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Suboptimal reliable guaranteed cost control for continuous-time systems with multi-criterion constraints*

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Abstract: The suboptimal reliable guaranteed cost control (RGCC) with multi-criterion constraints is investigated for a class of uncertain continuous-time systems with sensor faults. A fault model in sensors, which considers outage or partial degradation of sensors, is adopted. The influence of the disturbance on the quadratic stability of the closed-loop systems is analyzed. The reliable state-feedback controller is developed by a linear matrix inequalities (LMIs) approach, to minimize the upper bound of a quadratic cost function under the conditions that all the closed-loop poles be placed in a specified disk, and that the prescribed level of H_∞ disturbance attenuation and the upper bound constraints of control inputs' magnitudes be guaranteed. Thus, with the above multi-criterion constraints, the resulting closed-loop system can provide satisfactory stability, transient property, a disturbance rejection level and minimized quadratic cost performance despite possible sensor faults.

Key words: Sensor faults, Multi-criterion constraints, Reliable guaranteed cost control (RGCC), Linear matrix inequality (LMI)
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INTRODUCTION

In the past decades, reliable guaranteed cost control (RGCC) of uncertain systems has attracted considerable attention (Veillette, 1995; Yang *et al*., 2000; Yee *et al*., 2000; Hsieh, 2002; Wu and Zhang, 2005; Liu *et al*., 2006; Pujol *et al*., 2007). It is concerned with the design of a reliable controller such that for possible faults the closed-loop system is stable and the upper bound of a quadratic cost function is guaranteed or minimized (Veillette, 1995; Yang *et al*., 2000; Liu *et al*., 2006; Pujol *et al*., 2007). However, many existing RGCC approaches deal with the quadratic cost performance merely in the sense of stability. In practice, the performance requirements are usually multi-objective and such multiple performance constraints are often expected in faulty systems (Elia and Dahleh, 1997; Wang and Ho, 2005; Zhang G. *et al*., 2006; Zhang D. *et al*., 2007). For instance, the regional closed-loop pole constraints describe the stability and transient behavior of closed-loop systems (Garcia, 1997; Chilali *et al*., 1999; Wang and Ho, 2005) and the H_∞ disturbance attenuation performance is against external disturbances for many systems (Doyle *et al*., 1989; Wu, 2007). On the other hand, due to the saturation restriction of actuators in practical applications, the magnitudes of control input signals are usually constrained to a certain limit. In this case the expected performance may not be guaranteed (Yu *et al*., 2004). Unfortunately few results have been reported on such RGCC problems with multi-criterion constraints for continuous-time systems with tolerant faults.

Therefore it is our objective to investigate the suboptimal RGCC problem with multi-criterion constraints, namely regional closed-loop poles, H_∞ norm bound and control input constraints, for a class of

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uncertain continuous-time systems subject to sensor faults. In view of possible sensor faults as well as the saturation of actuators, we first prove that the quadratic stability is not affected by disturbance signals. Then a state-feedback RGCC law with the above multi-criterion constraints is derived by linear matrix inequality (LMI) techniques. Finally, based on the parameterized representation, a convex optimization problem is formulated to minimize the upper bound of the quadratic cost performance index, and the corresponding suboptimal RGCC controller is obtained.

The paper is organized as follows. After the problem formulation in Section 2, the sufficient conditions and corresponding suboptimal RGCC controller design strategy with the multiple constraints are presented in Section 3. Simulative examples are provided in Section 4 to illustrate the validity of the proposed method. Finally, the concluding remarks are given in Section 5.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n\times m$ real matrices. The superscript 'T' denotes matrix transpose. $\boldsymbol{I}$ and $\boldsymbol{0}$ are unit matrix and zero matrix of appropriate dimensions, respectively. diag{…} denotes a block-diagonal matrix. The notation $\boldsymbol{P}>\boldsymbol{0}$ (respectively, $\boldsymbol{P}\geq\boldsymbol{0}$ and $\boldsymbol{P}<\boldsymbol{0}$) for $\boldsymbol{P}\in\mathbb{R}^{n\times m}$ means that the matrix $\boldsymbol{P}$ is real symmetric positive definite (respectively, nonnegative definite and negative definite). $\Lambda(\boldsymbol{A})$ represents the eigenvalues of matrix $\boldsymbol{A}$. In symmetric block matrices, we use '*' as an ellipsis for terms that are induced by symmetry.

PROBLEM FORMULATION

Consider a class of uncertain continuous-time systems with faults:

$$\begin{cases}\dot{\boldsymbol{x}}(t)=(\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}^{\mathrm{f}}(t)+\boldsymbol{D}_1\boldsymbol{w}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}_2\boldsymbol{w}(t),\end{cases}\quad(1)$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state vector, $\boldsymbol{u}^{\mathrm{f}}(t)=[u_1^{\mathrm{f}}(t), u_2^{\mathrm{f}}(t), \ldots, u_i^{\mathrm{f}}(t), \ldots, u_p^{\mathrm{f}}(t)]^{\mathrm{T}}\in\mathbb{R}^p$ is the control input vector indicating possible sensor faults, $\boldsymbol{y}(t)\in\mathbb{R}^d$ is the controlled output vector, and $\boldsymbol{w}(t)\in\mathbb{R}^k$ is the norm-bounded exogenous disturbance and is integrable; $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are known real constant matrices of appropriate dimensions. The matrix $\Delta\boldsymbol{A}(t)$ represents parametric perturbations of the following form (Wang and Ho, 2005):

$$\Delta\boldsymbol{A}(t)=\boldsymbol{H}\boldsymbol{F}(t)\boldsymbol{E},\quad \boldsymbol{F}(t)\boldsymbol{F}^{\mathrm{T}}(t)\leq\boldsymbol{I},\quad(2)$$

where $\boldsymbol{H}$ and $\boldsymbol{E}$ are known constant matrices describing the structure of the uncertainty. For clarity, set $\Delta\boldsymbol{A}:=\Delta\boldsymbol{A}(t)$, $\boldsymbol{F}:=\boldsymbol{F}(t)$ throughout the rest of this paper.

For the simplicity of research and without loss of generality, we assume that the states of system Eq.(1) are measurable by sensors for state-feedback control, the measurement of the sensor is equal to the corresponding state in the normal case, and the dynamics of actuators is neglected except for the saturation.

