Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Relation among C-curve characterization diagrams*

CAO Juan†, WANG Guo-zhao

(*Institute of Computer Graphics and Image Processing, Department of Mathematics, Zhejiang University, Hangzhou 310027, China*)

†E-mail: cccjjqm@163.com

Received Jan. 25, 2007; revision accepted Apr. 3, 2007

Abstract: As three control points are fixed and the fourth control point varies, the planar cubic C-curve may take on a loop, a cusp, or zero to two inflection points, depending on the position of the moving point. The plane can, therefore, be partitioned into regions labelled according to the characterization of the curve when the fourth point is in each region. This partitioned plane is called a "characterization diagram". By moving one of the control points but fixing the rest, one can induce different characterization diagrams. In this paper, we investigate the relation among all different characterization diagrams of cubic C-curves based on the singularity conditions proposed by Yang and Wang (2004). We conclude that, no matter what the C-curve type is or which control point varies, the characterization diagrams can be obtained by cutting a common 3D characterization space with a corresponding plane.

Key words: Spline, C-curve, Characterization diagram, Singularity
doi:10.1631/jzus.2007.A1663 **Document code:** A **CLC number:** TP391.72

INTRODUCTION

A curve with a loop is self-intersecting and one with a cusp has a point where the unit tangent vector is discontinuous. A planar curve with an inflexion has a point where the curvature vanishes and the sign of the curvature changes in its neighborhood. In practical applications, it is often necessary to detect cusps, loops and inflection points of the curve. This problem has been studied before by other researchers from different points of view. Wang (1981) presented the conditions which lead to an inflection point or singularity for parametric cubic curves. Su and Liu (1983) and Sakai (1999) obtained the distribution of inflection points and a singularity of planar cubic Bézier curves and rational cubic Bézier curves respectively. Manocha and Canny (1992) studied this problem for polynomial and rational parametric curves of arbitrary degree, while Li and Cripps (1997) focused on rational curves. Monterde (2001) concluded that if an nth degree rational Bézier curve has a singular point, then this singular point also belongs to the two $(n-1)$th degree rational Bézier curves defined in the $(n-1)$th step of the de Casteljau algorithm. Yang and Wang (2004) proposed the conditions leading to inflections or singularities for planar cubic C-curves (Pottmann, 1993; Zhang, 1996; 1997; 1999; Zhang and Krause, 2005; Zhang *et al*., 2005).

In addition to the work mentioned above, there are two publications of Stone and DeRose (1989) and Juhász (2006), which are closely related to the work of this paper. The former studied the geometric characterization of planar cubic polynomial curves based on the observation that the moving control point determines the shape of a cubic curve when the other three control points are fixed. By contrast, the latter set the problem into a bit more general context. Juhász (2006) examined parametric curves that can be described by combination of control points of arbitrary dimension and basis functions. Specially, Juhász pointed out that a cubic polynomial curve or C-curve contains a singularity, one or two inflection points only in planar cases.

* Project supported by the National Natural Science Foundation of China (No. 60473130) and the National Basic Research Program (973) of China (No. 2004CB318000)

As we know, the locus of the moving control point that yields a cusp, a loop, one inflection point, two inflection points or none of the above, segments the plane into different regions. This segmented plane is called a characterization diagram. The different designations of the moving control point yield different characterization diagrams. An interesting consideration arising naturally is the relation among all these different characterization diagrams. It has been studied by Stone and DeRose (1989) for cubic polynomial parametric curves. However, the relation among the characterization diagrams of planar cubic C-curves is still unknown. This is just what we will discuss in this paper.

Our work is based on the conditions leading to inflection points and singularities of the planar cubic C-curves obtained by Yang and Wang (2004). We first investigate the relation among all characterization diagrams of C-Bézier curves. Then we extend the conclusion to all kinds of C-curves. We conclude that each characterization diagram is a 2D slice through a common 3D characterization space, i.e., it can be obtained by intersecting the 3D characterization space with an appropriate plane. Additionally, the implicit expression of the cutting plane can be easily obtained from its known parametric form.

This paper is organized as follows. In Section 2, we restate the conditions leading to inflection points and singularities of the planar cubic C-curves. Section 3 is devoted to obtaining the characterization diagrams. We reveal the relation among all the characterization diagrams in Section 4. Section 5 concludes the paper.

