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Generalized bounds on the partial periodic correlation of complex roots of unity sequence set*

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Abstract: In this paper, the generalized bounds are derived on the partial periodic correlation of complex roots of unity sequence set with zero or low correlation zone (ZCZ/LCZ) as the important criteria of the sequence design and application. The derived bounds are with respect to family size, subsequence length, maximum partial autocorrelation sidelobe, maximum partial cross-correlation value and the ZCZ/LCZ. The results show that the derived bounds include the previous periodic bounds, such as Sarwate bound, Welch bound, Peng-Fan bound and Paterson-Lothian bound, as special cases.

Key words: Partial correlation, Zero correlation zone (ZCZ), Low correlation zone (LCZ), CDMA
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INTRODUCTION

The partial correlation properties of sets of sequences, where correlations are computed over only subsequences of sequences, are important in CDMA system as well as in ranging, channel estimation and synchronization applications. The periodic auto- and crosscorrelation functions have come to much attention but the partial correlations of sequences are much less understood (Welch, 1974; Pursley, 1977; Pursley and Sarwate, 1977; Sarwate, 1979). Sarwate *et al*.(1984) analyzed the influence of partial correlation on the multiple access interference (MAI) performance under different sequence lengths and spreading gains. Therefore, the sequence sets having low absolute values of partial correlation are fundamentally important in communication systems.

Paterson and Lothian (1998) derived a lower partial periodic bound based on Welch's technique (Welch, 1974). But this bound cannot derive the lower partial correlation bounds on generalized orthogonality [GO, or zero correlation zone (ZCZ) and low correlation zone (LCZ)] sequences, which can be employed in quasi-synchronous CDMA (QS-CDMA) to eliminate the multiple access interference and multipath interference (Fan and Hao, 2000; Fan, 2004; Appuswamy and Chaturvedi, 2006; Tang and Mow, 2006; Zhou and Tang, 2006) derived bounds for binary and complex roots of unity sequences. However, these bounds also cannot derive the lower partial correlation bounds on GO sequences. It is therefore important to derive the theoretical limits among the subsequence length l, sequence family size M, maximum partial autocorrelation sidelobe value Pl_A, maximum partial crosscorrelation value Pl_C, and low correlation zone L_{CZ} or zero correlation zone Z_{CZ}.

It is the objective of this paper to derive generalized bounds on the partial periodic correlations of complex roots of unity sequences. It is shown that all the previous periodic sequence bounds, such as Welch bound, Paterson-Lothian bound and Peng-Fan bound, etc., are the special cases of the proposed bounds. In this paper, our attention will be paid only to the partial periodic correlation bounds on complex roots of unity sequences.

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DEFINITIONS AND PROPERTIES

Let q be an arbitrary positive integer greater than 1, $Z_q=\{0, 1, \ldots, q-1\}$, $\mathrm{i}=\sqrt{-1}$, $\omega=\exp(\mathrm{i}2\pi/q)$, $E=\{1, \omega^1, \ldots, \omega^{q-1}\}$. The $\boldsymbol{x}=(x_0, x_1, \ldots, x_{n-1})\in E^n$ is called a complex roots of unity sequence of length n. When q=2, the complex roots of unity sequence becomes the binary sequence. For any two such sequences $\boldsymbol{x}=(x_0, x_1, \ldots, x_{n-1})$ and $\boldsymbol{y}=(y_0, y_1, \ldots, y_{n-1})$, the partial inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_l = \sum_{i=0}^{l-1} x_i y_i^*. \tag{1}$$

For any sequence $\boldsymbol{x}=(x_0, x_1, \ldots, x_{n-1})\in E^n$, let T denote the operator which shifts sequence cyclically to the left by one place, that is $T\boldsymbol{x}=(x_1, \ldots, x_{n-1}, x_0)$, and let $T_0\boldsymbol{x}=\boldsymbol{x}$, $T_{i+1}\boldsymbol{x}=T(T_i\boldsymbol{x})$ for positive integer i. The partial periodic correlation function $Pl(\boldsymbol{x}, \boldsymbol{y}; s, t)$ of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$Pl(\boldsymbol{x},\boldsymbol{y};s,t) = \sum_{i=0}^{l-1} x_{i+s} y_{i+t}^* = \langle T_s\boldsymbol{x}, T_t\boldsymbol{y}\rangle_l, \quad s,t = 0,\ 1,\ \cdots,\ n-1 \tag{2}$$

where subscripts are performed modulo n, l denotes the length of subsequence, and $\boldsymbol{y}^*$ denotes the complex conjugate of $\boldsymbol{y}$.

