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Mathematical treatment of wave propagation in acoustic waveguides with *n* curved interfaces*

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Abstract: There are some curved interfaces in ocean acoustic waveguides. To compute wave propagation along the range with some marching methods, a flattening of the internal interfaces and a transforming equation are needed. In this paper a local orthogonal coordinate transform and an equation transformation are constructed to flatten interfaces and change the Helmholtz equation as a solvable form. For a waveguide with a flat top, a flat bottom and *n* curved interfaces, the coefficients of the transformed Helmholtz equation are given in a closed formulation which can be thought of as an extension of the formal work related to the equation transformation with two curved internal interfaces. In the transformed horizontally stratified waveguide, the one-way reformulation based on the Dirichlet-to-Neumann (DtN) map is then used to reduce the boundary value problem to an initial value problem. Numerical implementation of the resulting operator Riccati equation uses a large range step method to discretize the range variable and a truncated local eigenfunction expansion to approximate the operators. This method is particularly useful for solving long range wave propagation problems in slowly varying waveguides. Furthermore, the method can also be applied to wave propagation problems in acoustic waveguides associated with varied density.

Key words: Helmholtz equation, Local orthogonal transform, Dirichlet-to-Neumann (DtN) reformulation, Marching method, Curved interface, Multilayer medium

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THEORY

Consider the 2D Helmholtz equation (Tessmer *et al.*, 1992; Jensen *et al.*, 1994; Lu and McLaughlin, 1996; Evans, 1998; Lu *et al.*, 2001; Lu and Zhu, 2004; Zhu and Li, 2007) with *n* ($n\geq 2$) curved internal interfaces

$$u_{xx}+u_{zz}+\kappa^2(x,z)u=0 \tag{1}$$

for $-\infty<x<+\infty$, $0<z<D$, $L\gg D\gg 1/k$, where *L* is the range distance, *D* is the depth, *u* represents a Fourier transform of acoustic pressure and κ is called the wavenumber. The internal interfaces divide the waveguide into *n*+1 layered media where the first layer with density ρ_1 is located in $0<z<h_1(x)$, the *i*th ($2\leq i\leq n$) layer with density ρ_i is located in $h_{i-1}(x)<z<h_i(x)$, and the (*n*+1)th layer with density ρ_{n+1} is located in $h_n(x)<z<D$. The internal interfaces are *n* curves $z=h_i(x)$ (i=1, 2, ..., *n*). The sketch map of the waveguide is shown in Fig.1. We also assume the waveguide is range independent (i.e., κ and h_i are independent of *x*) for $x\leq 0$ and $x\geq L$, that is,

$$h_i(x)=\begin{cases}h_{i,0}, & x\leq 0,\\ h_{i,\infty}, & x\geq L,\end{cases}\quad i=1,2,...,n,$$

$$\kappa(x,z)=\begin{cases}\kappa_0(z), & x\leq 0,\\ \kappa_\infty(z), & x\geq L.\end{cases}$$

The top and bottom boundary conditions are supposed as $u|_{z=0}=0$ and $u|_{z=D}=0$. The interface conditions mean that

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$$\begin{cases} \lim\limits_{z\to(h_i(x))^-} u(x,z) = \lim\limits_{z\to(h_i(x))^+} u(x,z), \\ \dfrac{1}{\rho_i}\lim\limits_{z\to(h_i(x))^-}\dfrac{\partial u(x,z)}{\partial \boldsymbol{n}} = \dfrac{1}{\rho_{i+1}}\lim\limits_{z\to(h_i(x))^+}\dfrac{\partial u(x,z)}{\partial \boldsymbol{n}}, \end{cases}$$

where $\boldsymbol{n}$ is a normal vector on the interface $z=h_i(x)$ at x, i=1, 2, …, n.