With respect to the control input $\boldsymbol{u}^{\mathrm{f}}(t)$, when the state-feedback control law is applied to system Eq.(1), the following fault model similar to that in (Zhang D. *et al*., 2007) is adopted for this study:

$$\begin{cases}\boldsymbol{u}^{\mathrm{f}}(t)=\boldsymbol{G}\boldsymbol{x}^{\mathrm{f}}(t)=\boldsymbol{G}\boldsymbol{M}(t)\boldsymbol{x}(t),\\ \boldsymbol{x}^{\mathrm{f}}(t)=[x_1^{\mathrm{f}}(t), x_2^{\mathrm{f}}(t),\cdots,x_i^{\mathrm{f}}(t),\cdots,x_n^{\mathrm{f}}(t)]^{\mathrm{T}}\\ \qquad=\boldsymbol{M}(t)\boldsymbol{x}(t),\end{cases}\quad(3)$$

where $\boldsymbol{G}\in\mathbb{R}^{p\times n}$ is the designed feedback gain matrix, $\boldsymbol{M}(t)=\mathrm{diag}\{m_1(t), m_2(t), \cdots, m_n(t)\}\in\Theta_{\mathrm{s}}$ denotes the sensor fault matrix. $\Theta_{\mathrm{s}}=\{\boldsymbol{M}(t)|\boldsymbol{M}(t)\neq\boldsymbol{0}, m_{\mathrm{L}i}\leq m_i(t)\leq m_{\mathrm{U}i}, i=1, 2, \cdots, n\}$ is the set of sensor fault matrices, where $0\leq m_{\mathrm{L}i}\leq 1$, $1\leq m_{\mathrm{U}i}<\infty$ are known constants and assumed to be known *a priori*. $x_i^{\mathrm{f}}(t)=m_i(t)x_i(t)$, $u_i^{\mathrm{f}}(t)=\boldsymbol{g}_i\boldsymbol{x}^{\mathrm{f}}(t)=\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{x}(t)$, where $\boldsymbol{g}_i$ is the ith row vector of matrix $\boldsymbol{G}$. Then the closed-loop system with faults is given by

$$\begin{cases}\dot{\boldsymbol{x}}(t)=[\boldsymbol{A}+\Delta\boldsymbol{A}+\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}(t)]\boldsymbol{x}(t)+\boldsymbol{D}_1\boldsymbol{w}(t)\\ \qquad=[\boldsymbol{A}_{\mathrm{c}}+\Delta\boldsymbol{A}]\boldsymbol{x}(t)+\boldsymbol{D}_1\boldsymbol{w}(t)\\ \qquad=\bar{\boldsymbol{A}}_{\mathrm{c}}\boldsymbol{x}(t)+\boldsymbol{D}_1\boldsymbol{w}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}_2\boldsymbol{w}(t),\end{cases}\quad(4)$$

where $\boldsymbol{A}_{\mathrm{c}}=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}(t)$, $\bar{\boldsymbol{A}}_{\mathrm{c}}=\boldsymbol{A}_{\mathrm{c}}+\Delta\boldsymbol{A}$.

Remark 1 The above fault element $m_i(t)$ represents the time-varying sensor fault. $m_i(t)=1$ means that the ith sensor is normal, and corresponds to the normal case $x_i^{\mathrm{f}}(t)=x_i(t)$ and $u_i^{\mathrm{f}}(t)=u_i(t)$. If $m_i(t)=0$ or has other values, the corresponding sensor would be a total failure or partial degradation failure case.

The decomposition of fault function $\boldsymbol{M}(t)$ in the fault model Eq.(3) is given below, which will be used for our main results.

Define $\boldsymbol{M}_0=\text{diag}\{m_{01}, m_{02}, \cdots, m_{0n}\}$, $\boldsymbol{L}=\text{diag}\{l_1, l_2, \cdots, l_n\}$, $\boldsymbol{Z}(t)=\text{diag}\{z_1(t), z_2(t), \cdots, z_n(t)\}$, where $m_{0i}=(m_{\text{U}i}+m_{\text{L}i})/2$, $l_i=(m_{\text{U}i}-m_{\text{L}i})/(m_{\text{U}i}+m_{\text{L}i})$, $z_i(k)=[m_i(t)-m_{0i}]/m_{0i}$, $i=1, 2, \cdots, n$. We then have

$$\begin{cases} \boldsymbol{M}(\boldsymbol{t}) = \boldsymbol{M}_0 + \boldsymbol{M}_0\boldsymbol{Z}(\boldsymbol{t}), \\ \boldsymbol{M}_0 > \boldsymbol{0}, \ \ \boldsymbol{L} > \boldsymbol{0}, \ \ \boldsymbol{Z}^{\text{T}}(\boldsymbol{t})\boldsymbol{Z}(\boldsymbol{t}) \le \boldsymbol{L}^{\text{T}}\boldsymbol{L} \le \boldsymbol{I}. \end{cases} \tag{5}$$

For the closed-loop system Eq.(4), the quadratic cost function associated with this system is

$$\begin{aligned} J &= \int_0^\infty [\boldsymbol{x}^{\text{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t) + (\boldsymbol{u}^{\text{f}}(t))^{\text{T}}\boldsymbol{R}\boldsymbol{u}^{\text{f}}(t)]\,\text{d}t \\ &= \int_0^\infty \boldsymbol{x}^{\text{T}}(t)[\boldsymbol{Q} + (\boldsymbol{G}\boldsymbol{M}(t))^{\text{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t)]\boldsymbol{x}(t)\,\text{d}t \\ &= \int_0^\infty \boldsymbol{x}^{\text{T}}(t)\tilde{\boldsymbol{R}}\boldsymbol{x}(t)\,\text{d}t, \end{aligned} \tag{6}$$

where $\tilde{\boldsymbol{R}} = \boldsymbol{Q} + (\boldsymbol{G}\boldsymbol{M}(t))^{\text{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t)$, $\boldsymbol{Q}\geq\boldsymbol{0}$ and $\boldsymbol{R}>\boldsymbol{0}$ are given weighting matrices.

In our study, first of all we design the gain matrix $\boldsymbol{G}$ to assign all the closed-loop poles into an expected circular region $\Phi(-q, r)$ centered at the point $(-q, 0)$ with radius $0<r\leq q$, i.e., $\Lambda(\boldsymbol{A}_\text{c}+\Delta\boldsymbol{A})\subset\Phi(-q, r)$. Thus the system Eq.(4) would be stable and have a certain transient behavior. Then, taking the saturation of actuators and the energy limit into account, the ith control input $u_i^{\text{f}}(t)$ is restricted by

$$|u_i^{\text{f}}(t)| \leq \sqrt{\alpha_i}\,, \quad 0 < \alpha_i < \infty, \ \ i = 1, 2, ..., p, \tag{7}$$

where α_i is a known constant. Next, to attenuate the impact of exogenous disturbance, the level of H_∞ disturbance rejection of the closed-loop system is constrained within a given bound. To this end, the suboptimal RGCC problem considered in this paper is to determine the gain matrix $\boldsymbol{G}$ such that the following performance constraints hold for the closed-loop system Eq.(4) with sensor faults:

(a) The closed-loop poles are constrained to lie within the specified disk, i.e., $\Lambda(\boldsymbol{A}_\text{c}+\Delta\boldsymbol{A})\subset\Phi(-q, r)$;

(b) The H_∞ norm of the disturbance transfer matrix $\boldsymbol{H}(s)$ from $\boldsymbol{w}(t)$ to $\boldsymbol{y}(t)$ satisfies the constraint

$$\|\boldsymbol{H}(s)\|_\infty \leq \gamma, \tag{8}$$

where $\|\boldsymbol{H}(s)\|_\infty := \sup\{\sigma_{\max}(\boldsymbol{H}(s))\}$ and $\sigma_{\max}(\boldsymbol{H}(s))$ denotes the largest singular value of $\boldsymbol{H}(s)$, γ is a given positive constant;

(c) The control input constraint Eq.(7) is met;

(d) The upper bound of quadratic cost function Eq.(6) is minimized.