C-CURVES AND SHAPE CLASSIFICATION

C-curves with basis containing trigonometric functions are one kind of more flexible curves than polynomial curves. They are shape adjustable and can represent precisely the circular (or elliptical) arc, the cylinder, the helix, the cycloid and so on. A planar cubic C-curve, including a C-Bézier curve, a segment of CB-spline and a C-Ferguson curve, can be represented as (Zhang, 1996):

$$\boldsymbol{P}(t)=\sum_{i=0}^{3}\boldsymbol{p}_i d_i(t), \quad 0\le t\le \alpha, \tag{1}$$

where α=arccos C_α with $-1\le C_\alpha<1$, $\boldsymbol{p}_i$ and $d_i(t)$ are respectively the planar control point and basis function. For convenience, we write it in a more general form:

$$\boldsymbol{P}(t)=\boldsymbol{P}_0\sin t+\boldsymbol{P}_1\cos t+\boldsymbol{P}_2 t+\boldsymbol{P}_3,\ 0\le t\le\alpha, \tag{2}$$

where $\boldsymbol{P}_i$ is a linear combination of control points $\boldsymbol{p}_i$.

We introduce some notations as follows:

$$A=\det(\boldsymbol{P}_1,\boldsymbol{P}_2),\ B=\det(\boldsymbol{P}_2,\boldsymbol{P}_0),\ C=\det(\boldsymbol{P}_0,\boldsymbol{P}_1), \tag{3}$$
$$\Delta=A^2+B^2-C^2,$$
$$b=|C|\cdot(A^2+B^2)^{-1/2},\ \ c=(A^2+B^2)^{-1/2},$$
$$\beta=\arccos b,\ \ b\le 1,\ \ a=\cot\beta=|C|/\sqrt{\Delta},$$

and choose ϖ such that $\tan\varpi=B/A$ and $\varpi\in[-\pi/2,\pi/2]$ or $\varpi\in(\pi/2,3\pi/2)$ depending on whether $AC\ge0$ or $AC<0$. When $A=B=0$, the curve degenerates into a straight line or an ellipse. So we ignore these cases and assume that $A^2+B^2\ne0$ in this section.

Yang and Wang (2004) investigated the occurrence of inflection points and singularities of C-curves based on the observation that the C-curves are affine images of trochoids or sine curves. Apart from minor changes in notation and wording, the proposed conditions leading to inflection points and singularities are as follows:

Lemma 1 Curve (1) with $t\in(0,\alpha)$ has two inflection points if and only if $b<1$ and $\varpi\in(\beta-\alpha,-\beta)$. There is one inflection point if and only if $b<1$ and

$$\varpi>2\pi-\beta-\alpha,$$

or

$$\begin{cases}-\beta-\alpha<\varpi<-\beta,\\ \varpi\le\beta-\alpha,\end{cases}$$

or

$$\begin{cases}\beta-\alpha<\varpi<\beta,\\ \varpi\ge-\beta.\end{cases}$$

There is a cusp if and only if $b=1$ and $\varpi\in(-\alpha,0)\cup(2\pi-\alpha,3\pi/2)$. There is a loop if and only if $b>1$ and $\varpi\in(\tau-\alpha,\ -\tau)\cup(2\pi+\tau-\alpha,\ 2\pi-\tau)$, where $\tau>0$ is the smallest zero of function $g(t)=t/\sin t-b$.

Note that the conditions in Lemma 1 are expressed in the form of angle relations. As a preparation work for the later discussion, we translate these angle relations into the relations of trigonometric function values. At the same time, we split Lemma 1 into four propositions, respectively corresponding to the conditions leading to a cusp, a loop, one inflection point and two inflection points. The local monotonicity of the trigonometric functions gives the equivalence of the transformations. We only provide the proof of Proposition 4, and proofs of others are omitted since they are similar. The propositions are listed as follows, where the condition presented before the parenthesis can be replaced by the one within it:

Proposition 1 For there to be one inflection point on curve (1) with $t\in(0, \alpha)$, the conditions are $\Delta>0$ and one of the following conditions holds:

(1) $AC<0$: $c\sqrt{\Delta}>\max\{\cos\alpha,0\}$, $\dfrac{B}{A}>\dfrac{B+A\tan\alpha}{B\tan\alpha-A}$;

(2) $AC\geq 0$, $b>\cos(\alpha/2)$:

$$-\frac{1}{a}<\frac{B+A\tan\alpha}{A-B\tan\alpha}\leq\frac{1}{a}\quad\left(-\frac{1}{a}\leq B/A<\frac{1}{a}\right);$$

(3) $AC\geq 0$, $b\leq\cos(\alpha/2)$:

$$\frac{B+aA}{A-aB}<-\cot\alpha\quad\left(\frac{B-aA}{A+aB}<-\cot\alpha\right).$$

Proposition 2 For there to be two inflection points on curve (1) with $t\in(0,\alpha)$, the conditions are $\Delta>0$, $AC\geq 0$ and

(1) $\alpha\in(0,\pi/2]$: $-a<\dfrac{A}{B}<\dfrac{\tan\alpha+a}{a\tan\alpha-1}$;

(2) $\alpha\in(\pi/2,\pi]$: $b\leq\sin\alpha$, $-a<\dfrac{A}{B}<\dfrac{\tan\alpha+a}{a\tan\alpha-1}$,

or $b>\sin\alpha$, $-a<A/B\leq 0$.

Proposition 3 For there to be cusp on curve (1) with $t\in(0,\alpha)$, the conditions are $\Delta=0$ and

(1) $\alpha\in(0,\pi/2]$: $AC>0$, $-\tan\alpha<B/A<0$;

(2) $\alpha\in(\pi/2,\pi]$: $AC\geq 0$, $B/A<0$,

or $AC<0$, $-\tan\alpha<B/A$.

Proposition 4 For there to be loop on curve (1) with $t\in(0,\alpha)$, the conditions are $\Delta<0$, $b<\dfrac{\alpha/2}{\sin(\alpha/2)}$ and

(1) $\alpha\in(0,\pi/2]$: $AC\geq 0$, $B/A<0$ and

$$b<\min\left\{\frac{\alpha+\arctan(B/A)}{\sin(\alpha|A|)-\cos(\alpha|B|)},-\frac{\arctan(B/A)}{|B|}\right\}c;$$

(2) $\alpha\in(\pi/2,\pi]$: $AC\geq 0$, $B/A<0$ and

$$b<\min\left\{\frac{\pi/2-\alpha}{\cos\alpha},-\frac{\arctan(B/A)}{|B|}\right\}c,\quad\frac{\pi/2-\alpha}{\cos\alpha}\leq b<$$

$$\min\left\{-\frac{\pi+\arctan(B/A)}{|B|},\frac{\pi+\alpha+\arctan(B/A)}{\sin(\alpha|A|)-\cos|B|}\right\}c;$$

or $AC<0$, $B/A>0$,

$$\frac{|C|}{A^2+B^2}<\min\left\{\frac{\pi/2-\alpha}{\cos\alpha},\frac{-\pi+\alpha+\arctan(B/A)}{\sin(\alpha|A|)+\cos|B|}\right\}c.$$

Proof (of Proposition 4) Without loss of generality, we suppose $\alpha\in(0,\pi/2]$. Recall Lemma 1 that there exists a loop if and only if $b>1$ ($\Leftrightarrow\Delta<0$) and

$$\varpi\in(\tau-\alpha,-\tau)\cup(2\pi+\tau-\alpha,2\pi-\tau)\,. \tag{4}$$

Note that $t/\sin t$ is a monotonic function in $(0,\pi)$ and $(-\pi,0)$. Hence, in order to ensure that the intervals in Eq.(4) make sense, we have:

$$\tau-\alpha<-\tau\Leftrightarrow 0<\tau<\frac{\alpha}{2}\Leftrightarrow b<\frac{\alpha/2}{\sin(\alpha/2)}.$$