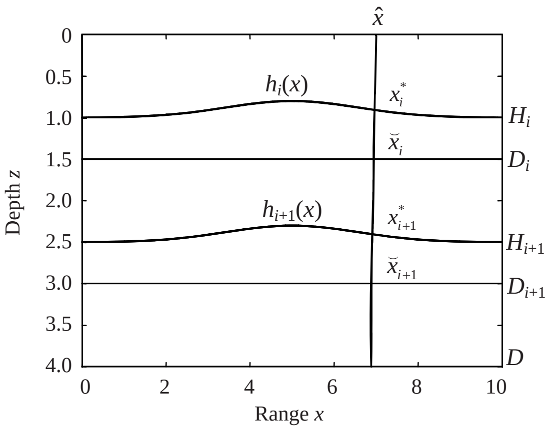

Fig.1 Sketch map of waveguide with *n* curved internal interfaces

Since the Helmholtz equation can be easily solved by a separable variable method for $x\le 0$ or $x\ge L$, we only need to solve the equation for $0\le x\le L$. If there are no waves coming from $+\infty$, the exact boundary condition (radiation condition) at $x=L$ is $u_x = \mathrm{i}\sqrt{\partial_z^2 + \kappa_\infty^2(z)}u$, where $\mathrm{i}=\sqrt{-1}$ and the square root operator is defined in (Lu and McLaughlin, 1996). The simplest boundary condition at x=0 is $u=u_0(z)$, where $u_0(z)$ is a given function of z.

Here we suppose that there are horizontal straight lines $z=D_i$ between $h_i(x)$ and $h_{i+1}(x)$ (i=1, 2, ..., n−1), and that there is a unique solution for Eq.(1) with the boundary and interface conditions.

LOCAL ORTHOGONAL TRANSFORM

The acoustic waveguide is assumed to be $2n+1$ parts separated by $z=D_i$ (i=1, 2, ..., n). We transform the $2n+1$ parts in physical coordinates into new coordinates $(\hat{x}, \hat{z})$, respectively. In the new coordinate system, $z=D_i$ is transformed into $\hat{z}=D_i$, $z=h_i(x)$ is transformed into $\hat{z}=H_i$ (i=1, 2, ..., n); z=0 and $z=D$ are transformed into $\hat{z}=0$ and $\hat{z}=D$, respectively. Here $z=D_n$ is used to avoid squeezing the coordinate net in the new coordinate system into a narrow coordinate net in the old system (x, z). Let $\breve{x}_i$ be the point of intersection for the curve $\hat{x}=\hat{x}_j$ and the added interface $z=D_i$, and x_i^* be the point of intersection for $\hat{x}=\hat{x}_j$ and the interface $z=h_i(x)$, where $\hat{x}_j$ is a dividing point along the range, i=1, 2, ..., n.

It is noticed that the computing of the corresponding old coordinate according to the new coordinate is often necessary when the marching method is used on the new coordinates. The following numerical scheme realizes $(\hat{x}, \hat{z}) \to (x, z)$ using the Newton method and we give only the relationship between $(\hat{x}, \hat{z})$ and (x, z). In the following, i varies from 1 to n, and the detailed transform scheme is as follows:

Step 1: The $(2i-1)$th part $D_{i-1}\le z\le h_i(x)$ in a medium with density ρ_i (Lu and Zhu, 2004) where D_0=0 and $\breve{x}_0=\hat{x}$. Let

$$\begin{cases} \hat{x} = f(x,z), \\ \hat{z} = g(x,z) = \dfrac{z-D_{i-1}}{h_i(x)-D_{i-1}}(H_i - D_{i-1}) + D_{i-1} \end{cases}$$

satisfy

$$\{(x,z)\,|\,0\le x\le L,\ D_{i-1}\le z\le h_i(x)\} \xleftrightarrow{f,g} \{(\hat{x},\hat{z})\,|\,0\le \hat{x}\le L,\ D_{i-1}\le \hat{z}\le H_i\},$$

where the function f is to be determined, $f(0, z)$=0, $f(x, 0)=x$ and $f(L, z)=L$. The transform is required to be orthogonal. Therefore,

$$\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z} = 0.$$

If $h_i'(x)\ne 0$, then the relationship between (x, z) and $(\hat{x}, \hat{z})$ is represented by