For $C\subseteq E^n$, $M=|C|$, $Pl_{\mathrm{A}}\geq 0$, $Pl_{\mathrm{C}}\geq 0$, $Pl_M=\max\{Pl_{\mathrm{A}}, Pl_{\mathrm{C}}\}$, the periodic low correlation zone L_{CZ}, the periodic low autocorrelation zone L_{ACZ} and the periodic low crosscorrelation zone L_{CCZ} of C are defined, respectively, as follows:

$$L_{\mathrm{CZ}} = \min\{L_{\mathrm{ACZ}}, L_{\mathrm{CCZ}}\}, \tag{3a}$$

$$L_{\mathrm{ACZ}} = \max\{T \,\|\, |Pl(\boldsymbol{x},\boldsymbol{x};s,t)| \leq Pl_{\mathrm{A}},\ \forall \boldsymbol{x}\in C,\ 0<|s-t|\leq T\}, \tag{3b}$$

$$L_{\mathrm{CCZ}} = \max\{T \,\|\, |Pl(\boldsymbol{x},\boldsymbol{y};s,t)| \leq Pl_{\mathrm{C}},\ \forall \boldsymbol{x},\boldsymbol{y}\in C,\ \boldsymbol{x}\neq\boldsymbol{y},\ |s|\leq T,\ |t|\leq T\}. \tag{3c}$$

A sequence set C with $L_{\mathrm{CZ}}>0$ is called periodic low correlation zone (LCZ) set if $Pl_M>0$, or zero correlation zone (ZCZ) sequence set if $Pl_M=0$.

Lemma 1 For any sequence $\boldsymbol{x}\in E^n$, and any integer $s, t=0, 1, \ldots, n-1$, we have

$$\sum_{\boldsymbol{y}\in E^n} |Pl(\boldsymbol{x},\boldsymbol{y};s,t)|^2 = lq^n. \tag{4}$$

Proof Let $\boldsymbol{x}=(\omega^{u_0}, \omega^{u_1}, \cdots, \omega^{u_{n-1}})$, $\boldsymbol{y}=(\omega^{v_0}, \omega^{v_1}, \ldots, \omega^{v_{n-1}})$, where $u_i, v_i\in Z_q$ (i=0, 1, …, $n-1$). We have

$$\sum_{\boldsymbol{y}\in E^n} |Pl(\boldsymbol{x},\boldsymbol{y};s,t)|^2 = \sum_{\boldsymbol{y}\in E^n} Pl(\boldsymbol{x},\boldsymbol{y};s,t) Pl^*(\boldsymbol{x},\boldsymbol{y};s,t)$$

$$= \sum_{\boldsymbol{y}\in E^n} \left(\sum_{i=0}^{l-1} \omega^{u_{i+s}-v_{i+t}}\right)\left(\sum_{j=0}^{l-1} \omega^{-u_{j+s}+v_{j+t}}\right) = \sum_{i,j=0}^{l-1} \omega^{u_{i+s}-u_{j+s}} H(i,j,t),$$

where $H(i,j,t) = \sum_{\boldsymbol{y}\in E^n} \omega^{-v_{i+t}+v_{j+t}}$. We have

$$H(i,j,t) = \begin{cases} q^{n-2} \sum_{r,s=0}^{q-1} \omega^{-r+s} = 0, & \text{if } i\neq j, \\ q^{n-1} \sum_{r=0}^{q-1} \omega^{-r+r} = q^n, & \text{if } i=j. \end{cases}$$

Therefore,

$$\sum_{\boldsymbol{y}\in E^n} |Pl(\boldsymbol{x},\boldsymbol{y};s,t)|^2 = q^n \sum_{i=0}^{l-1} \omega^{u_{i+s}-u_{i+s}} = lq^n.$$

Throughout this paper, it is assumed that $w_i\geq 0$, i=0, 1, …, L_{CZ}, $\sum_{i=0}^{L_{\mathrm{CZ}}} w_i = 1$ and $\boldsymbol{w}=(w_0, w_1, \ldots, w_{L_{\mathrm{CZ}}})$.

