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A quadratic programming method for optimal degree reduction of Bézier curves with G^1-continuity*

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Abstract: This paper presents a quadratic programming method for optimal multi-degree reduction of Bézier curves with G^1-continuity. The L_2 and l_2 measures of distances between the two curves are used as the objective functions. The two additional parameters, available from the coincidence of the oriented tangents, are constrained to be positive so as to satisfy the solvability condition. Finally, degree reduction is changed to solve a quadratic problem of two parameters with linear constraints. Applications of degree reduction of Bézier curves with their parameterizations close to arc-length parameterizations are also discussed.

Key words: Degree reduction, Bézier curves, Optimal approximation, G^1-continuity, Quadratic programming
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INTRODUCTION

Optimal degree reduction of Bézier curves is an important task in Computer Aided Geometric Design (CAGD). Such a process is often required in geometric modeling, due to its capability to exchange, convert or reduce data, and to compare geometric entities. It consists of approximating a given curve by another one of lower degree, and it is frequently required to preserve some continuity conditions (called constraints) at the endpoints.

Many methods have been proposed for degree reduction of Bézier curves. Watkins and Worsey (1988) used Chebyshev economization to produce the best L_∞-approximation of degree $n-1$ to a given degree n curve, but without considering endpoints interpolation. Then Ahn (2003) presented a good degree reduction in L_∞-norm with constraints of endpoints continuity by using Jacobi polynomials. Eck (1995) used constrained Legendre polynomials to minimize the L_2-norm between the two curves. Multi-degree reduction at one time avoiding stepwise computing was investigated in (Zheng and Wang, 2003; Ahn *et al.*, 2004), which showed that the optimal approximation in L_2-norm can be obtained by different approaches and that the results are equivalent. Furthermore, one can also find some other methods based on basis transformations (Lee *et al.*, 2002; Rababah *et al.*, 2006) and on the active contour model (Pottmann *et al.*, 2002).

Inspired by geometric Hermite interpolation [see e.g. (Degen, 2005)], we showed that the problem of degree reduction can be solved with constraints of G^1- and G^2-continuity (Lu and Wang, 2006a; 2006b). The main advantage is that the approximation can be further optimized by the additional parameters provided by geometric continuity. So we can obtain the approximating curve with a smaller approximation error.

In this paper, we propose to use the quadratic programming method to solve degree reduction of Bézier curves with G^1-continuity. We express the L_2-distance and the l_2-distance between two curves as a quadratic polynomial of two parameters. To meet

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the requirement of G^1-continuity, we impose the constraints of positivity on the two parameters. Then, optimal G^1-constrained degree reduction is equivalent to solving a quadratic programming problem (Gill *et al.*, 1981). The new methods are more stable and simple than previous methods. Furthermore, we also discuss applications of degree reduction by using more linear constraints.

In digital motion control, the impossible "ideal" of arc-length parameterization is desirable, since it facilitates the realization of prescribed (constant or variable) speeds along curved paths. Optimal reparameterization was studied by Farouki (1997) who introduced an optimality criterion of closeness to the arc-length parameterization. The key idea is to optimize the freedoms of reparameterization of polynomial or rational curves to achieve the approximating curve, with the first derivatives having magnitudes very close to a constant value. Costantini *et al.*(2001) developed the technique using piecewise rational curves. They showed that, for fixed knots, the optimal piecewise rational reparameterization is defined by a simple recursion relation, but that this representation is only C^0 continuous. To obtain the optimal C^1 reparameterization, the objective function is then highly nonlinear and hence does not admit a closed-form solution. Different from their approaches, our methods will generate (piecewise) C^1 continuous polynomial curves.

PRELIMINARIES

Definitions and notations

In this paper, Π_n^d denotes the space of all polynomial curves of degree n and $\|\cdot\|$ denotes the Euclidean norm $\|\boldsymbol{v}\|=\sqrt{<\boldsymbol{v},\boldsymbol{v}>}$.

A degree n Bézier curve is defined by the control points $\boldsymbol{p}_i\in\mathbb{R}^d$ in the form

$$\boldsymbol{P}(t)=\sum_{i=0}^{n}B_i^n(t)\boldsymbol{p}_i,\quad t\in[0,1],\tag{1}$$

where $B_i^n(t)=\binom{n}{i}(1-t)^{n-i}t^i$ are the Bernstein polynomials. Denoting $\boldsymbol{B}_n=[B_0^n(t),\ldots,B_n^n(t)]$ and $\boldsymbol{P}_n=[\boldsymbol{p}_0,\ldots,\boldsymbol{p}_n]^{\mathrm{T}}$, we may express Eq.(1) as $\boldsymbol{P}(t)=\boldsymbol{B}_n\boldsymbol{P}_n$.