$$\int_{\breve{x}_{i-1}}^{x} \frac{h_i(t)-D_{i-1}}{h_i'(t)}\mathrm{d}t + \frac{1}{2}(z-D_{i-1})^2 = 0,$$

where x can be easily obtained using the Newton method with the initial guess $\breve{x}_{i-1}$ and

$$z = \frac{\hat{z}-D_{i-1}}{H_i - D_{i-1}}(h_i(x) - D_{i-1}) + D_{i-1}.$$

If $h_i'(x)=0$, then let $x=\breve{x}_{i-1}$.

Remark 1 $\breve{x}_{i-1}$ is determined by the upper layers.

Step 2: The (2i)th part $h_i(x)\le z\le D_i$ in a medium with density ρ_{i+1} (Lu and Zhu, 2004). Let

$$\begin{cases}\hat{x}=f(x,z),\\ \hat{z}=g(x,z)=\dfrac{z-h_i(x)}{D_i-h_i(x)}(D_i-H_i)+D_i\end{cases}$$

satisfy

$$\{(x,z)\mid 0\le x\le L,\ h_i(x)\le z\le D_i\}\xleftrightarrow{f,g}\{(\hat{x},\hat{z})\mid 0\le \hat{x}\le L,\ H_i\le \hat{z}\le D_i\}.$$

If $h_i'(x)\ne 0$, then the relationship between (x, z) and $(\hat{x},\hat{z})$ is represented by

$$\int_{x_i^*}^{x}\frac{D_i-h_i(t)}{h_i'(t)}\mathrm{d}t-\frac{1}{2}\Big((z-D_i)^2-(h_i(x_i^*)-D_i)^2\Big)=0,$$

where x can be obtained by using the Newton method with the initial guess x_i^*, and

$$z=\frac{\hat{z}-D_i}{D_i-H_i}(D_i-h_i(x))+h_i(x).$$

If $h_i'(x)=0$, then let $x=x_i^*$.

Remark 2 $(x_i^*, h_i(x_i^*))\xleftrightarrow{1-1}(\hat{x}, H_i)$, x_i^* is determined by the upper layers.

Step 3: The last part $D_n\le z\le D$ in a medium with density ρ_{n+1}. Obviously the orthogonal transform is chosen by

$$\begin{cases}\hat{x}=f(x,D_n),\\ \hat{z}=g(x,z)=z.\end{cases}$$

Remark 3 When the interfaces are of weak smoothness, they can be approximated by some smooth ones, such as spline functions. Thus the problem can be approximately solved by our method.

EQUATION TRANSFORMATION

Because Eq.(1) is expected to be transformed as

$$V_{\hat{x}\hat{x}}+\alpha(\hat{x},\hat{z})V_{\hat{z}\hat{z}}+\beta(\hat{x},\hat{z})V_{\hat{z}}+\gamma(\hat{x},\hat{z})V=0, \qquad (2)$$

we let $u(x, z)=W(x, z)V(x, z)$, where W is determined by

$$2W_z f_z+Wf_{zz}+2W_x f_x+Wf_{xx}=0.$$

The coefficients of Eq.(2) are obtained as follows:

$$\begin{cases}\alpha(\hat{x},\hat{z})=\dfrac{g_z^2+g_x^2}{f_z^2+f_x^2},\\ \beta(\hat{x},\hat{z})=\dfrac{2W_zg_z+Wg_{zz}+2W_xg_x+Wg_{xx}}{W(f_z^2+f_x^2)},\\ \gamma(\hat{x},\hat{z})=\dfrac{W_{xx}+W_{zz}+\kappa^2W}{W(f_z^2+f_x^2)}.\end{cases} \qquad (3)$$

For a multilayered waveguide with n curved internal interfaces ($n\ge 2$), the n curved internal interfaces divide the waveguide into n+1 layers. Every layer is composed of two parts except the first layer, that is, $0\le z\le h_1(x)$, $h_i(x)\le z\le D_i$, $D_i\le z\le h_{i+1}(x)$, i=1, 2, ..., n−1, $h_n(x)\le z\le D_n$ and $D_n\le z\le D$. We denote $(\breve{x}_i,\breve{z}_i)$ as the point of intersection between $z=D_i$ and $\hat{x}=\hat{x}_j$, and (x_i^*, z_i^*) as the point of intersection between $z=h_i(x)$ and $\hat{x}=\hat{x}_j$ in physical coordinates.