Note that, according to the way we choose ϖ, the relation $\varpi\in(\tau-\alpha,-\tau)$ requires $AC\geq 0$ with $\varpi\in[-\pi/2,\pi/2]$. Together with the inequality $\tau-\alpha>-\pi/2$, we have

$$\varpi\in(\tau-\alpha,-\tau)=(\tau-\alpha,-\tau)\cap[-\pi/2,\pi/2],$$

which is equivalent to

$$B/A<0\ \text{and}\ \tau<\min\{\alpha+\varpi,\varpi\}\,. \tag{5}$$

Because of the monotonicity of $t/\sin t$, the second inequality of Eq.(5) is equivalent to

$$b<\min\left\{\frac{\alpha+\arctan(B/A)}{\sin(\alpha|A|)-\cos(\alpha|B|)},-\frac{\arctan(B/A)}{|B|}\right\}c.$$

GEOMETRIC CHARACTERIZATIONS OF CUBIC C-CURVES

This section shows how to obtain characterization diagrams of C-curves. For convenience, we take the case of C-Bézier curves as an example.

Generally, for a planar cubic curve, three of the control points can be affinely mapped to three specific locations on the plane. The characterization of this curve is completely determined by the position of its fourth control point. This image curve is called the canonical curve. Since the singularity of a C-curve is independent of the affine map, our analysis is applied to the canonical curves instead of the original for simplicity. The later discussion will imply that the choice of canonical curves instead of original ones by no means affects the generalization of the conclusion.

We can generally get the canonical curves by mapping the three fixed control points to (0,0), (0,1), (1,1). However, if the control points are collinear or coincident, the affine map becomes invalid. For completeness, these degenerate cases are briefly discussed in the second subsection.

Characterization diagrams of C-Bézier curves

A planar C-Bézier curve is defined as (Zhang, 1996)

$$P(t)=\frac{1}{\alpha-S}[\sin t \quad \cos t \quad t \quad 1] \cdot \begin{bmatrix} C_\alpha & 1-C_\alpha-M & M & -1 \\ -S & (\alpha-K)M & -KM & 0 \\ -1 & M & -M & 1 \\ \alpha & -(\alpha-K)M & KM & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}, \tag{6}$$

where $0 \le t \le \alpha$, $\boldsymbol{p}_i$ (i=0,1,2,3) are four control points in the plane,

$$\alpha=\arccos C_\alpha,\ S=\sin\alpha,\ K=\frac{\alpha-S}{1-C_\alpha},\ -1\le C_\alpha<1,$$

$$M=\begin{cases} 1, & C_\alpha=0, \\ \dfrac{S}{\alpha-2K}=\dfrac{S(1-C_\alpha)}{2S-\alpha-\alpha C_\alpha}, & \text{otherwise.} \end{cases}$$

We can choose an appropriate affine map such that $\boldsymbol{p}_0$=(0,0), $\boldsymbol{p}_1$=(0,1), $\boldsymbol{p}_2$=(1,1). Let the fourth moving control point be $\boldsymbol{p}_3=(x,y)$. Then, $\boldsymbol{p}_i$ of the general form Eq.(2), and A, B, C defined in Eq.(3) can be obtained by direct computation as:

$$A=m_1x+m_2y+m_3, \tag{7a}$$

$$B=m_4x+m_5, \tag{7b}$$

$$C=m_1x+m_2y+m_3+m_6, \tag{7c}$$

where

$$m_1=-(\alpha-2K)Mm,\ m_2=-KMm,\ m_3=-m_1M,$$
$$m_4=(1-C_\alpha)m,\ m_5=-m_4M,\ m_6=m_4Km,$$
$$m=(\alpha-S)^{-2}.$$

Obviously, A, B and C are linear with respect to parameters x and y.

Without loss of generality, we suppose $\alpha\in(0,\pi/2]$. Based on the propositions in Section 2 and Eqs.(7a)~(7c), we can get the conditions for there to be a cusp, a loop or one to two inflection points on this canonical curve. For example, if Eqs.(7a)~(7c) satisfy $\Delta=0$, $AC>0$ and $-\tan\alpha<B/A<0$, there is a cusp on the curve. Substituting Eqs.(7a)~(7c) into $\Delta=0$, we get a parabola (the red curve in Fig.1, see P1668):

$$a_0x^2+a_1x+a_2y+a_3=0, \tag{8}$$

where

$$\begin{cases} a_0=1-C_\alpha, \\ a_1=2M(-2K^2M+\alpha KM+C_\alpha-1), \\ a_2=2K^2M^2, \\ a_3=-M^2[(-4M-C+1)K^2+2\alpha MK+C_\alpha-1]. \end{cases}$$