Lemma 1 Let $\boldsymbol{M}_{m,n}=(m_{ij})$ be an $(m+1)\times(n+1)$ matrix with the elements given by

$$m_{ij}=\int_0^1 B_i^m(t)B_j^n(t)\mathrm{d}t=\frac{1}{m+n+1}\frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}},\tag{2}$$

then $\boldsymbol{M}_{m,m}$ is a real symmetric positive definite matrix.

Proof See (Lu and Wang, 2006b).

Problem statement

Given a degree n Bézier curve $\boldsymbol{P}(t)=\boldsymbol{B}_n\boldsymbol{P}_n$, the problem of G^1-constrained degree reduction is to find control points $\boldsymbol{Q}_m$, which define the approximating curve $\boldsymbol{Q}(t)=\boldsymbol{B}_m\boldsymbol{Q}_m$ of lower degree m ($3\le m<n$), such that the following two conditions are satisfied:

(1) $\boldsymbol{Q}(t)$ and $\boldsymbol{P}(t)$ are G^1 continuous at t=0,1, i.e.,

$$\boldsymbol{Q}(v)=\boldsymbol{P}(v),\ \boldsymbol{Q}'(v)/\|\boldsymbol{Q}'(v)\|=\boldsymbol{P}'(v)/\|\boldsymbol{P}'(v)\|,\ v=0,1.\tag{3}$$

(2) $\boldsymbol{Q}(t)$ minimizes a suitable distance function $d(\boldsymbol{P},\boldsymbol{Q})$ for all possible curves in Π_n^d.

We consider the L_2-distance between the two curves in the L_2-norm form as follows:

$$D_2^2(\boldsymbol{P},\boldsymbol{Q})=\int_0^1\|\boldsymbol{P}(t)-\boldsymbol{Q}(t)\|^2\,\mathrm{d}t.\tag{4}$$

We also consider the l_2-distance based on the coefficients (control points) of the two curves. In order to compare the coefficients, $\boldsymbol{Q}(t)$ has to be represented in terms of $\boldsymbol{B}_n$, i.e.,

$$\boldsymbol{Q}(t)=\boldsymbol{B}_n\hat{\boldsymbol{Q}}_n=\boldsymbol{B}_n\boldsymbol{T}_{n,m}\boldsymbol{Q}_m,$$

the degree raising operator $\boldsymbol{T}_{n,m}$ is an $(n+1)\times(m+1)$ matrix with the elements given by (Farin, 2001)

$$\boldsymbol{T}_{n,m}(i,j)=\binom{m}{j}\binom{n-m}{i-j}\Big/\binom{n}{i},\quad i=0,\ldots,n;\ j=0,\ldots,m.$$

We then express the l_2-distance in the l_2-norm form as

$$d_2^2(\boldsymbol{P},\boldsymbol{Q})=\|\boldsymbol{P}_n-\boldsymbol{T}_{n,m}\boldsymbol{Q}_m\|^2=\sum_{i=0}^{n}\|\boldsymbol{p}_i-\hat{\boldsymbol{q}}_i\|^2.\tag{5}$$

The L_2-distance measures the distance between the two curves. So, by minimizing Eq.(4), one can obtain the solution for degree reduction problem. And the l_2-distance measures the sum of the squared Euclidean distances of the corresponding coefficients. So, by minimizing Eq.(5), one solves the problem with an alternative approach. Ahn *et al.*(2004) proved that there exists a weight matrix so that the weighted l_2-distance equals the L_2-distance in C^α-constrained degree reduction of polynomial curves. However, it is obvious that this result cannot be generalized to the case of G^α-constrained degree reduction.