Letting

$$\begin{cases}W_{1,i-1}(x,z)=\sqrt{\dfrac{h_i'(x)(h_i(\breve{x}_{i-1})-D_{i-1})}{h_i'(\breve{x}_{i-1})(h_i(x)-D_{i-1})^2}},\\ W_{2,i-1}(x,z)=\sqrt{\dfrac{h_i(\breve{x}_{i-1})-D_{i-1}}{h_i'(\breve{x}_{i-1})}\dfrac{D_i-h_i(x_i^*)}{h_i(x_i^*)-D_{i-1}}\dfrac{h_i'(x)}{(D_i-h_i(x))^2}},\\ W_{3,n}(x,z)=\sqrt{\dfrac{h_n(\breve{x}_n)-D_{n-1}}{h_n'(\breve{x}_i)}\dfrac{D_n-h_n(x_n^*)}{h_n(x_n^*)-D_{n-1}}\dfrac{h_n'(x)}{D_n-h_n(x)}},\end{cases} \qquad (4)$$

we have

$$W(x,z)=\begin{cases}P_{i-1}(\breve{x}_{i-1})W_{1,i-1}(x,z), & D_{i-1}\le z\le h_i(x),\\ P_{i-1}(\breve{x}_{i-1})W_{2,i-1}(x,z), & h_i(x)\le z\le D_i,\\ P_{n-1}(\breve{x}_{n-1})W_{3,n}(x,z), & D_n\le z\le D,\end{cases} \qquad (5)$$

where

$$P_i(\breve{x}_i)=\left(\frac{\mathrm{d}\hat{x}}{\mathrm{d}\breve{x}_i}\right)^{-1/2}=\left(\frac{\mathrm{d}\hat{x}}{\mathrm{d}\breve{x}_{i-1}}\frac{\mathrm{d}\breve{x}_{i-1}}{\mathrm{d}\breve{x}_i}\right)^{-1/2}=\left(\frac{h_i'(\breve{x}_i)}{h_i'(\breve{x}_{i-1})}\right)^{1/2}\cdot\left(\frac{D_i-h_i(x_i^*)}{D_i-h_i(\breve{x}_i)}\right)^{1/2}\left(\frac{h_i(\breve{x}_{i-1})-D_{i-1}}{h_i(x_i^*)-D_{i-1}}\right)^{1/2}P_{i-1}(\breve{x}_{i-1}),$$

and $P_0(\breve{x}_0)=1$, i=1, 2, ..., n−1.

Let $M_i(\breve{x}_i)=(P_i(\breve{x}_i))^4$, then

$$M_{i-1}(\breve{x}_{i-1})P_{i-1,\,\breve{x}_{i-1}\breve{x}_{i-1}}\big/P_{i-1}(\breve{x}_{i-1})=M_{i-2}(\breve{x}_{i-2})\gamma_{3,i}(\hat{x},\hat{z})+M_{i-2}(\breve{x}_{i-2})P_{i-2,\,\breve{x}_{i-2}\breve{x}_{i-2}}\big/P_{i-2}(\breve{x}_{i-2}), \tag{6}$$

where

$$M_0(\breve{x}_0)P_{0,\,\breve{x}_0\breve{x}_0}\big/P_{i-1}(\breve{x}_0)=0.$$

In the (2*i*−1)th part,

$$\begin{cases}\alpha(\hat{x},\hat{z})=M_{i-1}(\breve{x}_{i-1})\alpha_1(\hat{x},\hat{z}),\\ \beta(\hat{x},\hat{z})=M_{i-1}(\breve{x}_{i-1})\beta_1(\hat{x},\hat{z}),\\ \gamma(\hat{x},\hat{z})=M_{i-1}(\breve{x}_{i-1})\gamma_1(\hat{x},\hat{z})\\ \qquad +M_{i-1}(\breve{x}_{i-1})P_{i-1,\,\breve{x}_{i-1}\breve{x}_{i-1}}\big/P_{i-1}(\breve{x}_{i-1}).\end{cases} \tag{7}$$

