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Exact solutions for different vorticity functions of couple stress fluids*

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Abstract: In this paper, inverse solutions are obtained for the class of 2D steady incompressible couple stress fluid flows. This class consists of flows for which the vorticity distribution is given by $\nabla^2\psi=\psi+f(x,y)$. The solutions are obtained by applying the inverse method, which makes certain hypotheses regarding the form of the velocity field and pressure but without making any regarding the boundaries of the domain occupied by the fluid. Inverse solutions are derived for three different forms of $f(x,y)$.

Key words: Exact solutions, Vorticity functions, Beltrami flow, Couple stress fluid

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INTRODUCTION

In spite of the mathematical complexity of Navier-Stokes equations, there exist very few exact solutions. Exact solutions are very important, because they serve a dual purpose. First, they provide a solution to be a flow that has some physical meaning. Second, such solutions can be used as checks against complicated numerical codes that have been developed for much more complex flows. Some exact solutions to Navier-Stokes equations have been obtained in cases where these equations can be linear and in cases where the partial differential equations can be reduced to ordinary differential equations, for which the solution is possible, in addition to flows in which the vorticity distribution is chosen such that the governing equations written in terms of the stream function become linear. Turning to the second fluids, the non-linearities occur not only in the inertial part but also in the viscous part of these equations, and the equations of motion become more highly non-linear partial differential equations.

These hypotheses are often made about the velocity field and rarely about the pressure. By considering the vorticity to be proportional to stream function alone, Taylor (1923) obtained the solution for a double infinite array of vortices decaying exponentially with time. In a more general case, Kovaznay (1948) and Lin and Tobak (1986) assumed the vorticity to be proportional to the stream function perturbed by a uniform stream. This enabled Kovaznay to obtain the grid flow solution and Lin and Tobak to obtain the reversed flow solution with suction. By the same assumption, Hui (1987) and Wang (1990) presented several new classes of exact analytical solutions for steady and unsteady flows. Hui also showed that the Lin and Tobak solution of wavy flow, with periodic suction and blowing, is incorrect. Chandna and Oku-Ukpong (1994) were also able to linearize the Navier-Stokes equations for the chosen vorticity functions and some of the results established in (Wang, 1990) can be obtained from their findings as a special case. Recently, Oleg (2003), Siddiqui *et al.*

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(2006) and Wu *et al*.(2007) obtained the exact solutions for the Navier-Stokes equations in different contexts.

In the case of non-Newtonian fluids, namely the Rivilin-Erikson fluids of second grade, these problems become even difficult to solve, because the non-linearity not only occurs in the inertial part of these equations but also in the viscous part and the inverse solutions of these equations become even rarer. For this reason, inverse methods have become attractive in the study of non-Newtonian second grade fluids. Rajagopal (1980; 1882) found that the non-linear part, which occurs in the viscous part in the equations of motion of second grade fluid, also vanishes for the specific problems studied in (Taylor, 1923; Kovaznay, 1948). Kaloni and Huschilt (1984), and Siddiqui (1986) found inverse and semi-inverse solutions for non-Newtonian fluids using the inverse method. By assuming a certain form of the stream function, inverse solutions for such fluids for the planar case were obtained by Siddiqui *et al*.(2001), Labropulu (2000), Islam and Zhou (2007a) and many others.