Similarly, substituting Eqs.(7a)~(7c) into $AC>0$ and $-\tan\alpha<B/A<0$ induces a blue region in Fig.1. It implies that when point $\boldsymbol{p}_3$, which moves along the parabola Eq.(8), is located in the blue region, there will be a cusp on the curve. The regions under other circumstances (a loop, one inflection point or two inflection points) can be determined in a similar fashion. So we only describe the result in Fig.2 (see P1668) but omit the deduction process. In addition, the resulting characterization diagrams given by designating a different moving control point can be obtained analogously. Likewise, we can compute the characterization diagrams for other kinds of C-curves, such as a segment of a CB-spline or a C-Ferguson curve.

Degenerate case

If all control points are collinear or some of them are coincident, not one of the three can be mapped to the canonical form. The mapping therefore becomes degenerate and the singularity of the curve cannot be analyzed based on the characterization diagram.

If all the control points are collinear but no two points coincide, the curve is a straight line. However, with the variation of t, the corresponding point on the line varies with the order of the control points. That is to say, if the four control points are not arranged in the order of their subscripts along a straight line, then there may be some folds on this line (see Fig.3).

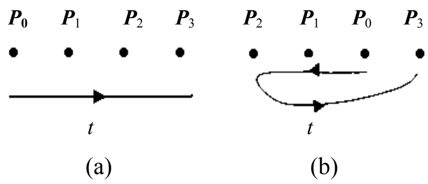

Fig.3 The motion of *t* of two cases for collinear cubic C-curve. (a) Monotonic motion of *t*; (b) *t* changes the direction and the line has a fold on it

If $\boldsymbol{p}_1=\boldsymbol{p}_2$ but they do not coincide with the others, this curve has an inflection point at each end. Otherwise, if $\boldsymbol{p}_0=\boldsymbol{p}_1$ (resp. $\boldsymbol{p}_2=\boldsymbol{p}_3$) but is disjoint to the others, this curve has a zero length tangent vector and zero curvature at $\boldsymbol{p}_0$ (resp. $\boldsymbol{p}_3$). If $\boldsymbol{p}_0=\boldsymbol{p}_1=\boldsymbol{p}_3$ (resp. $\boldsymbol{p}_0=\boldsymbol{p}_2=\boldsymbol{p}_3$), this curve becomes a straight line, with t varying non-monotonically along it. If the first (last) three control points are coincident, this curve also behaves as a straight line, and hence has a zero length tangent vector and zero curvature at corresponding ends. If $\boldsymbol{p}_0=\boldsymbol{p}_1$ and $\boldsymbol{p}_2=\boldsymbol{p}_3$, the curve is still a line and t monotonically varies along this line. Finally, if all control points are coincident, the curve collapses to a single point.

RELATION AMONG CHARACTERIZATION DIAGRAMS

In order to find out how the characterization diagrams of C-curves are related, we first focus our discussion on C-Bézier curves, and then extend the conclusion to C-curves of all types.

Relation among characterization diagrams of C-Bézier curves

All the diagrams in Section 3 are two dimensinal and the relation among them is ambiguous. In this subsection, we investigate the relation among them in a common 3D space, making the conclusion more intuitive.

Recall an observation in Section 2 that the characterization of the curve can be completely determined by A, B and C. By direct calculation, the curvature of C-Bézier curves can be obtained as

$$\kappa(t)=[A,B,C]\cdot[-\cos t,\sin t,1]\big/|\boldsymbol{P}'(t)|^3.$$

A, B and C essentially determine the sign of the curvature. As can be seen, A, B and C are three important quantities of the cubic C-curves. Thus, we associate every planar cubic C-curve with a point (A, B, C) in a 3D space. As stated above, the curve has a cusp if $\Delta=0$, $AC>0$ and $-\tan\alpha<B/A<0$. Therefore, geometrically, we say that the 3D point corresponding to this curve in the common space is at the intersection of the surface defined by $A^2+B^2-C^2=0$ (a cone, see Fig.4) and volumes defined by $AC>0$ and $-\tan\alpha<B/A<0$. In Fig.4, the portion of the cone which is located in the volume defined by $AC>0$ and $B/A<0$ is colored orange. The purple plane is defined by $B+A\tan\alpha=0$.