For $\boldsymbol{x}\in E^n$, $A, B\subseteq E^n$, $|A||B|>0$, let $W(\boldsymbol{x})=\{T_i\boldsymbol{x} \mid i=0, 1, \ldots, L_{\mathrm{CZ}}\}$, $W(A)=\bigcup_{\boldsymbol{x}\in A} W(\boldsymbol{x})$ and

$$F(A,B) := \frac{1}{|A||B|} \sum_{\boldsymbol{x}\in A}\sum_{\boldsymbol{y}\in B}\sum_{s=0}^{L_{\mathrm{CZ}}}\sum_{t=0}^{L_{\mathrm{CZ}}} \left|\langle T_s\boldsymbol{x}, T_t\boldsymbol{y}\rangle_l\right|^2 w_s w_t.$$

Lemma 2 For any $\boldsymbol{x}\in E^n$, $A\subseteq E^n$,

$$F(\{\boldsymbol{x}\}, E^n) = F(E^n, E^n) = l. \tag{5}$$

Proof It is noted that $\langle T_s\boldsymbol{x}, T_t\boldsymbol{y}\rangle_l = Pl(\boldsymbol{x},\boldsymbol{y};s,t)$. Thus we have

$$F(\{\boldsymbol{x}\}, E^n) = \frac{1}{q^n} \sum_{\boldsymbol{y}\in E^n}\sum_{s=0}^{L_{\mathrm{CZ}}}\sum_{t=0}^{L_{\mathrm{CZ}}} \left|\langle T_s\boldsymbol{x}, T_t\boldsymbol{y}\rangle_l\right|^2 w_s w_t$$

$$= \frac{1}{q^n}\left[\sum_{s=0}^{L_{\mathrm{CZ}}}\sum_{t=0}^{L_{\mathrm{CZ}}}\sum_{\boldsymbol{y}\in E^n} |Pl(\boldsymbol{x},\boldsymbol{y};s,t)|^2 w_s w_t\right]$$

$$= \frac{1}{q^n}\left[\sum_{s=0}^{L_{\mathrm{CZ}}}\sum_{t=0}^{L_{\mathrm{CZ}}} lq^n w_s w_t\right] = l.$$

This gives the formula for $F(\{\boldsymbol{x}\}, E^n)$, which is independent of $\boldsymbol{x}$, and thus completes the proof.

Lemma 3 For $C\subseteq E^n$

$$F(C,C)\geq F(E^n,E^n). \quad (6)$$

Proof For any i, j=0, 1, …, n−1; s=0, 1, …, L_{CZ}, define function $f(i, j; \boldsymbol{X})$ on $W(E^n)$ as follows:

If $\boldsymbol{X}=T_s\boldsymbol{x}=(x_s, x_{s+1}, \ldots, x_{s+n-1})\in W(E^n)$, where $\boldsymbol{x}\in E^n$, then

$$f(i,j;\boldsymbol{X})=x_{s+i}x_{s+j}^{*}w_s.$$

We can verify that

$$F(A,B)=\frac{1}{|A||B|}\sum_{\boldsymbol{X}\in W(A)}\sum_{\boldsymbol{Y}\in W(B)}\sum_{i,j=0}^{l-1}f(i,j;\boldsymbol{X})f^{*}(i,j;\boldsymbol{Y}). \quad (7)$$