In numerous technological applications, the fluids in use do not obey the commonly assumed linear relationship between the stress and the rate of strain at a point. Such fluids are recognized as non-Newtonian fluids. In particular, the interest in non-Newtonian fluids has grown considerably, due largely to the demand of such diverse areas as biorheology, geophysics and chemical and petroleum industries. For this reason several models have been proposed to predict the non-Newtonian behavior of various types of material. One class of fluids which has gained considerable attention in recent years is the couple stress fluid. Couple stresses are a consequence of the assumption that the numerical action of one part of a body on another, across a surface, is equivalent to a force and moment distribution. Couple stress fluids consist of rigid, randomly oriented particles suspended in a viscous medium such as blood fluids, electro-rheological fluids and synthetic fluids. The main feature of couple stress fluid is that the stress tensor is anti-symmetric and their accurate flow behavior cannot be predicted by the classical Newtonian theory. Stokes (1966) generalized the classical model to include the effect of the presence of the couple stresses and this couple stress model has been widely used because of its relative mathematical simplicity compared with the other models developed for the couple stress fluid. Recently, the study of couple stress fluid flows has been the subject of great interest, due to its widespread industrial and scientific applications in pumping fluids, such as synthetic fluids, polymer-thickened oils, liquid crystals and animal bloods. Other important fields where couple stress fluids have applications are squeezing and lubrication theory (Wang *et al*., 2002; Sinha and Singh, 1981; Ramanaiah, 1979; Gupta and Sharma, 1988; Mindlin and Tierstea, 1962; Nabil *et al*., 1995; Islam and Zhou, 2007b).

Wang *et al.*(2002) conducted a numerical study of the performance of a dynamically loaded journal bearing lubricated with couple stress fluid where the dimensionless generalized equation based on the Stokes' couple stress fluid model was solved using a finite difference scheme. Sinha and Singh (1981) did research on the effects of couple stresses on the lubrication of rolling contact bearings with a couple stress fluid as lubricant. Ramanaiah (1979) carried out a theoretical study of squeeze film between finite plates of various shapes lubricated by fluids with couple stress and obtained an expression for squeeze time for a general shape. Gupta and Sharma (1988) also conducted a study of the influence of couple stresses on a lubricating film using the Stokes couple stress fluid model where an approximate expression for the pressure distribution was obtained by using an "energy integral method". A weaker flow rate and higher load capacity were observed for small values of the couple stress parameter. Some numerical studies were also carried out. For the elastic case, Mindlin and Tierstea (1962) obtained the constitutive equation for a linearly perfectly elastic solid using the couple stresses. Nabil *et al*.(1995) studied the unsteady magnetohydrodynamic flow of an incompressible electrically conducting fluid between two parallel plates, taking into account the couple stresses. More recently, Islam and Zhou (2007b) studied the Stokes couple stress model and obtained a class of exact solutions and showed that the results obtained in (Taylor, 1923; Kovaznay, 1948; Lin and Tobak, 1986; Hui, 1987) can be obtained from their findings as special cases.

In this paper, we study the generalized Beltrami flows, $\nabla^2\psi=\psi+Ay^2+Bxy+Cx+Dy$, $\nabla^2\psi=\psi+Ay^2+Cx+$

Dy, and $\nabla^2\psi=\psi+Cx+Dy$ for 2D incompressible couple stress fluids, and obtain a class of exact solutions in an unbounded domain which do not require additional boundary conditions. Inverse solutions are derived in each case. The rest of the article is divided into three sections. The mathematical formulation of governing equations is presented in Section 2. Exact solutions for three different forms (as already discussed by Chandna and Oku-Ukpong (1994) for viscous case) of $\nabla^2\psi=\psi+f(x,y)$, are given in Section 3. Conclusions are given in Section 4.

FLOW EQUATIONS

The basic equations governing the motion of an incompressible, steady couple stress fluids in the absence of body force and body moments are given by the following equations (Chandna and Oku-Ukpong, 1994; Stokes, 1966; Mindlin and Tierstea, 1962; Nabil *et al*., 1995; Islam and Zhou, 2007b)

$$\nabla\cdot\boldsymbol{V}=0, \tag{1}$$

$$\rho((\boldsymbol{V}\cdot\nabla)\boldsymbol{V})=-\nabla p+\mu\nabla^2\boldsymbol{V}-\eta\nabla^4\boldsymbol{V}, \tag{2}$$

where $\boldsymbol{V}$ is the velocity vector, ρ is constant density, p is the pressure, μ is the coefficient of viscosity and η material constant responsible for couple stress parameter.