In this paper, we will solve the RGCC problem according to the multi-objective optimization strategy and LMI techniques (Boyd *et al*., 1994).

MAIN RESULTS

Proposition 1 The system Eq.(4) is said to be reliable and quadratically stable for the cost function Eq.(6), if there exists a control gain matrix $\boldsymbol{G}$ associated with matrix $\boldsymbol{P}>\boldsymbol{0}$ such that the following matrix inequality

$$\overline{\boldsymbol{A}}_\text{c}^{\text{T}}\boldsymbol{P} + \boldsymbol{P}\overline{\boldsymbol{A}}_\text{c} + \boldsymbol{Q} + (\boldsymbol{G}\boldsymbol{M}(t))^{\text{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t) < \boldsymbol{0} \tag{9}$$

holds. Meanwhile, the closed-loop quadratic cost function Eq.(6) satisfies

$$J < \boldsymbol{x}^{\text{T}}(0)\boldsymbol{P}\boldsymbol{x}(0), \tag{10}$$

where $\boldsymbol{x}(0)$ is the initial state.

Proof The Lyapunov function candidate

$$V(t) = \boldsymbol{x}^{\text{T}}(t)\boldsymbol{P}\boldsymbol{x}(t) + \int_0^t \delta\boldsymbol{w}^{\text{T}}(\tau)\boldsymbol{w}(\tau)\text{d}\tau$$

is positive definite, where $\delta>0$ is a small enough scalar. Then we have $V(0)=\boldsymbol{x}^{\text{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)$. The corresponding differential of the Lyapunov function is given by

$$\begin{aligned} \dot{V}(t) &= \dot{\boldsymbol{x}}^{\text{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)+\boldsymbol{x}^{\text{T}}(t)\boldsymbol{P}\dot{\boldsymbol{x}}(t)+\delta\boldsymbol{w}^{\text{T}}(t)\boldsymbol{w}(t) - \delta\|\boldsymbol{w}(0)\|^2 \\ &= \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{w}\end{bmatrix}^{\text{T}} \begin{bmatrix}\overline{\boldsymbol{A}}_\text{c}^{\text{T}}\boldsymbol{P}+\boldsymbol{P}\overline{\boldsymbol{A}}_\text{c} & \boldsymbol{P}\boldsymbol{D}\\ \boldsymbol{D}^{\text{T}}\boldsymbol{P} & \delta\boldsymbol{I}\end{bmatrix} \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{w}\end{bmatrix} - \delta\|\boldsymbol{w}(0)\|^2. \end{aligned}$$

If Eq.(9) holds, then

$$\overline{\boldsymbol{A}}_\text{c}^{\text{T}}\boldsymbol{P}+\boldsymbol{P}\overline{\boldsymbol{A}}_\text{c} - \boldsymbol{P}\boldsymbol{D}\boldsymbol{D}^{\text{T}}\boldsymbol{P}/\delta+\boldsymbol{Q}+(\boldsymbol{G}\boldsymbol{M}(t))^{\text{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t) < \boldsymbol{0}$$

is true. It follows from the Schur complement (Pujol *et al.*, 2007) that

$$\begin{bmatrix} \bar{\boldsymbol{A}}_c^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\bar{\boldsymbol{A}}_c+\boldsymbol{Q}+(\boldsymbol{G}\boldsymbol{M}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t) & \boldsymbol{P}\boldsymbol{D} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & \delta\boldsymbol{I} \end{bmatrix}<\boldsymbol{0}$$

$$\Rightarrow \begin{bmatrix} \bar{\boldsymbol{A}}_c^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\bar{\boldsymbol{A}}_c & \boldsymbol{P}\boldsymbol{D} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & \delta\boldsymbol{I} \end{bmatrix}<\begin{bmatrix} -\boldsymbol{Q}-(\boldsymbol{G}\boldsymbol{M}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

That means

$$\begin{aligned} \dot{V}(t) &< \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\boldsymbol{Q}-(\boldsymbol{G}\boldsymbol{M}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix} \\ &= -\boldsymbol{x}^{\mathrm{T}}[\boldsymbol{Q}+(\boldsymbol{G}\boldsymbol{M}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t)]\boldsymbol{x}<\boldsymbol{0}. \end{aligned} \quad (11)$$

According to the Lyapunov theory and quadratic stability concept mentioned in (Garcia, 1997), system Eq.(4) is reliable and quadratically stable and $V(t)$ is monotonously convergent with $V(t)\le V(0)=\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)$ as well as $\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)\le\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)$.

Moreover, from Eq.(11) we have $-\dot{V}(t)>\boldsymbol{x}^{\mathrm{T}}[\boldsymbol{Q}+(\boldsymbol{G}\boldsymbol{M}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{G}\boldsymbol{M}(t)]\boldsymbol{x}$. Integrating both sides of the above inequality from 0 to ∞ and using the system stability yield

$$J=\int_0^{\infty}\boldsymbol{x}^{\mathrm{T}}(t)\tilde{\boldsymbol{R}}\boldsymbol{x}(t)\,\mathrm{d}t<V(0)=\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0). \quad (12)$$

This ends the proof.

Remark 2 Many existing results (Garcia, 1997; Yu *et al.*, 2004; Pujol *et al.*, 2007) seldom discuss the quadratic stability of systems with disturbance input. The above proof indicates that the disturbance $\boldsymbol{w}(t)$ does not affect the quadratic stability of the closed-loop system.