In fact, for any $\boldsymbol{x}\in A$, $\boldsymbol{y}\in B$, let $\boldsymbol{X}=T_s\boldsymbol{x}=(x_s, x_{s+1}, \ldots, x_{s+n-1})$, $\boldsymbol{Y}=T_t\boldsymbol{y}=(y_t, y_{t+1}, \ldots, y_{t+n-1})$, we have

$$\begin{aligned}
&\sum_{\boldsymbol{X}\in W(\boldsymbol{x})}\sum_{\boldsymbol{Y}\in W(\boldsymbol{y})}\sum_{i,j=0}^{l-1}f(i,j;\boldsymbol{X})f^{*}(i,j;\boldsymbol{Y})\\
&=\sum_{s,t=0}^{L_{\mathrm{CZ}}}\sum_{i,j=0}^{l-1}f(i,j;T_s\boldsymbol{x})f^{*}(i,j;T_t\boldsymbol{y})\\
&=\sum_{s,t=0}^{L_{\mathrm{CZ}}}\sum_{i,j=0}^{l-1}(x_{s+i}x_{s+j}^{*}w_s)(y_{t+i}y_{t+j}^{*}w_t)^{*}\\
&=\sum_{s,t=0}^{L_{\mathrm{CZ}}}\left(\sum_{i=0}^{l-1}(x_{s+i}y_{t+i}^{*})\right)\left(\sum_{j=0}^{l-1}(x_{s+j}y_{t+j}^{*})^{*}\right)w_s w_t\\
&=\sum_{s,t=0}^{L_{\mathrm{CZ}}}\left|\sum_{i=0}^{l-1}x_{s+i}y_{t+i}^{*}\right|^2 w_s w_t=\sum_{s,t=0}^{L_{\mathrm{CZ}}}\left|\langle T_s\boldsymbol{x},T_t\boldsymbol{y}\rangle_l\right|^2 w_s w_t.
\end{aligned}$$

And

$$\begin{aligned}
&\sum_{\boldsymbol{x}\in A}\sum_{\boldsymbol{y}\in B}\sum_{s,t=0}^{L_{\mathrm{CZ}}}\left|\langle T_s\boldsymbol{x},T_t\boldsymbol{y}\rangle_l\right|^2 w_s w_t\\
&=\sum_{\boldsymbol{x}\in A}\sum_{\boldsymbol{y}\in B}\sum_{\boldsymbol{X}\in W(\boldsymbol{x})}\sum_{\boldsymbol{Y}\in W(\boldsymbol{y})}\sum_{i,j=0}^{l-1}f(i,j;\boldsymbol{X})f^{*}(i,j;\boldsymbol{Y})\\
&=\sum_{\boldsymbol{X}\in W(A)}\sum_{\boldsymbol{Y}\in W(B)}\sum_{i,j=0}^{l-1}f(i,j;\boldsymbol{X})f^{*}(i,j;\boldsymbol{Y}).
\end{aligned}$$

Therefore, the verification holds.

Using the Cauchy inequality, we have

$$\begin{aligned}
&\{|A||B|F(A,B)\}^2\\
&=\left\{\sum_{i,j=0}^{l-1}\left(\sum_{\boldsymbol{X}\in W(A)}f(i,j;\boldsymbol{X})\right)\left(\sum_{\boldsymbol{Y}\in W(B)}f^{*}(i,j;\boldsymbol{Y})\right)\right\}^2\\
&\leq\sum_{i,j=0}^{l-1}\left|\sum_{\boldsymbol{X}\in W(A)}f(i,j;\boldsymbol{X})\right|^2\cdot\sum_{i,j=0}^{l-1}\left|\sum_{\boldsymbol{Y}\in W(B)}f(i,j;\boldsymbol{Y})\right|^2\\
&=|A|^2F(A,A)|B|^2F(B,B).
\end{aligned}$$

Therefore, $\left|F(A,B)\right|^2\leq F(A,A)F(B,B)$.