For steady plane flows in Cartesian coordinates, $\boldsymbol{V}$ can be expressed as

$$\boldsymbol{V}=(u(x,y),v(x,y),0), \tag{3}$$

and the generalized pressure p^* and vorticity ω functions are defined as

$$p^*=\frac{\rho}{2}\left|\boldsymbol{V}^2\right|+p, \tag{4}$$

$$\omega=v_x-u_y, \tag{5}$$

By substituting Eqs.(3), (4) and (5) into Eqs.(1) and (2), the equations of motion become

$$u_x+v_y=0. \tag{6}$$

The x-component of momentum equation is

$$p_x^*-\rho(v\omega)=-\mu\omega_y+\eta(\omega_{xxy}+\omega_{yyy}). \tag{7}$$

The y-component of momentum equation is

$$p_y^*+\rho(u\omega)=\mu\omega_x-\eta(\omega_{xxx}+\omega_{xyy}). \tag{8}$$

The system of Eqs.(6)~(8) has three unknowns.

Remark 1 If $\eta=0$, we recover the Chandna and Oku-Ukpong (1994) case. Eq.(6) implies the existence of a stream function $\psi(x,y)$ such that

$$u_x=\psi_y,\ \ u_y=-\psi_x. \tag{9}$$

Substitution of Eq.(9) into Eqs.(7) and (8) and elimination of pressure from the resulting equations using $p_{xy}^*=p_{yx}^*$ yields

$$\nu_1\nabla^4\psi-\nu_2\nabla^6\psi+\partial(\psi,\nabla^2\psi)/\partial(x,y)=0, \tag{10}$$

where $\nu_1=\mu/\rho$ is the kinematic viscosity, and $\nu_2=\eta/\rho$ is the couple stress viscosity. Eq.(10) is a steady non-linear and represents the planar motion of a steady incompressible couple stress fluid in the absence of body force. When a solution of Eq.(10) has been obtained, the velocity components are given by Eq.(9) and the pressure can be found by integrating Eqs.(7) and (8).

Exact solutions for $\nabla^2\psi=\psi+Ay^2+Bxy+Cx+Dy$

In this section we will consider the case of the motion (Chandna and Oku-Ukpong, 1994) when

$$\nabla^2\psi=\psi+Ay^2+Bxy+Cx+Dy, \tag{11}$$

where A, B, C and D are constants. Substituting Eq.(11) into Eq.(10), we obtain

$$\begin{aligned}&R(\psi+Ay^2+Bxy+Cx+Dy+2A)\\&\quad+(2Ay+Bx+D)\psi_x-(By+C)\psi_y=0,\end{aligned} \tag{12}$$

where

$$\partial(\psi,\nabla^2\psi)/\partial(x,y)=(2Ay+Bx+D)\psi_x-(By+C)\psi_y,$$

$$\nabla^4\psi=\psi+Ay^2+Bxy+Cx+Dy+2A,$$

$$\nabla^6\psi = \psi + Ay^2 + Bxy + Cx + Dy + 2A,$$
$$\nu_1 - \nu_2 = R \neq 0.$$

Two cases arise from Eq.(12) and these are

$$(1)\ \nu_1 - \nu_2 = R > 0; \quad (2)\ \nu_1 - \nu_2 = R < 0.$$

Case 1 If $\nu_1-\nu_2=R>0$

First introduce the canonical coordinates of the form

$$\xi = Ay^2 + Bxy + Cx + Dy, \quad \tau = y. \tag{13}$$

In view of Eq.(13), Eq.(12) becomes

$$-(By+C)\psi_y + R(\psi + \xi + 2A) = 0, \tag{14}$$

where $(By+C)\neq 0$.

Solution of Eq.(14) is given as

$$\psi = \phi(\xi)(By+C)^{R/B} - (Ay^2 + Bxy + Cx + Dy + 2A), \tag{15}$$

where ϕ is an arbitrary function.