To remove the dependence on the initial state in Eq.(12), we adopt the deterministic approach (Chen *et al.*, 2004) in this paper. Suppose that the initial state of the system Eq.(1) is arbitrary but belongs to the set $N=\{\boldsymbol{x}(0)\in\mathbb{R}^n|\boldsymbol{x}(0)=\boldsymbol{U}\boldsymbol{v},\ \boldsymbol{U}\ne\boldsymbol{0},\ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}\le 1\}$, where $\boldsymbol{U}$ is a given matrix. The quadratic cost bound in Eq.(12) then leads to

$$\begin{cases} J<\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)\le\lambda_{\max}(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{U}), \\ \lambda:=\sup\limits_{\boldsymbol{P}}[\lambda_{\max}(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{U})], \end{cases} \quad (13)$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of the matrix. Then we have $\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)\le\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)\le\lambda$ based on Proposition 1.

As for the constraint Eq.(7) of the ith control input signal, we have

$$|u_i^{\mathrm{f}}(t)|\le\sqrt{\alpha_i}\Rightarrow|\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{x}(t)|\le\sqrt{\alpha_i}$$

when the sensor failure occurred. Then it follows that

$$\begin{aligned} |u_i^{\mathrm{f}}(t)|^2 &= |\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{x}(t)|^2=|\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{P}^{-1/2}\boldsymbol{P}^{1/2}\boldsymbol{x}(t)|^2 \\ &\le\|\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{P}^{-1/2}\|^2\cdot\|\boldsymbol{P}^{1/2}\boldsymbol{x}(t)\|^2 \\ &\le\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{P}^{-1}\boldsymbol{M}^{\mathrm{T}}(t)\boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t) \\ &\le\lambda\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{P}^{-1}\boldsymbol{M}^{\mathrm{T}}(t)\boldsymbol{g}_i^{\mathrm{T}}. \end{aligned} \quad (14)$$

So, if the inequality $\lambda\boldsymbol{g}_i\boldsymbol{M}(t)\boldsymbol{P}^{-1}\boldsymbol{M}^{\mathrm{T}}(t)\boldsymbol{g}_i^{\mathrm{T}}\le\alpha_i$ holds, the constraint Eq.(7) is met. Therefore, we have the following inequality on the control input $\boldsymbol{u}^{\mathrm{f}}(t)$ of faulty systems:

$$\lambda\boldsymbol{G}\boldsymbol{M}(t)\boldsymbol{P}^{-1}\boldsymbol{M}^{\mathrm{T}}(t)\boldsymbol{G}^{\mathrm{T}}<\boldsymbol{X}, \quad (15)$$

where matrix $\boldsymbol{X}>\boldsymbol{0}$ with $X_{ii}\le\alpha_i$ (i=1, 2, …, p), X_{ii} denotes the ith diagonal elements of $\boldsymbol{X}$. The inequality Eq.(15) ensures that the variation of control inputs' amplitudes is within the given constraints. $\boldsymbol{X}$ can be determined by many methods such as the LMI technique (Boyd *et al.*, 1994) and the technique introduced in (Zhang D. *et al.*, 2007).

Lemma 1 (Cao *et al.*, 2004) Given matrices $\boldsymbol{A}$, $\boldsymbol{H}$ and $\boldsymbol{E}$ of appropriate dimensions and $\boldsymbol{A}$ is symmetric, then $\boldsymbol{A}+\boldsymbol{H}\boldsymbol{F}\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}<\boldsymbol{0}$ for all $\boldsymbol{F}$ satisfying $\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}\le\boldsymbol{I}$, if and only if there exists a scalar $\varepsilon>0$ such that $\boldsymbol{A}+\varepsilon\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}}+\varepsilon^{-1}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}<\boldsymbol{0}$.

Lemma 2 (Zhang D. *et al.*, 2007) Suppose that $\boldsymbol{S}=\mathrm{diag}\{s_1, s_2, \ldots, s_n\}$ is a time-varying diagonal matrix with $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S}\le\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W}$, where $\boldsymbol{W}>\boldsymbol{0}$ is a known diagonal matrix. $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are matrices of appropriate dimensions. Then for any scalar $\varepsilon>0$, the inequality $\boldsymbol{R}_1\boldsymbol{S}\boldsymbol{R}_2+\boldsymbol{R}_2^{\mathrm{T}}\boldsymbol{S}\boldsymbol{R}_1^{\mathrm{T}}\le\varepsilon\boldsymbol{R}_1\boldsymbol{W}\boldsymbol{R}_1^{\mathrm{T}}+\varepsilon^{-1}\boldsymbol{R}_2^{\mathrm{T}}\boldsymbol{W}\boldsymbol{R}_2$ holds.

Theorem 1 Consider the system Eq.(1) and cost function Eq.(6) in the sensor fault case, for the given constraints (a), (b) and (c), if there exist matrices $\boldsymbol{S}>\boldsymbol{L}>\boldsymbol{0}$, $\boldsymbol{Y}$ and scalars $\varepsilon>0$, $\lambda>0$, such that the following LMIs Eqs.(16)~(18) hold,

$$\begin{bmatrix} -\boldsymbol{I} & \boldsymbol{U}^{\mathrm{T}} \\ \boldsymbol{U} & -\boldsymbol{S} \end{bmatrix} < \boldsymbol{0}, \tag{16}$$

$$\begin{bmatrix} -\boldsymbol{X} & * & * & * \\ \boldsymbol{Y}^{\mathrm{T}} & -\boldsymbol{S} & * & * \\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{0} & -\boldsymbol{S} & * \\ \boldsymbol{0} & \boldsymbol{S} & \boldsymbol{0} & -\boldsymbol{L}^{-1} \end{bmatrix} < \boldsymbol{0}, \tag{17}$$

$$\begin{bmatrix} -r^2\boldsymbol{S} & * & * & * & * & * & * & * & * \\ \lambda\boldsymbol{D}_1^{\mathrm{T}} & -\lambda\gamma^2\boldsymbol{I} & * & * & * & * & * & * & * \\ \boldsymbol{CS} & \lambda\boldsymbol{D}_2 & -\lambda\boldsymbol{I} & * & * & * & * & * & * \\ (\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{S}+\boldsymbol{BY} & \boldsymbol{0} & \boldsymbol{0} & \varepsilon\boldsymbol{HH}^{\mathrm{T}}-\boldsymbol{S} & * & * & * & * & * \\ \boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{Q})^{-1} & * & * & * & * \\ \boldsymbol{Y} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{R})^{-1} & * & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{Y}^{\mathrm{T}} & -\boldsymbol{S} & * & * \\ \boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{L}^{-1} & * \\ \boldsymbol{ES} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\varepsilon\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \tag{18}$$

where $\boldsymbol{U}$, $\boldsymbol{X}>\boldsymbol{0}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are known constant matrices. Then for all admissible uncertainties and possible faults $\boldsymbol{M}(t)$, the closed-loop system Eq.(4) with a reliable guaranteed cost controller

$$\boldsymbol{u}^{\mathrm{f}}(t) = \boldsymbol{Gx}^{\mathrm{f}}(t) = \boldsymbol{YS}^{-1}\boldsymbol{M}_0^{-1}\boldsymbol{x}^{\mathrm{f}}(t) \tag{19}$$

is quadratically stable with constraints (a), (b) and (c), and the corresponding closed-loop quadratic cost function J is with $J<(q/r^2)\lambda$.