In the (2*i*)th part,

$$\begin{cases}\alpha(\hat{x},\hat{z})=M_{i-1}(\breve{x}_{i-1})\alpha_2(\hat{x},\hat{z}),\\ \beta(\hat{x},\hat{z})=M_{i-1}(\breve{x}_{i-1})\beta_2(\hat{x},\hat{z}),\\ \gamma(\hat{x},\hat{z})=M_{i-1}(\breve{x}_{i-1})\gamma_2(\hat{x},\hat{z})\\ \qquad +M_{i-1}(\breve{x}_{i-1})P_{i-1,\,\breve{x}_{i-1}\breve{x}_{i-1}}\big/P_{i-1}(\breve{x}_{i-1}).\end{cases} \tag{8}$$

In the (2*n*+1)th part,

$$\begin{cases}\alpha(\hat{x},\hat{z})=M_{n-1}(\breve{x}_{n-1})\alpha_3(\hat{x},\hat{z}),\\ \beta(\hat{x},\hat{z})=M_{n-1}(\breve{x}_{n-1})\beta_3(\hat{x},\hat{z}),\\ \gamma(\hat{x},\hat{z})=M_{n-1}(\breve{x}_{n-1})\gamma_{3,n}(\hat{x},\hat{z})\\ \qquad +M_{n-1}(\breve{x}_{n-1})P_{n-1,\,\breve{x}_{n-1}\breve{x}_{n-1}}\big/P_{n-1}(\breve{x}_{n-1}),\end{cases} \tag{9}$$

where α_j, β_j, γ_j (j=1, 2), α_3, β_3, $\gamma_{3,i}$ and P_i are listed in Appendices A and B.

INTERFACE CONDITIONS

Range discretization and matrix approximations are the same as those in (Lu and Zhu, 2004). We omit them here for simplicity and only list the boundary and interface conditions.

For Eq.(2) the top and bottom boundary conditions are $V|_{\hat{z}=0}=0$ and $V|_{\hat{z}=D}=0$. The interface conditions at $\hat{z}=D_{i-1}$ between the parts $H_{i-1}\le\hat{z}\le D_{i-1}$ and $D_{i-1}\le\hat{z}\le H_i$ are

$$WV|_{\hat{z}\to D_{i-1}^-}=WV|_{\hat{z}\to D_{i-1}^+}$$

and

$$W\frac{H_{i-1}-D_{i-1}}{h_{i-1}(x)-D_{i-1}}V_{\hat{z}}|_{\hat{z}\to D_{i-1}^-}=W\frac{H_i-D_i}{h_i(x)-D_i}V_{\hat{z}}|_{\hat{z}\to D_{i-1}^+}.$$

The interface conditions at $\hat{z}=H_i$ between the parts $D_{i-1}\le\hat{z}\le H_i$ and $H_i\le\hat{z}\le D_i$ become

$$WV|_{\hat{z}=H_i^-}=WV|_{\hat{z}=H_i^+}$$

and

$$\begin{aligned}&\frac{1}{\rho_i}W\left[\frac{1}{2}\left(h_i''(x)-2\frac{(h_i'(x))^2}{h_i(x)-D_{i-1}}\right)V\right.\\&\left.\left.-P_{i-1}(\breve{x}_{i-1})\frac{H_i-D_{i-1}}{h_i(x)-D_{i-1}}\left(1+(h_i'(x))^2\right)V_{\hat{z}}\right]\right|_{\hat{z}\to H_i^-}\\&=\frac{1}{\rho_{i+1}}W\left[\frac{1}{2}\left(h_i''(x)+2\frac{(h_i'(x))^2}{D_i-h_i(x)}\right)V\right.\\&\left.\left.-P_{i-1}(\breve{x}_{i-1})\frac{D_i-H_i}{D_i-h_i(x)}\left(1+(h_i'(x))^2\right)V_{\hat{z}}\right]\right|_{\hat{z}\to H_i^+}.\end{aligned} \tag{10}$$