Substituting Eq.(15) into Eq.(11) gives

$$\begin{aligned}&\{[C^2(C^2+D^2)+2BCD\xi+B^2\xi^2]\phi''(\xi)\\&+2[C(AC+RD)+RB\xi]\phi'(\xi)+[R^2-RB-C^2]\phi(\xi)\}\\&+\{4C[(BC^2+AD)+AB\xi]\phi''(\xi)\\&+4AC(B+R)\phi'(\xi)-2BC\phi(\xi)\}\tau\\&+\{2[C^2(2A^2+3B^2)+ABCD+AB^2\xi^2]\phi''(\xi)\\&+2AB(B+R)\phi'(\xi)-B^2\phi(\xi)\}\tau^2\\&+[4BC(A^2+B^2)\phi''(\xi)]\tau^3\\&+[B^2(A^2+B^2)\phi''(\xi)]\tau^4=0.\end{aligned} \tag{16}$$

As ξ and τ are independent variables and $\{1,\tau,\tau^2,\tau^3,\tau^4\}$ is a linearly independent set, so from Eq.(16) we get

$$\phi(\xi) = c_1\xi + c_2, \tag{17}$$
$$2AB(B+R)c_1 - B^2\xi c_1 - B^2 c_2 = 0, \tag{18}$$

where c_1 and c_2 are arbitrary constants. Since $\{1,\xi\}$ is linearly independent set, so from Eq.(18) we have

$$2AB(B+R)c_1 - B^2c_2 = 0,\ B^2c_1 = 0, \tag{19}$$

where $c_1=c_2=0$, and Eq.(17) becomes

$$\phi(\xi) = 0. \tag{20}$$

In view of Eq.(15), the resultant stream function and velocity components take the form

$$\psi = (-Ay^2 + Bxy + Cx + Dy + 2A), \tag{21}$$
$$u = -(2Ay + Bx + D),\ v = (By + C). \tag{22}$$

In order to find the pressure field, we substitute the velocity components Eq.(22) into Eqs.(7) and (8), and then integrate the resulting equations to obtain

$$\begin{aligned}p = p_0 - \rho[&B^2(x^2+y^2)+(C^2+D^2)\\&-2(2AC-BD)+2BCy]/2,\end{aligned} \tag{23}$$

where p_0 is an arbitrary constant.

Solution Eq.(21) shows an impingement of two constant vorticity oblique flows. Fig.1 represents the stream lines for stream function Eq.(21).

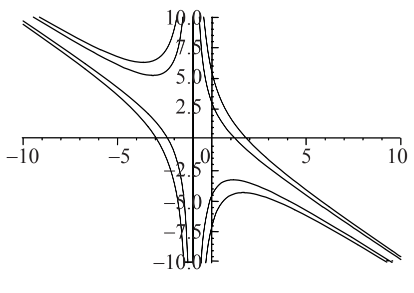

Fig.1 Investigation of stream lines for $\psi(x,y)=-(Ay^2+Bxy+Cx+Dy+2A)$, $A=B=C=D=2$

Case 2 If $\nu_1-\nu_2=R<0$, we have the same solutions for the stream function, velocity components and pressure field and the presence of negative sign do not bring any drastic change in structure of the solutions.

Remark 2 The solution Eq.(21) remains the same as the solution obtained by Chandna and Oku-Ukpong (1994), and the couple stress parameter does not bring any change in structure of the solution.

Exact solutions for $\nabla^2\psi=\psi+Ay^2+Cx+Dy$

In this section we consider the case of the motion (Chandna and Oku-Ukpong, 1994) together with the compatibility equation when

$$\nabla^2\psi=\psi+Ay^2+Cx+Dy, \tag{24}$$

where A, C and D are constants and, ν_1 and ν_2 are already defined in Section 2. Using Eq.(24) into Eq.(10), we find that

$$R(\psi+Ay^2+Cx+Dy+2A)+(2Ay+D)\psi_x-C\psi_y=0, \tag{25}$$

where

$$\nabla^4\psi=\psi+Ay^2+Cx+Dy+2A,$$
$$\nabla^6\psi=\psi+Ay^2+Cx+Dy+2A,$$

where $R=\nu_1-\nu_2>0$. If $\nu_2=0$ we recover the Chandna and Oku-Ukpong (1994) case.

