meek and Walton, 1990; Manocha and Canny, 1992; Li and Cripps, 1997; Sakai, 1999; Monterde, 2001).

Curves that can be obtained by means of the combination of control points and basis function, e.g. Bézier, B-spline or NURBS, are essential for computer aided geometric design. Juhász (2006) provided a general and simple method for the singularity detection of this class of curves. The basic idea of the method is that we fix all control points but one, and determine the locus of those positions of the moving control point that result in singularities on the curve. The discriminant curve of the studied curve with respect to the moving control point plays a significant role in the determination of these loci.

In this paper we examine points of vanishing torsion of curves defined by control points and basis functions. We show that the torsion is zero at the point $\bar{u}$ of such a curve iff the moving control point is on the osculating plane of the discriminant curve at its point of the same parameter value.

DISCRIMINANT CURVE

Let us consider curves that are combinations of control points $\boldsymbol{d}_j$ and basis functions $F_j(u)$:

$$\boldsymbol{g}(u)=\sum_{j=0}^{n}F_j(u)\boldsymbol{d}_j,\quad u\in[a,b]. \tag{1}$$

The only restriction for the basis functions is that they have to be at least twice continuously differentiable.

We let control point $\boldsymbol{d}_i$, $i\in\{0, 1, ..., n\}$ vary and fix the rest. Separating the fixed and varying parts of Eq.(1), we obtain the form

$$\boldsymbol{g}(u)=F_i(u)\boldsymbol{d}_i+\boldsymbol{r}_i(u),\ \boldsymbol{r}_i(u)=\sum_{j=0,j\neq i}^{n}F_j(u)\boldsymbol{d}_j. \tag{2}$$

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Definition 1 The curve $\boldsymbol{c}_i(u) = -\dot{\boldsymbol{r}}_i(u)/\dot{F}_i(u)$, $u\in[a, b]$ is the discriminant curve (or simply discriminant) of Curve (1) with respect to its control point $\boldsymbol{d}_i$, $i\in\{0, 1, \ldots, n\}$.

Based on the moving control point concept, in (Juhász, 2006) we have proved the following properties.

Theorem 1 Curve (1) has a cusp at its point $\bar{u}\in[a, b]$ iff $\boldsymbol{d}_i = \boldsymbol{c}_i(\bar{u})$.

Theorem 2 The curvature of Curve (1) vanishes at $\bar{u}\in[a,b]$ iff the control point $\boldsymbol{d}_i$ is on the tangent line of the discriminant at its point $\boldsymbol{c}_i(\bar{u})$.

Theorem 3 Points $\boldsymbol{g}(\bar{u})$, $\bar{u}\in[a,b]$ and $\boldsymbol{g}(\bar{u}+\delta)$, $\delta\in(0, b-\bar{u}]$ of Curve (1) coincide iff

$$\boldsymbol{d}_i = -\frac{\boldsymbol{r}_i(\bar{u}+\delta)-\boldsymbol{r}_i(\bar{u})}{F_i(\bar{u}+\delta)-F_i(\bar{u})}.$$

Those positions of $\boldsymbol{d}_i$ that yield self intersection of Curve (1) form a triangular surface patch, a boundary curve of which is the discriminant $\boldsymbol{c}_i(u)$.

In the preceding theorems any control point and the corresponding discriminant can be used, i.e., $i\in\{0, 1, \ldots, n\}$.

VANISHING TORSION

Our objective is to find those positions of control point $\boldsymbol{d}_i$, $i\in\{0, 1, \ldots, n\}$ that yield zero torsion at $\boldsymbol{g}(\bar{u})$, $\bar{u}\in[a, b]$ of Curve (1). Hereafter, we assume that basis functions $F_j(u)$ are at least three times continuously differentiable. Torsion

$$\tau(u) = \frac{(\dot{\boldsymbol{g}}(u), \ddot{\boldsymbol{g}}(u), \dddot{\boldsymbol{g}}(u))}{\left|\dot{\boldsymbol{g}}(u)\times\ddot{\boldsymbol{g}}(u)\right|^2}$$