Proof First of all, from the Bound Real Lemma (Doyle *et al.*, 1989) and Proposition 1, we can prove that if the following inequalities

$$\begin{cases} (\bar{\boldsymbol{A}}_{\mathrm{c}}+q\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}(\bar{\boldsymbol{A}}_{\mathrm{c}}+q\boldsymbol{I})-r^2\boldsymbol{P}+r^2\tilde{\boldsymbol{R}}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} \\ +(\boldsymbol{PD}_1+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{D}_2)(\gamma^2\boldsymbol{I}-\boldsymbol{D}_2^{\mathrm{T}}\boldsymbol{D}_2)^{-1}(\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{D}_2^{\mathrm{T}}\boldsymbol{C})<\boldsymbol{0}, \\ \gamma^2\boldsymbol{I}-\boldsymbol{D}_2^{\mathrm{T}}\boldsymbol{D}_2>\boldsymbol{0}, \end{cases} \tag{20}$$

hold for all possible faults $\boldsymbol{M}(t)$, then system Eq.(4) is quadratically stable meeting the constraints (a) and (b), and the closed-loop cost function Eq.(6) satisfies $J<\boldsymbol{x}^{\mathrm{T}}(0)(q/r^2)\boldsymbol{Px}(0)$. Based on Eq.(13), if inequality $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{PU}-\lambda\boldsymbol{I}<\boldsymbol{0}$ holds, we can conclude that $J<(q/r^2)\lambda_{\max}(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{PU})<(q/r^2)\lambda$. With the Schur complement, Eq.(16) is equivalent to $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{PU}-\lambda\boldsymbol{I}<\boldsymbol{0}$ by letting $\boldsymbol{S}=\lambda\boldsymbol{P}^{-1}$.

Next, according to the Schur complement, the inequalities in Eq.(20) are equivalent to

$$\begin{bmatrix} r^2\tilde{\boldsymbol{R}}-r^2\boldsymbol{P} & * & * & * \\ \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{P} & -\gamma^2\boldsymbol{I} & * & * \\ \boldsymbol{C} & \boldsymbol{D}_2 & -\boldsymbol{I} & * \\ \boldsymbol{P}(\bar{\boldsymbol{A}}_{\mathrm{c}}+q\boldsymbol{I}) & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{P} \end{bmatrix} < \boldsymbol{0}. \tag{21}$$

Pre- and post-multiplying the left hand side of the above inequality by matrix diag$\{\sqrt{\lambda}\boldsymbol{P}^{-1}, \sqrt{\lambda}\boldsymbol{I}, \sqrt{\lambda}\boldsymbol{I}, \sqrt{\lambda}\boldsymbol{P}^{-1}\}$, it follows by the Schur complement that Eq.(22) (see the next page) holds. With the sensor fault decomposition Eq.(5) and the Schur complement, Eq.(22) can be rewritten as Eq.(23). By Lemma 2, if $\lambda\boldsymbol{P}^{-1}>\boldsymbol{L}$ we have Eq.(24). Substituting Eq.(24) into Eq.(23), we can obtain that Eq.(23) must be true if Eq.(25) holds. Dealing with the uncertainty $\Delta\boldsymbol{A}$ in Eq.(25) by Lemma 1, Eq.(26) is equivalent to Eq.(25) for a scalar $\varepsilon>0$. Let $\lambda\boldsymbol{P}^{-1}=\boldsymbol{S}$, $\boldsymbol{GM}_0\boldsymbol{S}=\boldsymbol{Y}$, so Eq.(26) is the inequality Eq.(18).

In terms of the constraint (c), with fault decomposition Eq.(5), converting Eq.(15) into LMI form by the Schur complement and Lemma 2, yields Eq.(27) for $\lambda\boldsymbol{P}^{-1}>\boldsymbol{L}$.

$$\begin{bmatrix} -\lambda r^2\boldsymbol{P}^{-1} & * & * & * & * & * \\ \lambda\boldsymbol{D}_1^{\mathrm{T}} & -\lambda r^2\boldsymbol{I} & * & * & * & * \\ \lambda\boldsymbol{C}\boldsymbol{P}^{-1} & \lambda\boldsymbol{D}_2 & -\lambda\boldsymbol{I} & * & * & * \\ \lambda(\bar{\boldsymbol{A}}_{\mathrm{c}}+q\boldsymbol{I})\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & -\lambda\boldsymbol{P}^{-1} & * & * \\ \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{Q})^{-1} & * \\ \lambda\boldsymbol{G}\boldsymbol{M}(t)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{R})^{-1} \end{bmatrix} < \boldsymbol{0} \tag{22}$$

$$\begin{bmatrix} -\lambda r^2\boldsymbol{P}^{-1} & * & * & * & * & * \\ \lambda\boldsymbol{D}_1^{\mathrm{T}} & -\lambda r^2\boldsymbol{I} & * & * & * & * \\ \lambda\boldsymbol{C}\boldsymbol{P}^{-1} & \lambda\boldsymbol{D}_2 & -\lambda\boldsymbol{I} & * & * & * \\ \lambda(\boldsymbol{A}+\Delta\boldsymbol{A}+q\boldsymbol{I}+\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & -\lambda\boldsymbol{P}^{-1} & * & * \\ \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{Q})^{-1} & * \\ \lambda\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{R})^{-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & * & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{0} & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & * & * & * \\ \lambda\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{Z}(t)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & * \\ \lambda\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{Z}(t)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} < \boldsymbol{0} \tag{23}$$

$$\begin{bmatrix} \boldsymbol{0} & * & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{0} & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & * & * & * \\ \lambda\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{Z}(t)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & * \\ \lambda\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{Z}(t)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} < \begin{bmatrix} \lambda^2\boldsymbol{P}^{-1}\boldsymbol{L}\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0\lambda\boldsymbol{P}^{-1}(\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0)^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0\lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}\boldsymbol{M}_0\lambda\boldsymbol{P}^{-1}(\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0)^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{G}\boldsymbol{M}_0\lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} -\lambda r^2\boldsymbol{P}^{-1} & * & * & * & * & * & * & * \\ \lambda\boldsymbol{D}_1^{\mathrm{T}} & -\lambda r^2\boldsymbol{I} & * & * & * & * & * & * \\ \lambda\boldsymbol{C}\boldsymbol{P}^{-1} & \lambda\boldsymbol{D}_2 & -\lambda\boldsymbol{I} & * & * & * & * & * \\ \lambda(\boldsymbol{A}+\Delta\boldsymbol{A}+q\boldsymbol{I}+\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & -\lambda\boldsymbol{P}^{-1} & * & * & * & * \\ \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{Q})^{-1} & * & * & * \\ \lambda\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{R})^{-1} & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0} & \lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} & -\lambda\boldsymbol{P}^{-1} & * \\ \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{L}^{-1} \end{bmatrix} < \boldsymbol{0} \tag{25}$$