Introducing the canonical coordinates

$$\xi=Ay^2+Cx+Dy,\quad \tau=y. \tag{26}$$

Eq.(25) takes the form

$$-C\psi_y+R(\psi+\xi+2A)=0. \tag{27}$$

Solving Eq.(27) gives

$$\psi=f(\xi)\mathrm{e}^{Ry/C}-(Ay^2+Cx+Dy+2A), \tag{28}$$

where f is an arbitrary function of ξ. To find $f(\xi)$, we substitute Eq.(28) into Eq.(24), and get

$$\begin{aligned}&[C^4f''(\xi)+2AC^2f'(\xi)+(R^2-C^2)f(\xi)]\\&\quad+2RC(2A\tau+D)f'(\xi)+C^2(2A\tau+D)^2f'(\xi)=0.\end{aligned} \tag{29}$$

Since ξ and τ are linearly independent and [1, $(2A\tau+D)$, $(2A\tau+D)^2$] is linearly independent set, so

$$f''(\xi)=0,\ f'(\xi)=0,\ (R^2-C^2)f(\xi)=0, \tag{30}$$

where $R^2=(\nu_1-\nu_2)^2\geq 0$. Here we will discuss the two possibilities from $(R^2-C^2)f(\xi)=0$:

(1) If $f(\xi)=0$, and $R^2\neq C^2$;

(2) If $f(\xi)\neq 0$, and $R^2=C^2$.

(1) If $f(\xi)=0$ and $R^2\neq C^2$, the stream function Eq.(31) and the corresponding velocity components are

$$\psi=-(Ay^2+Cx+Dy+2A), \tag{31}$$
$$u=-(2Ay+D),\ v=C. \tag{32}$$

Solution Eq.(32) may be realized on a plate positioned along $y=-D/(2A)$ with uniform suction, if $C>0$, and blowing if $C<0$, respectively, for blowing and suction at the plate. Fig.2 shows the flow $\psi=-(Ay^2+Cx+Dy+2A)$.

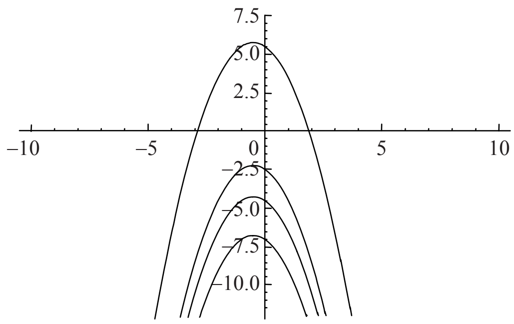

Fig.2 Investigation of stream lines for $\psi(x,y)=-(Ay^2+Cx+Dy+2A)$, $A=B=C=D=2$

Using Eq.(32) into Eqs.(7) and (8), the pressure field becomes

$$p=p_0-\rho[(C^2+D^2)-4ACx]/2. \tag{33}$$

(2) If $f(\xi)\neq 0$, and $R^2=C^2$, the corresponding stream function and velocity components are

$$\psi=K\mathrm{e}^{Ry/C}-(Ay^2+Cx+Dy+2A), \tag{34}$$
$$u=(KR/C)\cdot\mathrm{e}^{Ry/C}-(2Ay+D),\ v=C, \tag{35}$$

where $R=\nu_1-\nu_2$ and $K=f$ is a non-zero constant. The pressure field for the stream function Eq.(34) is

$$\begin{aligned}p=p_0-\rho[(C^2+D^2)-4ACx]/2+\\(\mu R^3/B^3-CR^3/B^3-\eta R^5/B^5)K\mathrm{e}^{Ry/C}.\end{aligned} \tag{36}$$

The $K=CD/R$ in Eq.(34) and Eq.(35), the velocity profile in Eq.(35) can be realized on a plate located along $y=0$ with uniform suction, and the velocity profile attains the form

$$u = De^{Ry/C} - (2Ay + D), \quad v = C, \tag{37}$$

which may be regarded as asymptotic suction profile. $C>0$ for blowing and $C<0$ for suction at the plate. Fig.3 presents the flow behaviour of Eq.(34).