vanishes at the point $\bar{u}\in[a, b]$ if one of the following conditions is fulfilled: (1) $\dot{\boldsymbol{g}}(\bar{u})=0$; (2) $\ddot{\boldsymbol{g}}(\bar{u})=0$; (3) $\dot{\boldsymbol{g}}(\bar{u})\times\ddot{\boldsymbol{g}}(\bar{u})=0$; (4) $\dddot{\boldsymbol{g}}(\bar{u})=0$; (5) vectors $\dot{\boldsymbol{g}}(\bar{u})$, $\ddot{\boldsymbol{g}}(\bar{u})$ and $\dddot{\boldsymbol{g}}(\bar{u})$ are coplanar (Certainly, this case includes the previous cases, but this separation facilitates the forthcoming explanations).

Assumptions (1)~(3) imply that control point $\boldsymbol{d}_i$ has to be on the tangent line of the discriminant

$$\boldsymbol{c}_i(u) = -\dot{\boldsymbol{r}}_i(u)/\dot{F}_i(u)$$

at $u=\bar{u}$, thus

$$\boldsymbol{d}_i = \frac{\lambda\ddot{\boldsymbol{r}}_i(\bar{u})-\dot{\boldsymbol{r}}_i(\bar{u})}{\dot{F}_i(\bar{u})-\lambda\ddot{F}_i(\bar{u})},\ \lambda\in\Re \ \text{(c.f. Theorems 1, 2).}$$

Assumption (4) implies $\boldsymbol{d}_i = -\dddot{\boldsymbol{r}}_i(\bar{u})/\dddot{F}_i(\bar{u})$.

Assumption (5) implies that $\exists\alpha,\ \beta\in\Re$, $\dot{\boldsymbol{g}}(\bar{u}) = \alpha\ddot{\boldsymbol{g}}(\bar{u})+\beta\dddot{\boldsymbol{g}}(\bar{u})$, from which we obtain

$$\begin{aligned}\boldsymbol{d}_i &= \frac{\alpha\ddot{\boldsymbol{r}}_i(\bar{u})+\beta\dddot{\boldsymbol{r}}_i-\dot{\boldsymbol{r}}_i(\bar{u})}{\dot{F}_i(\bar{u})-\alpha\ddot{F}_i(\bar{u})-\beta\dddot{F}_i(\bar{u})} \\ &= \frac{\dot{F}_i(\bar{u})[\alpha\ddot{\boldsymbol{r}}_i(\bar{u})+\beta\dddot{\boldsymbol{r}}_i]-[\alpha\ddot{F}_i(\bar{u})+\beta\dddot{F}_i(\bar{u})]\dot{\boldsymbol{r}}_i(\bar{u})}{\dot{F}_i(\bar{u})[\dot{F}_i(\bar{u})-\alpha\ddot{F}_i(\bar{u})-\beta\dddot{F}_i(\bar{u})]}-\frac{\dot{\boldsymbol{r}}_i(\bar{u})}{\dot{F}_i(\bar{u})}\end{aligned} \quad (3)$$

for the control point under consideration. Eq.(3) is a parametric surface with parameters α, β. This surface contains: (1) point $\boldsymbol{c}_i(\bar{u})$ if $\alpha=\beta=0$; (2) point $-\dddot{\boldsymbol{r}}_i(\bar{u})/\dddot{F}_i(\bar{u})$ if $\alpha=0, \beta\to\infty$ (this is a singularity of the parametrization); (3) point $-\ddot{\boldsymbol{r}}_i(\bar{u})/\ddot{F}_i(\bar{u})$ if $\beta=0$, $\alpha\to\infty$ (this is also a singularity of the parametrization).

It can also be seen that both α and β isoparametric lines of the above surface are straight lines.

Now, we show that Surface (3) is the osculating plane of the discriminant curve $\boldsymbol{c}_i(u)$ at $\bar{u}$. For conciseness, we drop the notation $(\bar{u})$ in the subsequent formulae.