Likewise, according to Proposition 4, the curve has a loop on it if its moving control point is located in the volumes defined by $AC\geq0$, $B/A<0$, and

$$b<\min\left\{\frac{\alpha+\arctan(B/A)}{\sin(\alpha|A|)-\cos(\alpha|B|)},-\frac{\arctan(B/A)}{|B|}\right\}c.$$

What is more, two special boundary surfaces of these volumes can be obtained by replacing "<" of the last inequality by "=", i.e.,

$$\begin{gathered}AC\geq0,\ \ B/A<0,\ \text{ and }\ \frac{|C|}{A^2+B^2}=\\ \min\left\{\frac{\alpha+\arctan(B/A)}{\sin(\alpha|A|)-\cos(\alpha|B|)},-\frac{\arctan(B/A)}{|B|}\right\}.\end{gathered}\tag{9}$$

It means that if the moving control point is located on curve (8), the curve has a loop at $t=0$ or $t=\alpha$. Similarly, the boundary surfaces of the volumes corresponding to cases of one inflection point and two inflection points can be easily computed from Propositions 1 and 2. We call these boundary surfaces the "characterization surfaces" of the common 3D space (Fig.4, three characterization surfaces are shown).

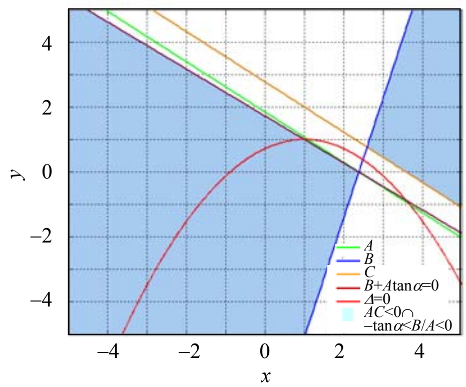

Fig.1 The cusp curve of cubic C-Bézier curve when α=1.5. The red curve is the image of Δ=0 and the blue regions are the zones where AC>0 and $-\tan\alpha<B/A<0$. If p_3 is not only located on the red curve but also in the blue region, there will be a cusp on the curve

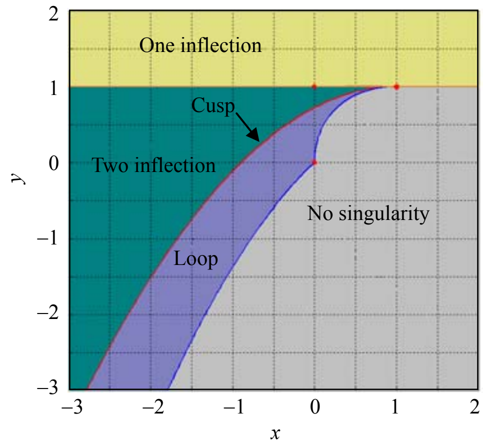

Fig.2 The characterization diagram of cubic C-Bézier curve in case of α=1.5. The curve has a cusp on it if the fourth control point is located on the red curve. There will be one inflection point on the curve if the fourth control point is located in the yellow region with an orange boundary; the curve will take on two inflection points if the fourth control point is located in the green region. There will be a loop on the curve if the fourth control point is located on the purple region with a blue boundary. The curve has no inflection point or singularity on it if the fourth control point is located on the gray region

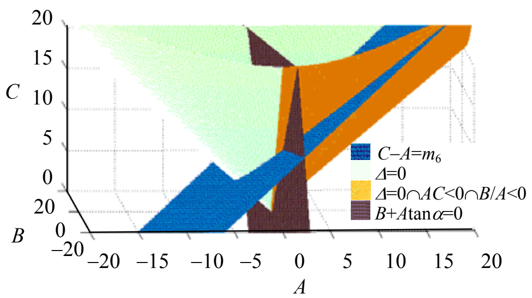

Fig.4 Characterization surfaces and cutting surface

These characterization surfaces segment the common 3D space into different volumes. The curve may possess a singularity, one or two inflection points or none of the above if its corresponding 3D point is located in these different volumes. We name this segmented 3D space the "characterization space".