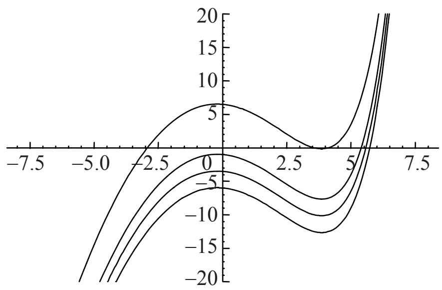

Fig.3 Investigation of stream lines for $\psi=Ke^{Ry/c}-(Ay^2+Cx+Dy+2A)$, $A=B=C=D=K=2$, and $R=3$

Remark 3 On setting $\nu_2=0$ in Eqs.(34) and (37), we recover the Chandna and Oku-Ukpong (1994) case.

Exact Solutions for $\nabla^2\psi=\psi+Cx+Dy$

Here we consider the case of the motion (Chandna and Oku-Ukpong, 1994) together with the compatibility equation when

$$\nabla^2\psi = \psi + Cx + Dy, \tag{38}$$

where C and D are arbitrary constants. Using of Eq.(38) into Eq.(10) gives

$$R(\psi + Cx + Dy) + D\psi_x - C\psi_y = 0, \tag{39}$$

where $R=\nu_1-\nu_2>0$. Introducing the canonical coordinates

$$\xi = Cx + Dy, \ \ \eta = y, \ \text{ where } C \neq 0, \tag{40}$$

Eq.(39) becomes

$$-C\psi_\eta + R(\psi + \xi) = 0. \tag{41}$$

Solution of Eq.(41) is given as

$$\psi = g(\xi)e^{Ry/C} - (Cx + Dy), \tag{42}$$

where $g(\xi)$ is an arbitrary function. Using of Eq.(42) into Eq.(38) gives

$$C^2(C^2 + D^2)g''(\xi) + 2CDRg'(\xi) + (R^2 - C^2)g(\xi) = 0. \tag{43}$$

Solution of Eq.(43) is

$$\text{I. } g(\xi) = A_1e^{\lambda_1\xi} + B_1e^{\lambda_2\xi}, \text{ when } (C^2 + D^2) - R^2 > 0; \tag{44}$$

$$\text{II. } g(\xi) = (A_2 + B_2\xi)e^{-DR\xi/(C(C^2+D^2))}, \quad \text{when } (C^2 + D^2) - R^2 = 0; \tag{45}$$

$$\text{III. } g(\xi) = A_3\cos(m\xi + B_3)e^{-DR\xi/(C(C^2+D^2))}, \quad \text{when } (C^2 + D^2) - R^2 < 0, \tag{46}$$

where
$\lambda_{1,2} = [DR \pm C\sqrt{(C^2 + D^2) - R^2}\,]/[C(C^2 + D^2)]$, and
$m = \sqrt{R^2 - (C^2 + D^2)}\,/(C^2 + D^2)$.

For different cases, here we give the solution for the stream functions.