G^1 condition

Clearly, for G^0-continuity, the endpoints of $\boldsymbol{Q}(t)$ should coincide with the endpoints of $\boldsymbol{P}(t)$. And for G^1-continuity, the coincidence of the oriented tangents is additionally required. From Eq.(3), we have

$$\begin{cases} \boldsymbol{q}_0 = \boldsymbol{p}_0, & \boldsymbol{q}_1 = \boldsymbol{p}_0 + n\delta_0(\boldsymbol{p}_1 - \boldsymbol{p}_0)/m, \\ \boldsymbol{q}_m = \boldsymbol{p}_n, & \boldsymbol{q}_{m-1} = \boldsymbol{p}_n - n\delta_1(\boldsymbol{p}_n - \boldsymbol{p}_{n-1})/m. \end{cases} \tag{6}$$

Noting that geometric continuity does not depend on the chosen parameterization, the control point $\boldsymbol{q}_1$ can thus move along the direction $\boldsymbol{p}_0\boldsymbol{p}_1$ without violating G^1 condition (Fig.1), and the same holds for $\boldsymbol{q}_{m-1}$. When replacing G^1-continuity with C^1-continuity, $\boldsymbol{q}_1$ and $\boldsymbol{q}_{m-1}$ are uniquely determined by $\delta_v=1$. Thus it provides two additional parameters to optimize the shape of the degree reduced curve. However, δ_v should obey the following rule:

$$\delta_v > 0,\ v = 0,1. \tag{7}$$

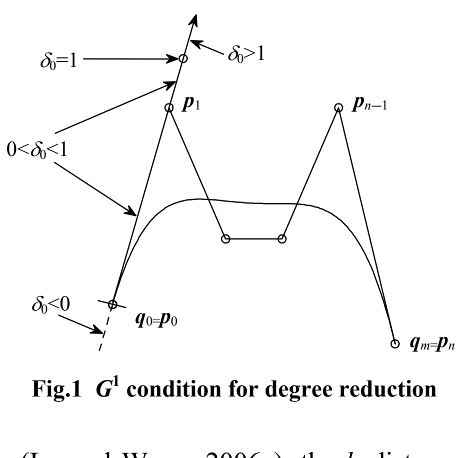

Fig.1 G^1 condition for degree reduction

In (Lu and Wang, 2006a), the l_2-distance combined with regularization terms is used as the objective function. Then δ_v are solved from a linear system and satisfy the condition (7). However, one has to adjust the balance factor once (7) is violated, which controls the ratio between the l_2-distance and regularization terms. So, it is often an experience-oriented and trial-and-error process, which is time-consuming. And in (Lu and Wang, 2006b), the L_2-distance is used as the objective function. The condition (7) is always satisfied since δ_v in Eq.(6) are replaced by δ_v^2. The role of regularization terms is only to avoid the singularities incurred when δ_v is (nearly) equal to zero. The only limitation of such strategy is that Eq.(4) turns out to be a quartic system with two parameters.

DEGREE REDUCTION BY L_2-NORM

Given a degree n Bézier curve $\boldsymbol{P}(t)$, the problem of G^1-constrained degree reduction can be solved through two stages. In the first stage, we construct a degree m Bézier curve $\boldsymbol{Q}(t)$ interpolating $\boldsymbol{P}(t)$ according to Eq.(6). More precisely, it can be written as

$$\boldsymbol{Q}(t) = B_0^m(t)\boldsymbol{q}_0 + B_1^m(t)\boldsymbol{q}_1 + \sum_{i=2}^{m-2} B_i^m(t)\boldsymbol{q}_i + B_{m-1}^m(t)\boldsymbol{q}_{m-1} + B_m^m(t)\boldsymbol{q}_m, \tag{8}$$

where $\boldsymbol{q}_1$ and $\boldsymbol{q}_{m-1}$ contain the unknown parameters δ_0 and δ_1, respectively.

The first stage is the same as that detailed in (Lu and Wang, 2006b), we report the main results as follows. We assume that the free parameters δ_0 and δ_1 are temporarily fixed, then solve the interior control points $\boldsymbol{q}_i$ (i=2,…,m−2) by minimizing the L_2-distance

$$\begin{aligned} D_2^2(\boldsymbol{P},\boldsymbol{Q}) &= \int_0^1 \left\| \boldsymbol{B}_n\boldsymbol{P}_n - \boldsymbol{B}_m\boldsymbol{Q}_m \right\|^2 \mathrm{d}t \\ &= \int_0^1 \left\| \boldsymbol{B}_n\boldsymbol{P}_n - \boldsymbol{B}_m^{\mathrm{c}}\boldsymbol{Q}_m^{\mathrm{c}} - \boldsymbol{B}_m^{\mathrm{f}}\boldsymbol{Q}_m^{\mathrm{f}} \right\|^2 \mathrm{d}t, \end{aligned} \tag{9}$$

where $\boldsymbol{Q}_m^{\mathrm{c}} = [\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_{m-1}, \boldsymbol{q}_m]^{\mathrm{T}}$ and $\boldsymbol{Q}_m^{\mathrm{f}}$ denotes the other control points of $\boldsymbol{Q}_m$.