Let $A=E^nB=C$, by Lemma 1 we have

$$\left|F(E^n,C)\right|^2\leq F(E^n,E^n)F(C,C),$$

$$F(C,C)\geq F(E^n,E^n)=l.$$

LOWER BOUNDS ON PERIODIC CORRELATION OF LCZ AND ZCZ SEQUENCES

Let C be a set of M complex roots of unity sequences of length n. Pl_{A} denotes the maximum partial periodic autocorrelation sidelobe, Pl_{C} denotes the maximum partial periodic crosscorrelation value, $Pl_M=\max\{Pl_{\mathrm{A}}, Pl_{\mathrm{C}}\}$, L_{CZ} denotes the low correlation zone, l denotes the subsequence length. Then we can derive the partial periodic correlation bounds of LCZ complex roots of unity sequences in this section. We have the following main result from Lemmas 2 and 3:

Theorem 1 For any $C\subseteq E^n$, $M=|C|>0$, we have

$$\frac{1}{M}\left(1-\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2\right)Pl_{\mathrm{A}}^2+\left(1-\frac{1}{M}\right)Pl_{\mathrm{C}}^2\geq l-\frac{l^2}{M}\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2. \quad (8)$$

Proof We have

$$\begin{aligned}
M^2F(C,C)=&\sum_{\boldsymbol{x}\in C}\sum_{s=0}^{L_{\mathrm{CZ}}}Pl^2(\boldsymbol{x},\boldsymbol{x};s,s)w_s w_s+\\
&\sum_{\boldsymbol{x}\in C}\sum_{s,t=0;s\neq t}^{L_{\mathrm{CZ}}}Pl^2(\boldsymbol{x},\boldsymbol{x};s,t)w_s w_t+\\
&\sum_{\boldsymbol{x},\boldsymbol{y}\in C;\boldsymbol{x}\neq\boldsymbol{y}}\sum_{s,t=0}^{L_{\mathrm{CZ}}}Pl^2(\boldsymbol{x},\boldsymbol{y};s,t)w_s w_t\\
\leq& Ml^2\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2+MPl_{\mathrm{A}}^2\left(1-\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2\right)+M(M-1)Pl_{\mathrm{C}}^2,
\end{aligned}$$

where $\sum_{s,t=0;s\neq t}^{L_{\mathrm{CZ}}}w_s w_t=1-\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2$. Thus the proof is completed.

In particular, we have the following lower bounds:

Corollary 1 For $C\subseteq E^n$, we have

$$Pl_M^2\geq\left(Ml-l^2\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2\right)\Bigg/\left(M-\sum_{s=0}^{L_{\mathrm{CZ}}}w_s^2\right). \quad (9)$$

For any integer $0\le L\le L_{CZ}$, let

$$w_s=\begin{cases}1/(L+1), & 0\le s\le L,\\ 0, & L<s\le L_{CZ},\end{cases}$$

and $\boldsymbol{w}=(w_0, w_1, \ldots, w_{L_{CZ}})$, then $\sum_{s=0}^{L_{CZ}} w_s^2=1/(L+1)$.

Corollary 2 For $C\subseteq E^n$, any integer $0\le L\le L_{CZ}$, we have by Corollary 1:

$$Pl_M^2 \ge \frac{ML+M-l}{ML+M-1}l. \tag{10}$$

Let $l=n$, we have the bound obtained by Peng and Fan (2003). When $q=2$, the result was derived by Peng and Fan (2002). Let $L=n-1$, we have the bound which was obtained by Paterson and Lothian (1998).

In addition, we have by Theorem 1:

Corollary 3 For $C\subseteq E^n$, any integer $0\le L\le L_{CZ}$,

$$\frac{1}{M}\left(1-\frac{1}{L+1}\right)Pl_A^2+\left(1-\frac{1}{M}\right)Pl_C^2 \ge l-\frac{l^2}{M(L+1)}. \tag{11}$$

In particular, let $L=n-1$, $l=n$, we can obtain the famous Sarwate bound (Sarwate, 1979). Further, let $Pl_M=\max\{Pl_A, Pl_C\}$, we can obtain the well-known Welch bound (Welch, 1974).

CONCLUSION

Generalized lower bounds on the partial periodic correlation of complex roots of unity sequence set are derived. Because the complex roots of unity sequences include binary sequences as special cases, all the previous periodic binary sequence bounds, such as Sarwate bounds, Welch bounds, Peng-Fan bounds, Kenneth-Paul bounds and so on, can be considered as the special cases of this paper.

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