Case I When $(C^2+D^2)-R^2>0$, the solution for the stream function and the velocity components are

$$\psi = (A_1e^{\lambda_1(Cx+Dy)} + B_1e^{\lambda_2(Cx+Dy)})e^{Ry/C} - (Cx + Dy), \tag{47}$$

$$u = e^{Ry/C}[(D\lambda_1 + R/C)A_1e^{\lambda_1(Cx+Dy)} + (D\lambda_2 + R/C)B_1e^{\lambda_2(Cx+Dy)}] - D,$$
$$v = -Ce^{Ry/C}(A_1\lambda_1e^{\lambda_1(Cx+Dy)} + B_1\lambda_2e^{\lambda_2(Cx+Dy)}) + C. \tag{48}$$

Solution Eq.(48) shows an impingement of an oblique uniform stream with an oblique rotational, divergent flow. Figs.4 and 5 show the stream lines for Eq.(47), which illustrate the behaviour of the flows

$$\psi = (A_1e^{\lambda_1(Cx+Dy)} + B_1e^{\lambda_2(Cx+Dy)})e^{Ry/C} - (Cx + Dy).$$

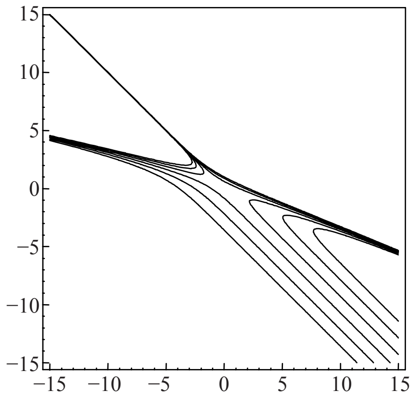

Fig.4 Investigation of stream lines for $\psi=(A_1\mathrm{e}^{\lambda_1(Cx+Dy)}+B_1\mathrm{e}^{\lambda_2(Cx+Dy)})\mathrm{e}^{Ry/C}-(Cx+Dy)$, when $A_1=2$, $B_1=-2$, $C=2$, $D=2$, $R=5/2$, $\lambda_1=(5+\sqrt{7})/16$, $\lambda_2=(5-\sqrt{7})/16$

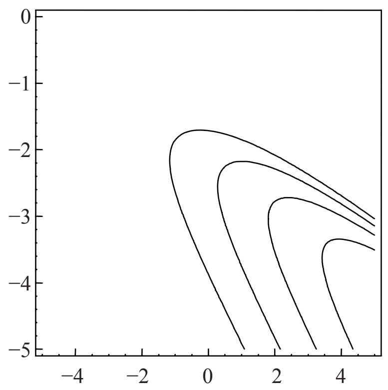

Fig.5 Investigation of stream lines for $\psi=(A_1\mathrm{e}^{\lambda_1(Cx+Dy)}+B_1\mathrm{e}^{\lambda_2(Cx+Dy)})\mathrm{e}^{Ry/C}-(Cx+Dy)$, when $A_1=70$, $B_1=40$, $C=2$, $D=2$, $R=5/2$, $\lambda_1=(5+\sqrt{7})/16$, $\lambda_2=(5-\sqrt{7})/16$

Case II When $(C^2+D^2)-R^2=0$, the solution for the stream function and the velocity components are

$$\psi=(A_2+B_2(Cx+Dy))\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-(Cx+Dy), \tag{49}$$

$$u=[DB_2+\gamma_1(A_2+B_2(Cx+Dy))]\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-D,$$
$$v=-[CB_2+\gamma_2(A_2+B_2(Cx+Dy))]\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-C, \tag{50}$$

where $\gamma_1=CR/(C^2+D^2)$ and $\gamma_2=-DR/(C^2+D^2)$.

Solution Eq.(49) represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, if B_2 is a positive real constant. In the case if B_2 shows an oblique uniform stream with an oblique rotational, convergent flow (Chandna and Oku-Ukpong, 1994). Figs.6 and 7 are reserved for the flows of the solution Eq.(49).