By the least squares method, Eq.(9) is minimized by choosing

$$\boldsymbol{Q}_m^{\mathrm{f}} = \boldsymbol{Q}_m^{\mathrm{f}}(\delta_0,\delta_1) = (\boldsymbol{M}_{m,m}^{\mathrm{f}})^{-1}(\boldsymbol{M}_{m,n}^{\mathrm{f}}\boldsymbol{P}_n - \boldsymbol{M}_{m,m}^{\mathrm{c}}\boldsymbol{Q}_m^{\mathrm{c}}), \tag{10}$$

where

$$\boldsymbol{M}_{m,m}^{\mathrm{c}} := \boldsymbol{M}_{m,m}(2,\ldots,m-2;\ 0,1,m-1,m),$$
$$\boldsymbol{M}_{m,m}^{\mathrm{f}} := \boldsymbol{M}_{m,m}(2,\ldots,m-2;\ 2,\ldots,m-2),$$
$$\boldsymbol{M}_{m,n}^{\mathrm{f}} := \boldsymbol{M}_{m,n}(2,\ldots,m-2;\ 0,\ldots,n).$$

Here, the notation $\boldsymbol{A}(\ldots;\ \ldots)$ denotes the submatrix of the matrix $\boldsymbol{A}$ obtained by extracting the specific rows and columns. Note that $\boldsymbol{q}_1$ and $\boldsymbol{q}_{m-1}$ are linear functions of the parameters δ_0 and δ_1, respectively. Therefore, by Eq.(10), the interior control points $\boldsymbol{q}_i$ (i=2,…,m−2) are linear functions of δ_v.

The second stage is then to determine the two free parameters δ_v such that Eq.(9) is minimized. Recall that $\boldsymbol{Q}_m = \boldsymbol{Q}_m^{\mathrm{c}} \cup \boldsymbol{Q}_m^{\mathrm{f}}$. By Eqs.(6) and (10), we can express the control points $\boldsymbol{q}_i$ (i=1,…,m−1) in linear functions of δ_v, rather than constraints. After substituting them into Eq.(9), it forms a quadratic polynomial with two parameters.

The problem is now to force δ_v to satisfy the condition (7). However, the approximating curve will become singular at the endpoint when δ_v is nearly equal to zero, which is obviously undesirable in shape designing. For this reason, we use the following constraints:

$$\delta_0 \geq l_0,\ \ \delta_1 \geq l_1, \tag{11}$$

where l_v (v=0,1) are two positive lower bounds and usually prescribed to small values to obtain the optimal approximation. And it is not necessary to impose upper bounds on δ_v, because the L_2-distance cannot reach the minimum when they are large enough.

Finally, we change the minimization of Eq.(9) with the constraints (11) to a quadratic programming problem with two parameters. We use the 'quadprog' procedure of MATLAB to solve it.

After replacing all the parameters δ_v in $\boldsymbol{Q}_m$ with the values solved above, we obtain the multi-degree reduced approximating curve $\boldsymbol{Q}(t)$ which preserves G^1-continuity at the endpoints. We now summarize the algorithm for degree reduction as follows:

Step 1: Express $\boldsymbol{q}_1$ and $\boldsymbol{q}_{m-1}$ by Eq.(6) and $\boldsymbol{q}_i$ (i=2,…,m−2) by Eq.(10) in terms of δ_v;

Step 2: Solve δ_0 and δ_1 from the minimum of Eq.(9) with the constraints (11) using the quadratic programming method;

Step 3: Compute $\boldsymbol{q}_i$ (i=1,…,m−1) by Eqs.(6) and (10) and the L_2-distance by Eq.(9).