$$\begin{bmatrix} -\lambda r^2\boldsymbol{P}^{-1} & * & * & * & * & * & * & * & * \\ \lambda\boldsymbol{D}_1^{\mathrm{T}} & -\lambda r^2\boldsymbol{I} & * & * & * & * & * & * & * \\ \lambda\boldsymbol{C}\boldsymbol{P}^{-1} & \lambda\boldsymbol{D}_2 & -\lambda\boldsymbol{I} & * & * & * & * & * & * \\ \lambda(\boldsymbol{A}+\Delta\boldsymbol{A}+q\boldsymbol{I}+\boldsymbol{B}\boldsymbol{G}\boldsymbol{M}_0)\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \varepsilon\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}}-\lambda\boldsymbol{P}^{-1} & * & * & * & * & * \\ \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{Q})^{-1} & * & * & * & * \\ \lambda\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{R})^{-1} & * & * & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0} & \lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} & -\lambda\boldsymbol{P}^{-1} & * & * \\ \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{L}^{-1} & * \\ \lambda\boldsymbol{E}\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\varepsilon\boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{26}$$

$$\begin{bmatrix} -\boldsymbol{X} & * & * & * \\ \lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} & -\lambda\boldsymbol{P}^{-1} & * & * \\ \lambda\boldsymbol{P}^{-1}\boldsymbol{M}_0^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} & \boldsymbol{0} & -\lambda\boldsymbol{P}^{-1} & * \\ \boldsymbol{0} & \lambda\boldsymbol{P}^{-1} & \boldsymbol{0} & -\boldsymbol{L}^{-1} \end{bmatrix} < \boldsymbol{0}. \quad (27)$$

Defining $\lambda\boldsymbol{P}^{-1}=\boldsymbol{S}$, $\boldsymbol{G}\boldsymbol{M}_0\boldsymbol{S}=\boldsymbol{Y}$, so Eq.(27) is Eq.(17). Thus, with the LMIs Eqs.(16)~(18), the controller Eq.(19) can be obtained for the multiple criteria (a)~(c). This completes the proof.

To solve the suboptimal RGCC problem in this paper, the desirable RGCC controller that minimizes the upper bound of the cost function Eq.(6) can be developed straightforwardly, based on Theorem 1. The design is formulated as the following optimization problem:

Corollary 1 Consider the system Eq.(1) and cost function Eq.(6) in the sensor fault case. For the given constraints (a), (b) and (c), if there exist matrices $\boldsymbol{S}>\boldsymbol{L}>\boldsymbol{0}$, $\boldsymbol{Y}$ and scalars $\varepsilon>0$, $\lambda>0$, such that the following optimization problem

$$\min_{\boldsymbol{S},\boldsymbol{Y},\lambda,\varepsilon} \lambda, \qquad \text{s.t. LMIs (16)~(18)} \qquad (28)$$

has solution ε_0, λ_0, $\boldsymbol{S}_0$ and $\boldsymbol{Y}_0$, then for all admissible uncertainties and possible faults $\boldsymbol{M}(t)$,

$$\boldsymbol{u}_0^{\mathrm{f}}(t)=\boldsymbol{G}_0\boldsymbol{x}^{\mathrm{f}}(t)=\boldsymbol{Y}_0\boldsymbol{S}_0^{-1}\boldsymbol{M}_0^{-1}\boldsymbol{x}^{\mathrm{f}}(t) \qquad (29)$$

is a suboptimal RGCC controller, such that the faulty closed-loop system Eq.(4) is quadratically stable meeting the constraints (a)~(d), and the corresponding closed-loop quadratic cost function J is with $J<(q/r^2)\lambda_0$.

It is clear that Eq.(28) is a convex optimization problem with LMI constraints and can be effectively solved by various existing LMI software. Thus, the minimization of λ implies the minimization of the upper bound of the quadratic cost function J in Eq.(6). The convexity of the optimization problem ensures that a global optimum, when it exists, is reachable.

Based on Theorem 1 and Corollary 1, the design procedure for the suboptimal RGCC controller with constraints (a)~(d) can be concluded as follows:

Step 1: Initialization. According to the description of the system, set the suitable parameters of the fault model and the uncertainty model, and the given multi-criterion constraints.

Step 2: For the constraint (c), with the technique of the LMI Toolbox (Boyd *et al.*, 1994) or that introduced in (Zhang D. *et al.*, 2007), select a matrix $\boldsymbol{X}>\boldsymbol{0}$ with $0<X_{ii}\le\alpha_i$ (i=1, 2, …, p); determine a proper matrix $\boldsymbol{U}$ by the initial state.

Step 3: Compute the optimal problem Eq.(28) by the LMI software.

Step 4: If the optimal solution is available, the desirable reliable controller Eq.(29) is obtained, stop; otherwise,

Step 4.1: Relax X_{ii} and go to Step 2.

Step 4.2: If for all possible $\boldsymbol{X}$ and $\boldsymbol{U}$, the optimal problem is not feasible, this method cannot get a feasible controller for the prescribed multi-criterion constraints, stop.

In a similar manner to Theorem 1 and Corollary 1, as for the system Eq.(1) in the normal case with constraints (a)~(d), we can get the suboptimal GCC controller without fault tolerance. It is described by the following corollary without proof:

Corollary 2 Consider the system Eq.(1) and cost function Eq.(6) without faults, i.e., $\boldsymbol{M}(t)=\boldsymbol{I}$. For the given constraints (a), (b) and (c), if there exist matrices $\boldsymbol{S}>\boldsymbol{0}$, $\boldsymbol{Y}$ and scalars $\lambda>0$, $\beta>0$, such that the following optimization problem

$$\min_{\boldsymbol{S},\boldsymbol{Y},\lambda,\beta} \lambda \quad \text{s.t. LMIs (16), (31) and (32)} \qquad (30)$$

has solution λ_{n}, β_{n}, $\boldsymbol{S}_{\mathrm{n}}$, and $\boldsymbol{Y}_{\mathrm{n}}$, where LMIs Eqs.(31) and (32) are given as follows:

$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{Y} \\ \boldsymbol{Y}^{\mathrm{T}} & -\boldsymbol{S} \end{bmatrix} < \boldsymbol{0}, \qquad (31)$$

$$\begin{bmatrix} -r^2\boldsymbol{S} & * & * & * & * & * & * \\ \lambda\boldsymbol{D}_1^{\mathrm{T}} & -\lambda\gamma^2\boldsymbol{I} & * & * & * & * & * \\ \boldsymbol{C}\boldsymbol{S} & \lambda\boldsymbol{D}_2 & -\lambda\boldsymbol{I} & * & * & * & * \\ (\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{S}+\boldsymbol{B}\boldsymbol{Y} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}}-\boldsymbol{S} & * & * & * \\ \boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{Q})^{-1} & * & * \\ \boldsymbol{Y} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda(r^2\boldsymbol{R})^{-1} & * \\ \boldsymbol{E}\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\beta\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \qquad (32)$$

then for all admissible uncertainties,

$$\boldsymbol{u}_{\mathrm{n}}(t)=\boldsymbol{G}_{\mathrm{n}}\boldsymbol{x}(t)=\boldsymbol{Y}_{\mathrm{n}}\boldsymbol{S}_{\mathrm{n}}^{-1}\boldsymbol{x}(t) \tag{33}$$

is a suboptimal GCC controller such that the closed-loop system Eq.(4) without faults is quadratically stable with constraints (a), (b), (c) and (d), and the corresponding closed-loop quadratic cost function J is with $J<(q/r^2)\lambda_{\mathrm{n}}$.

SIMULATIVE EXAMPLES

In this section two simple examples for sensor faults are presented to illustrate the proposed design method. To show the necessity of RGCC, the results are also compared with its corresponding GCC controller design without taking the possible fault into consideration.

Example 1 Consider an uncertain system Eq.(1) with parameters as follows:

$$\boldsymbol{A}=\begin{bmatrix}-2.6 & 1\\ 0.1 & 0.04\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}1.42\\ 1\end{bmatrix},\ \boldsymbol{H}=\begin{bmatrix}0.61\\ 0\end{bmatrix},$$

$$\boldsymbol{D}_1=\begin{bmatrix}0\\ -1\end{bmatrix},\ \boldsymbol{x}(0)=\begin{bmatrix}0.5\\ -0.4\end{bmatrix},\ \boldsymbol{E}=[0\ \ -0.3],\ \boldsymbol{C}=[-1\ \ 0],$$

$D_2=0.5$, $\boldsymbol{Q}=0.6\boldsymbol{I}_{2\times2}$, $R=1$,

Obviously, the open-loop system is unstable. Assume that the parameters of the sensor fault model are $l_{\mathrm{L}i}=0.6936$ and $l_{\mathrm{U}i}=1.5$ ($i=1,\ 2$), then $\boldsymbol{M}_0=\mathrm{diag}\{1.0968,\ 1.0968\}$, $\boldsymbol{L}=\mathrm{diag}\{0.3676,\ 0.3676\}$. Suppose that the fault occurred at $t=2.01$ s in the second channel of sensors for state-feedback control: $m_2(t)=1.4-0.045[1-\exp(-0.003t)]$. Let $\boldsymbol{U}=[0.5\ \ -0.4]^{\mathrm{T}}$ in the simulation calculation. The multi-criterion constraints are:

(a′) $\Lambda(\boldsymbol{A}_{\mathrm{c}}+\Delta\boldsymbol{A})\subset\Phi(-q,r)=\Phi(-2,\ 1.999)$;

(b′) $\|\boldsymbol{H}(s)\|_\infty\le\gamma=7$;

(c′) Control input constraint $|\boldsymbol{u}^{\mathrm{f}}(t)|\le\sqrt{\alpha}=0.4243$;

(d′) The upper bound for the quadratic cost function Eq.(34) is minimized:

$$J=\int_0^\infty\left[0.6x_1^2(t)+0.6x_2^2(t)+(\boldsymbol{u}^{\mathrm{f}}(t))^2\right]\mathrm{d}t. \tag{34}$$

We apply the design method developed in Section 3 and solve the associate LMIs by the LMI Toolbox of Matlab. With $X=0.18$, the corresponding optimal solution for reliable control design is $\boldsymbol{S}_0=\begin{bmatrix}2.5507 & -0.0064\\ -0.0064 & 0.3677\end{bmatrix}$, $\lambda_0=4.4073$, $\varepsilon_0=1.2909$, $\boldsymbol{Y}_0=[-0.06503\ \ -0.1609]$. Thus, a desired suboptimal RGCC controller I with constraint (c′) is $u_0^{\mathrm{f}}(t)=[-0.02425\ \ -0.3993]\boldsymbol{x}^{\mathrm{f}}(t)$, which makes the faulty closed-loop system meet the constraints (a′) and (b′), and the optimized upper bound of quadratic cost function Eq.(34) is $J<\boldsymbol{x}^{\mathrm{T}}(0)(q/r^2)\boldsymbol{P}\boldsymbol{x}(0)=\boldsymbol{x}^{\mathrm{T}}(0)(q/r^2)\cdot(\boldsymbol{S}_0/\lambda_0)^{-1}\boldsymbol{x}(0)=1.1702$. For comparison, a suboptimal GCC controller II without fault tolerance by Corollary 2 is developed as $u_{\mathrm{n}}(t)=[0.1045\ \ -0.7174]\boldsymbol{x}(t)$ for the same constraints (a′)~(d′). The optimized upper bound of quadratic cost function Eq.(34) is 0.2722. Figs.1 and 2 illustrate the distribution of the closed-loop poles and the magnitudes of control inputs with the two controllers for the normal case and fault case, respectively. The upper bound of H_∞ disturbance rejection level for controller I is 0.7904 in the normal case and 0.7263 in the fault case, while for controller II the bound is 0.7904 in the normal case and 0.7379 in the fault case.

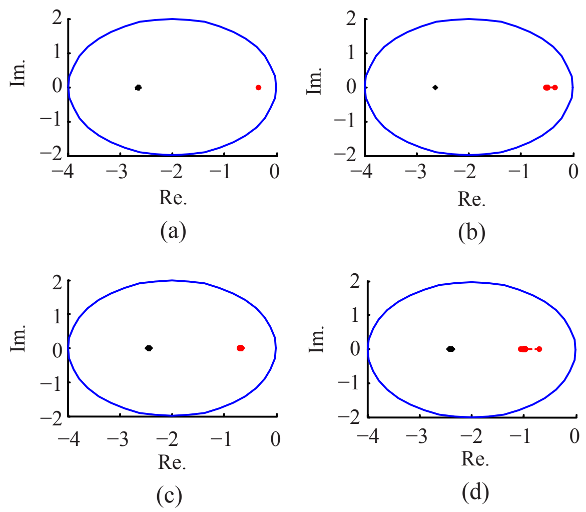

Fig.1 Distribution of closed-loop poles on the *s*-plane in normal and fault cases (black color: the first closed-loop pole; red color: the second closed-loop pole). (a) With controller I in the normal case; (b) With controller I in the fault case; (c) With controller II in the normal case; (d) With controller II in the fault case