So, on the one hand, the characterization diagram (Fig.2) can be obtained by taking out a plane section from the characterization space. In detail, Eqs.(7a)~(7c) indicate that point (A, B, C) traces out a plane in the characterization space as p_3 moves. And the plane's parametric description (with respect to x, y) is

$$(A,B,C)=(m_1x+m_2y+m_3, m_4x+m_5, m_1x+m_2y+m_3+m_6).$$

Namely, we can compute the characterization diagram (Fig.2) by intersecting the characterization space with a plane, whose implicit expression is

$$C-A=m_6\ \left(=KM(1-C_\alpha)/(\alpha-S)^2\right). \qquad (10)$$

On the other hand, the canonical plane (Fig.2) can be viewed as being embedded in the characterization space, with the embedding given by Eq.(10).

By fixing $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_3$ and moving $\boldsymbol{p}_2$ [we still use the canonical curve here, i.e., $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_3$ are mapped to (0,0), (0,1), (1,1) respectively and set $\boldsymbol{p}_2$=(x,y)], we obtain another characterization diagram different from Fig.2. It can be viewed geometrically as a different cut through characterization space. In particular, direct calculation for the dependence of A, B and C on $\boldsymbol{p}_2$ shows that the embedding for this case is given by

$$(A,B,C)=(n_1x+n_2y+n_3, n_4x+n_5, n_6x+n_2y+n_3), \quad (11)$$

where

$$\begin{cases} n_1=(-K-2KM+M\alpha)Mm,\ \ n_2=KMm, \\ n_3=(K-\alpha)Mm,\ \ n_4=(1-C_\alpha)Mm, \\ n_5=C_\alpha-1,\ \ n_6=(-KC_\alpha-2KM+M\alpha)Mm, \\ m=(\alpha-S)^{-2}, \end{cases}$$

which is a parametric description for the plane

$$A+KB-C=K(C_\alpha-1). \qquad (12)$$

It means that the characterization diagram in this case can be described by slicing plane (12) through the characterization space. The different characterization diagrams corresponding to varying $\boldsymbol{p}_0$ or $\boldsymbol{p}_1$ can be obtained similarly. So we conclude that the four different characterization diagrams of the C-Bézier curves in its canonical forms can be obtained by cutting the characterization space with corresponding four different planes. Implicitizating the parameter forms (A, B, C), we can easily compute the cutting planes.

Relation among characterization diagrams of C-curves

The conclusion obtained in the previous subsection, however, is particular with reference to the canonical form of C-Bézier curves. In fact, such a conclusion is also valid for all original C-Bézier curves, as well as C-curves of other types. In other words, the characterization diagram induced by moving a control point of original C-Bézier curves and other cubic planar C-curves such as a segment of CB-spline or a C-Ferguson curve, can be obtained by cutting the characterization space with an appropriate plane. Actually, for C-Bézier curves and cubic planar C-curves of other types, A, B and C have the form

$$\det(\boldsymbol{P}_s, \boldsymbol{P}_r) = \det\left(\sum_q m_{sq}\boldsymbol{p}_q, \sum_t m_{tr}\boldsymbol{p}_t\right)$$
$$= \sum_q \sum_t m_{sq} m_{tr} \det(\boldsymbol{p}_q, \boldsymbol{p}_t),$$

where the coefficients are either the elements of the matrix of the affine map or the elements of the transformation matrix between the basis and {sin$t$, cos$t$, t, 1}. It implies that if only one control point is moving, each of A, B and C is linear in the coordinate of the moving control points or remains constant. Thus we can easily compute the implicit description of the cutting plane in the characterization space.

In sum, for a given α, we get the characterization diagrams of C-curves by the following steps:

Step 1: Compute the characterization surfaces based on Propositions 1~4. The 3D common space is transformed to be a characterization space by these characterization surfaces.

Step 2: Compute A, B and C defined by Eq.(3) and take each of them as a point in the characterization space.

Step 3: Compute the implicit description of the cutting plane from its parametric description (A, B, C).

Step 4: Cut the characterization space with the plane to get the characterization diagram.