DEGREE REDUCTION BY l_2-NORM

Similar to that described in the above section, degree reduction by l_2-norm is also processed in two stages. Firstly, with the assumption that the free parameters (δ_0 and δ_1) are temporarily fixed, the problem is now to minimize the l_2-distance

$$\begin{aligned} d_2^2(\boldsymbol{P},\boldsymbol{Q}) &= \left\|\boldsymbol{P}_n - \boldsymbol{T}_{n,m}\boldsymbol{Q}_m\right\|^2 \\ &= \left\|\boldsymbol{P}_n - \boldsymbol{T}_{n,m}^{\mathrm{c}}\boldsymbol{Q}_m^{\mathrm{c}} - \boldsymbol{T}_{n,m}^{\mathrm{f}}\boldsymbol{Q}_m^{\mathrm{f}}\right\|^2, \end{aligned} \tag{12}$$

where

$$\boldsymbol{T}_{n,m}^{\mathrm{c}} := \boldsymbol{T}_{n,m}(0,\ldots,n;\ 0,1,m-1,m),$$
$$\boldsymbol{T}_{n,m}^{\mathrm{f}} := \boldsymbol{T}_{n,m}(0,\ldots,n;\ 2,\ldots,m-2).$$

From the result in (Lu and Wang, 2006a), Eq.(12) is minimized by choosing

$$\begin{aligned} \boldsymbol{Q}_m^{\mathrm{f}} &= \boldsymbol{Q}_m^{\mathrm{f}}(\delta_0,\delta_1) \\ &= [(\boldsymbol{T}_{n,m}^{\mathrm{f}})^{\mathrm{T}}\boldsymbol{T}_{n,m}^{\mathrm{f}}]^{-1}(\boldsymbol{T}_{n,m}^{\mathrm{f}})^{\mathrm{T}}(\boldsymbol{P}_n - \boldsymbol{T}_{n,m}^{\mathrm{c}}\boldsymbol{Q}_m^{\mathrm{c}}). \end{aligned} \tag{13}$$

Secondly, after substituting Eq.(13) into Eq.(12), we change the minimization of Eq.(12) with the constraints (11) to the quadratic programming problem with two parameters. The algorithm is similar to that in Section 3, except that the l_2-distance is used as the objective function instead of the L_2-distance. Thus, we omit it for simplicity.

LOCAL ADJUSTMENT NEAR THE ENDPOINTS

Besides the lower bounds (11) of δ_v, we can impose more constraints on them, e.g.,

$$l_0 \leq \delta_0 \leq u_0,\ \ l_1 \leq \delta_1 \leq u_1, \tag{14}$$

where the lower bounds l_v and the upper bounds u_v can be arbitrary positive values. Obviously, (14) implies the equality constraints when l_v=u_v.

By the constraints (14), the derivative magnitudes of the approximating curve at the endpoints can be limited in the specified ranges. That is, we have

$$\left\|\boldsymbol{Q}'(v)\right\| \in \left[l_v\left\|\boldsymbol{P}'(v)\right\|,\ u_v\left\|\boldsymbol{P}'(v)\right\|\right],\ \ v=0,1.$$

In practical applications, we may deal with the quadratic programming problem formed by the minimization of Eq.(9) or Eq.(12) with the linear constraints (11) and (14). It is very convenient to adjust the approximating curve and control the effect near the endpoints by regulating the parameters in the constraints. However, we should mention that the resulting approximation might be not optimal, and that the L_2- or l_2-distance might become bigger than that by using only the constraints (11).

APPLICATION EXAMPLES

We consider the planar quintic Bézier curve with the control points given by (0.5, 0), (0, 0.5), (2, 5), (5, 5), (8, 3), (5, 0).

Fig.2 compares G^1-constrained degree reduction with C^1-constrained one (Ahn *et al.*, 2004), with the quartic and cubic approximations shown in Figs.2a and 2b, respectively. We only use (11) as the constraints to meet G^1 condition and set $l_v=10^{-4}$. Clearly, our methods approximate the whole curve better, due to the G^1-continuity. More examples of comparison can also be found in (Lu and Wang, 2006a; 2006b).

For a given curve, two distinct approximating curves are generally generated by the two methods in this paper. Although it is hard to say which method always gives better approximation in any case, it is noted from our experiments that degree reduction by L_2-norm usually produces better result (cf. Fig.2).

Our methods can be applied to obtain approximating curves with improved distribution of derivative magnitudes, which is accomplished by regulating the lower and upper bounds in the constraints (14). First of all, we determine the bounds of δ_v according to practical applications. Then, the corresponding approximating curve will follow the change near the endpoints. And subdivision techniques can also be adopted to obtain closer approximations to the arc-length parameterization.