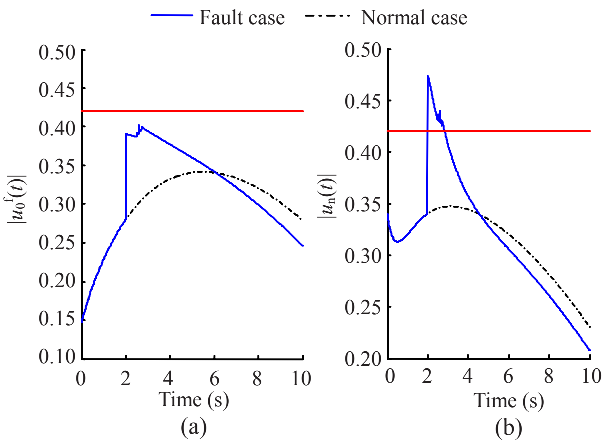

Fig.2 Magnitudes of control input signals (local zooming in). (a) With reliable controller I; (b) With controller II

The above simulation results indicate that both the controllers ensure the corresponding closed-loop system achieves the expected performance constraints (a′)~(d′) in the normal case. Whereas in the fault case it is easy to check from Fig.2b that the magnitude of the control input governed by controller II is greater than $\alpha^{1/2}$=0.4243 between t=2.01 s and t=2.84 s, so the performance constraint (c′) cannot be guaranteed. At that moment, the performance constraints (a′)~(d′) can still be guaranteed by controller I. The simulation verifies the validity of the proposed method.

Example 2 This example demonstrates the comparison between the optimal GCC design in (Yu *et al.*, 2004) and our reliable design with the constraints (c) and (d).

The system parameters are as follows:

$$\boldsymbol{A}=\begin{bmatrix}-0.1361 & -1.3454 & 0.5426\\ 0.1669 & -1.7522 & -0.1989\\ 0.8811 & -0.8681 & -1.3406\end{bmatrix},$$

$$\boldsymbol{B}=\begin{bmatrix}6.6422 & -6.4263\\ -0.6341 & -2.2412\\ 3.8187 & -7.3330\end{bmatrix},\ \boldsymbol{Q}=\boldsymbol{I}_{3\times3},\boldsymbol{R}=\boldsymbol{I}_{2\times2},\ \boldsymbol{x}(0)=\begin{bmatrix}1\\1\\1\end{bmatrix},$$

and other parameters are all zeros. Assume that the parameters of the sensor fault model are $0.5678\le m_1(t)\le 1.10$, $0.8064\le m_2(t)\le 1.90$, $0.6496\le m_3(t)\le 1.50$. Suppose that the fault signal is $\boldsymbol{M}(t)$=diag{1, 1.790, 1.40} and $\boldsymbol{U}=[1\ \ 1\ \ 1]^{\mathrm{T}}$ in the simulation. The multi-criterion constraints are:

(a″) The closed-loop system is asymptotically stable;

(b″) Control input constraint $|\boldsymbol{u}^{\mathrm{f}}(t)|\le\sqrt{\alpha}=0.83$;

(c″) The upper bound for the quadratic cost function Eq.(35) is minimized:

$$J=\int_0^{\infty}[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t)+(\boldsymbol{u}^{\mathrm{f}}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u}^{\mathrm{f}}(t)]\mathrm{d}t. \qquad (35)$$

By the reliable design method proposed in this paper, a desirable controller III is obtained as

$$\boldsymbol{u}_0^{\mathrm{f}}(t)=\begin{bmatrix}-0.11123 & 0.08741 & -0.02348\\ -0.008261 & 0.00325 & 0.0037325\end{bmatrix}\boldsymbol{x}^{\mathrm{f}}(t).$$

In contrast, an optimal GCC controller IV by the method in (Yu *et al.*, 2004) is

$$\boldsymbol{u}_{\mathrm{n}}(t)=\begin{bmatrix}-0.5463 & 0.3251 & -0.02967\\ -0.04279 & 0.4153 & 0.3485\end{bmatrix}\boldsymbol{x}(t).$$

Both the two controllers can guarantee the given constraints in the normal case, and the optimized upper bounds of quadratic cost index Eq.(35) with controllers III and IV are 191.89 and 0.3609, respectively. In the fault case, however, the index is 3.7973 for the latter, which is beyond the bound although the faulty closed-loop system is still stable. In contrast, the index is 133.65 for controller III. For the control input, Fig.3 illustrates that the magnitude of $u_{\mathrm{n}2}(t)$ with controller IV violates the constraint (b″). In this case, the reliable control guaranteed all the constraints, indicating the validity of our proposed method.

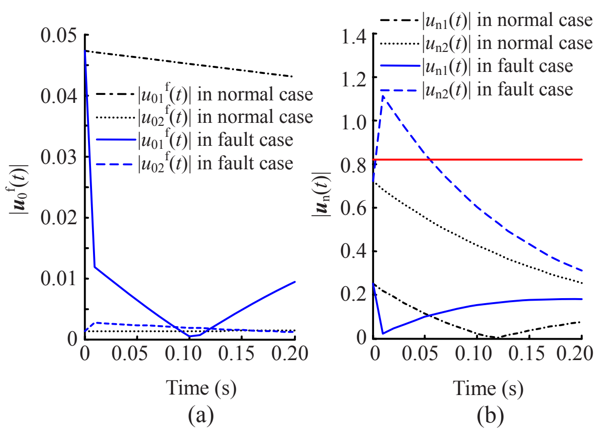

Fig.3 Magnitudes of control input signals (local zooming in). (a) With reliable controller III; (b) With controller IV

CONCLUSION

Taking the saturation of actuators into account, the suboptimal RGCC problem with multiple-criterion constraints has been investigated for a class of uncertain continuous-time systems subject to sensor failures. Attention has been paid to the design of a state-feedback RGCC controller with regional closed-loop poles, H_∞ disturbance attenuation level and control input constraints, which guarantees the resulting closed-loop system satisfies the pre-specified multiple performance constraints and has an optimized quadratic cost performance regardless of possible sensor faults. It is not difficult to extend the control strategy of this paper to the actuator fault case and output feedback control. In addition, we can observe from the simulation results in the normal case that the comprehensive performance by a controller without fault tolerance is generally better than the one by a reliable controller, which implies the conservatism of reliable control. Meanwhile, due to the LMI-based computation of the controller parameters, the main drawback associated with this strategy is that a single common solution set must work for all the LMI constraints, which may also lead to a very conservative controller design. It is our further objective to make the design method less conservative.

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