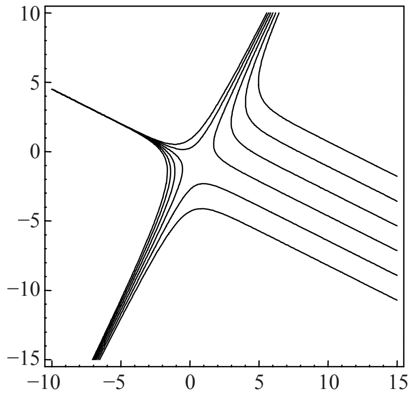

Fig.6 The stream lines for $\psi=(A_2+B_2(Cx+Dy))\cdot\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-(Cx+Dy)$, when $A_2=50$, $B_2=60$, $R=2$, $D=\sqrt{2}$, $C=\sqrt{2}$

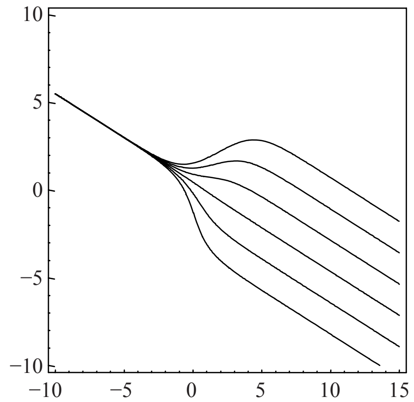

Fig.7 The stream lines for $\psi=(A_2+B_2(Cx+Dy))\cdot\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-(Cx+Dy)$, when $A_2=2$, $B_2=-2$, $R=2$, $D=\sqrt{2}$, $C=\sqrt{2}$

Case III When $(C^2+D^2)-R^2<0$, the solution for the stream function and the velocity components are

$$\psi=A_3\cos(m(Cx+Dy)+B_3)\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-(Cx+Dy), \tag{51}$$

$$u=A_3(\gamma_1\cos(m(Cx+Dy)+B_3))$$
$$-mD\sin(m(Cx+Dy)+B_3))\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-D,$$
$$v=A_3(-\gamma_2\cos(m(Cx+Dy)+B_3)$$
$$+mC\sin(m(Cx+Dy)+B_3))\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)}-C, \tag{52}$$

Fig.8 shows the streamlines for

$$\psi = A_3 \cos(m(Cx+Dy)+B_3)\mathrm{e}^{-DR\cdot(Cx+Dy)/(C(C^2+D^2))} - (Cx+Dy).$$

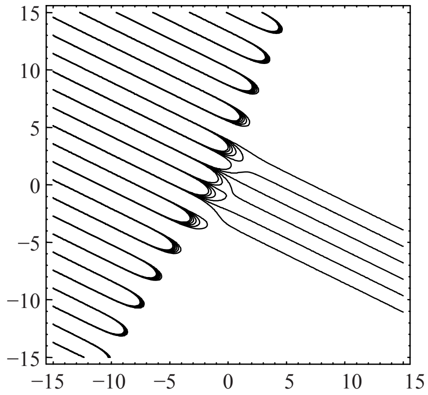

Fig.8 $\psi = A_3\cos(m(Cx+Dy)+B_3)\mathrm{e}^{R(Cy-Dx)/(C^2+D^2)} - (Cx+Dy)$, **where** A_3=2, B_3=0, C=1, D=2, R=3 **and** m=1

Remark 4 On setting ν_2=0, in solutions Eqs.(47), (49) and (51), we recover the Chandna and Oku-Ukpong (1994) case.

CONCLUSION

In this paper, inverse solutions are obtained for 2D incompressible couple stress fluid flows where the vorticity distribution is given by $\nabla^2\psi=\psi+f(x,y)$ for three different forms of $f(x,y)$. These solutions are although valid for all Reynolds number, but may be limited by instability. The expressions for velocity profile and streamline are constructed in each case while the pressure distribution is found for the first two cases. The present study has shown that the presence of the couple stress ν_2 may change the development of the solutions significantly. Also, the present analysis is more general and several results (Chandna and Oku-Ukpong, 1994) can be recovered in the limiting cases. Lastly, we should emphasized, however, that in order for an exact solution to be meaningful, it must have some possible physical application.

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