The above-mentioned osculating plane is parallel to the vectors $\dot{\boldsymbol{c}}_i = (\ddot{F}_i\dot{\boldsymbol{r}}_i-\dot{F}_i\ddot{\boldsymbol{r}}_i)/\dot{F}_i^2$, and

$$\ddot{\boldsymbol{c}}_i = \frac{(\dot{F}_i\dddot{F}_i-2\ddot{F}_i^2)\dot{\boldsymbol{r}}_i+2\dot{F}_i\ddot{F}_i\ddot{\boldsymbol{r}}_i-\dot{F}_i^2\dddot{\boldsymbol{r}}_i}{\dot{F}_i^3}.$$

Therefore its normal is

$$\dot{\boldsymbol{c}}_i\times\ddot{\boldsymbol{c}}_i = \frac{\dot{F}_i^2\dddot{F}_i(\dot{\boldsymbol{r}}_i\times\ddot{\boldsymbol{r}}_i)-\dot{F}_i^2\ddot{F}_i(\dot{\boldsymbol{r}}_i\times\dddot{\boldsymbol{r}}_i)+\dot{F}_i^3(\ddot{\boldsymbol{r}}_i\times\dddot{\boldsymbol{r}}_i)}{\dot{F}_i^5}.$$

With some algebra (the usage of a computer algebra program helps a lot), it can be shown that the scalar product of this normal and the vector

$$\boldsymbol{d}_i - \boldsymbol{c}_i = \frac{\dot{F}_i(\alpha\ddot{\boldsymbol{r}}_i + \beta\dddot{\boldsymbol{r}}_i) - (\alpha\ddot{F}_i + \beta\dddot{F}_i)\dot{\boldsymbol{r}}_i}{\dot{F}_i(\dot{F}_i - \alpha\ddot{F}_i - \beta\dddot{F}_i)}$$

is zero $\forall\alpha, \beta$, i.e., $\boldsymbol{d}_i$ is in the osculating plane. Certainly, this osculating plane contains the tangent line of the discriminant curve at $\bar{u}$, thus Conditions (1)~(5) are satisfied. Therefore, we proved the following theorem.

Theorem 4 The torsion of Curve (1) vanishes at the point $\boldsymbol{g}(\bar{u})$, $\bar{u} \in [a, b]$ iff control point $\boldsymbol{d}_i$ is on the osculating plane of the discriminant at its point $\boldsymbol{c}_i(\bar{u})$, $i \in \{0, 1, \ldots, n\}$.

This means that discriminant $\boldsymbol{c}_i(u)$ provides a proper characterization not only of curvature but of torsion as well.

Curve (1) has n+1 different discriminants. In principle we can use any of them for curve interrogation and design. However, in practice the suitable choice of the discriminant makes the interrogation/design process easier. The problem is that discriminants have point(s) at infinity in general, i.e., discriminants are composed of more than one branch in general. For instance, discriminants of Bézier curves are usually rational curves, however discriminants $\boldsymbol{c}_0$ and $\boldsymbol{c}_n$ are non-rational curves [its proof can be found in (Juhász, 2006)], and consist of a single branch. The situation is very similar in the case of C-Bézier curves, where discriminants $\boldsymbol{c}_0$ and $\boldsymbol{c}_3$ consist of one branch, while discriminants $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ have two branches [c.f. (Juhász, 2006)].

BASIS FUNCTIONS SUM TO 1

If Curve (1) is a barycentric combination of its control points, i.e. $\sum_{j=0}^{n} F_j(u) = 1$, then

$$\dot{F}_i(u) = -\sum_{j=0, j\neq i}^{n} \dot{F}_j(u).$$

Thus discriminant $\boldsymbol{c}_i(u)$ is also a barycentric combination of its control points. Therefore, in case of n=3 the discriminant is always planar, consequently the torsion of Curve (1) either vanishes everywhere (planar curve) or vanishes nowhere. This result is known for cubic parametric curves, the usage of which is widespread in computer aided geometric design.

CONCLUSION

We have studied the vanishing of torsion of parametric curves defined by combination of control points and basis functions. By applying the moving control point concept, we have shown that the locus of the moving control point that yields points of zero torsion on the curve is an osculating plane of the corresponding discriminant.

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