CONCLUSION AND FUTURE WORK

Stone and DeRose (1989) and Juhász (2006) described an approach to detect the singularity of the curves that can be represented by a combination of control points and basis functions. They observed the characterization of curves by moving one control point while leaving the others fixed. For cubic C-curves, only the planar case can take on singularity or inflection points. The locus of the moving control point that yields cusp, loop, one inflection or two inflections segments the plane into different regions. We obtain different plane partitions when different control points serve as the moving point. However, the relation among these different diagrams of C-curves has not been investigated yet. By exploiting the conclusions obtained by Yang and Wang (2004), we show that the relation among these diagrams is explained by a 3D characterization space. All the diagrams can be obtained by cutting the characterization space with a plane. This characterization space is a common 3D space segmented by some characterization surfaces. Furthermore, all the cutting slices have the parametric description already, and their implicit description can be obtained easily.

Our investigation was restricted to the planar cubic C-curves in this paper. For more general cases, such as planar polynomial curves and C-curves of order higher than 3, the characterization diagrams become even more complex and the relation among them is unclear. For space curves, the locus of the moving control point that yields cusps, loops or inflection points will also segment the space into different partitions. The relation among these different space partitions is also unclear up to now. These will be our work in the future.

References

Juhász, I., 2006. On the singularity of a class of parametric curves. *Computer Aided Geometric Design*, **23**(2):146-156. [doi:10.1016/j.cagd.2005.05.005]

Li, Y.M., Cripps, R.J., 1997. Indentification of inflection points and cusps on rational curves. *Computer Aided Geometric Design*, **14**(5):491-497. [doi:10.1016/S0167-8396(96)00041-6]

Manocha, D., Canny, J.F., 1992. Detecting cusps and inflection points in curves. *Computer Aided Geometric Design*, **9**(1):1-24. [doi:10.1016/0167-8396(92)90050-Y]

Monterde, J., 2001. Singularities of rational Bézier curves. *Computer Aided Geometric Design*, **18**(8):805-816. [doi:10.1016/S0167-8396(01)00072-3]

Pottmann, H., 1993. The geometry of Tchebycheffian spines. *Computer Aided Geometric Design*, **10**(3-4):181-210. [doi:10.1016/0167-8396(93)90036-3]

Sakai, M., 1999. Inflection points and singularities on planar rational cubic curve segments. *Computer Aided Geometric Design*, **16**(3):149-156. [doi:10.1016/S0167-8396(98)00036-3]

Stone, M.C., DeRose, T.D., 1989. A geometric characterization of parametric cubic curves. *ACM Trans. on Graph.*, **8**:147-163. [doi:10.1145/77055.77056]

Su, B.Q., Liu, D.Y., 1983. An affine invariant and its application in computational geometry. *Scientia Sinica Series A*, **24**(3):259-267.

Wang, C.Y., 1981. Shape classification of the parametric cubic curve and parametric B-spline cubic curve. *Computer-Aided Design*, **13**(4):199-206. [doi:10.1016/0010-4485(81)90141-X]

Yang, Q.Y., Wang, G.Z., 2004. Inflection points and singularities on C-curves. *Computer Aided Geometric Design*, **21**(2):207-213. [doi:10.1016/j.cagd.2003.11.002]

Zhang, J.W., 1996. C-curves: an extension of cubic curves. *Computer Aided Geometric Design*, **13**(3):199-217. [doi:10.1016/0167-8396(95)00022-4]

Zhang, J.W., 1997. Two different forms of CB-splines. *Computer Aided Geometric Design*, **14**(1):31-41. [doi:10.1016/S0167-8396(96)00019-2]

Zhang, J.W., 1999. C-Bézier curves and surfaces. *Graphical Models and Image Processing*, **61**(1):2-15. [doi:10.1006/gmip.1999.0490]

Zhang, J.W., Krause, F.L., 2005. Extend cubic uniform B-splines by unified trigonometric and hyberolic basis. *Graphical Models*, **67**(2):100-119. [doi:10.1016/j.gmod.2004.06.001]

Zhang, J.W., Krause, F.L., Zhang, H.Y., 2005. Unifying C-curves and H-curves by extending the calculation to complex numbers. *Computer Aided Geometric Design*, **22**(9):865-883. [doi:10.1016/j.cagd.2005.04.009]