In Fig.3a, we use the following constraints

$$c_v - 0.1c_v \le \delta_v \le c_v + 0.1c_v, \quad v = 0,1,$$

where the mean values c_v of the ranges are computed from the condition

$$c_v \|\boldsymbol{P}'(v)\| = \frac{1}{2}\left(\|\boldsymbol{P}'(0)\| + \|\boldsymbol{P}'(1)\|\right).$$

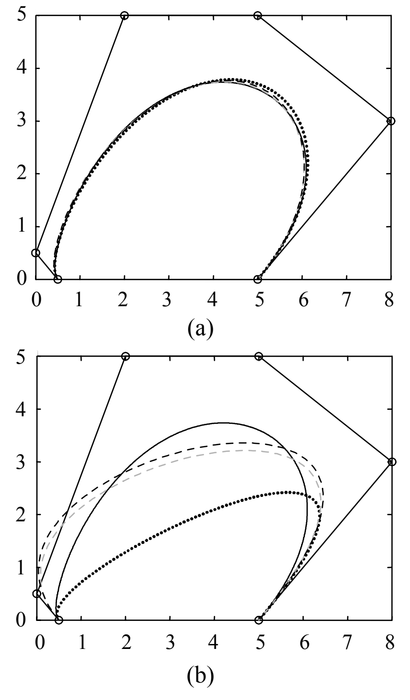

Fig.2 Degree reduction of the quintic Bézier curve (solid). Dashed: the method by L_2-norm; gray: the method by l_2-norm; dotted: the method in (Ahn *et al.*, 2004). (a) Degree 5 to degree 4; (b) Degree 5 to degree 3

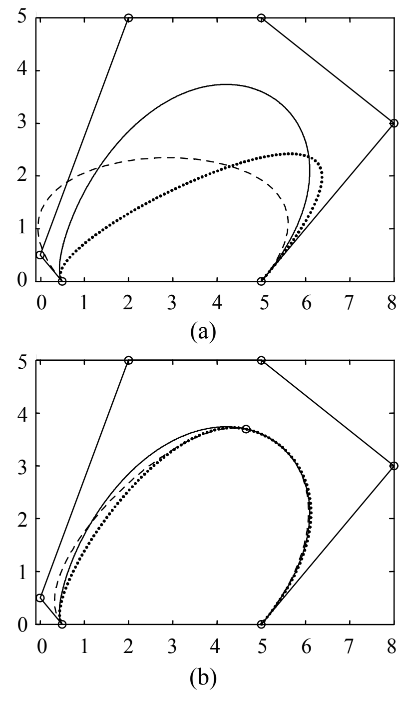

Fig.3 Cubic polynomial approximations with local adjustment near the endpoints. Dashed: the method by L_2-norm; dotted: the method in (Ahn *et al.*, 2004). (a) Without subdivision; (b) With one subdivision

The amplitudes $\pm 0.1c_v$ are available to optimize the approximation. Fig.4a illustrates the derivative magnitudes of the curves in Fig.3a.

To improve the approximation, we subdivide the given curve at t_s=0.6 and show the results in Figs.3b and 4b. The two intermediate endpoints are constrained by equality constraints so that the approximating curves will have the same first derivatives at the endpoints. The other two endpoints are constrained similarly as above. As remarked in (Lu and Wang, 2006b), the resulting curves will be G^1 (not C^1) continuous at the breakpoints, if C^1-constrained degree reduction methods are used with the combination of non-uniform subdivision schemes (see Fig.4b).

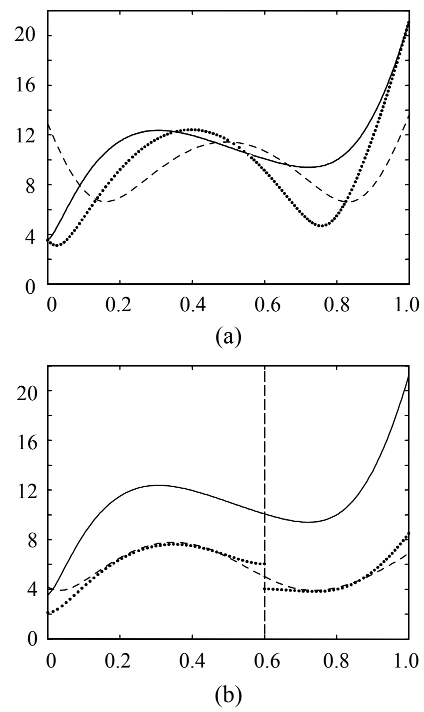

Fig.4 Magnitudes of the first derivatives of the curves shown in Fig.3. Dashed: the method by L_2-norm; dotted: the method in (Ahn *et al.*, 2004). (a) Without subdivision; (b) With one subdivision

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