The interface conditions at $\hat{z}=D_n$ between the parts $H_n\le\hat{z}\le D_n$ and $D_n\le\hat{z}\le D$ are

$$WV|_{\hat{z}\to D_n^-}=WV|_{\hat{z}\to D_n^+}$$

and

$$W\frac{H_n-D_n}{h_n(x)-D_n}V_{\hat{z}}|_{\hat{z}\to D_n^-}=WV_{\hat{z}}|_{\hat{z}\to D_n^+}.$$

NUMERICAL EXAMPLES

The method presented in the previous section has been tested on a number of examples of waveguides with three internal interfaces. Four of them, which represent some different situations in ocean acoustics, are given below. In the examples the waveguide is divided into four layers by three curved interfaces $z=h_1(x)$, $z=h_2(x)$ and $z=h_3(x)$. In the curved stratified structure, $z=D_1$, $z=D_2$ and $z=D_3$ divide the waveguide into seven parts as discussed in the previous section. After flattening the three curved interfaces by the orthogonal transform, we can use the marching scheme (Lu and Zhu, 2004) to compute the solution $u(x, z)$ at $x=L$, where τ represents the range step.

Example 1 Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x),\\ 0.7\times 16, & h_1(x)<z<h_2(x),\\ 0.2\times 16, & h_2(x)<z<h_3(x),\\ 0.2\times 16, & h_3(x)<z<D,\end{cases}$$

with L=10, n=30, H_1=0.5, D_1=1.0, H_2=1.5, D_2=2.0, H_3=2.5, D_3=3.0, D=4, N=400, ρ_1=1, ρ_2=1.7, ρ_3=2.7, ρ_4=2.8; $h_i(x)=H_i-\varepsilon_i\exp(-\sigma_i(x/L-0.5)^2)$ (i=1, 2, 3), $\varepsilon_1=\varepsilon_2=\varepsilon_3$=0.1 and $\sigma_1=\sigma_2=\sigma_3$=10, where N is the number of points to discretize the $\hat{z}$ variable, n is the number of points to truncate the $N\times N$ matrices that approximate the operators appearing in the marching process (Lu and Zhu, 2004). Here $V(0,\hat{z})$ is given by the eigenfunction whose corresponding eigenvalue is the largest one at x=0.

In this example we discuss a special situation in which the sound velocity in the third layer is the same as that in the bottom layer. The corresponding solution is shown in Fig.2.

Example 2 Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x),\\ 0.5\times 16, & h_1(x)<z<h_2(x),\\ 0.22\times 16, & h_2(x)<z<h_3(x),\\ 0.18\times 16, & h_3(x)<z<D,\end{cases}$$

with L=10, n=30, H_1=0.5, D_1=1.0, H_2=1.5, D_2=2.0, H_3=2.5, D_3=3.0, D=4, N=400, ρ_1=1, ρ_2=1.7, ρ_3=2.7, ρ_4=2.8; $h_i(x)=H_i-\varepsilon_i\exp(-\sigma_i(x/L-0.5)^2)$ (i=1, 2, 3), $\varepsilon_1=\varepsilon_2=\varepsilon_3$=0.1 and $\sigma_1=\sigma_2=\sigma_3$=20. Here $V(0,\hat{z})$ is given by the eigenfunction whose corresponding eigenvalue is the largest one at x=0.

Unlike Example 1, Example 2 discusses an ocean acoustic waveguide in which the sound velocity in the bottom layer is the largest. The corresponding solution is shown in Fig.3.

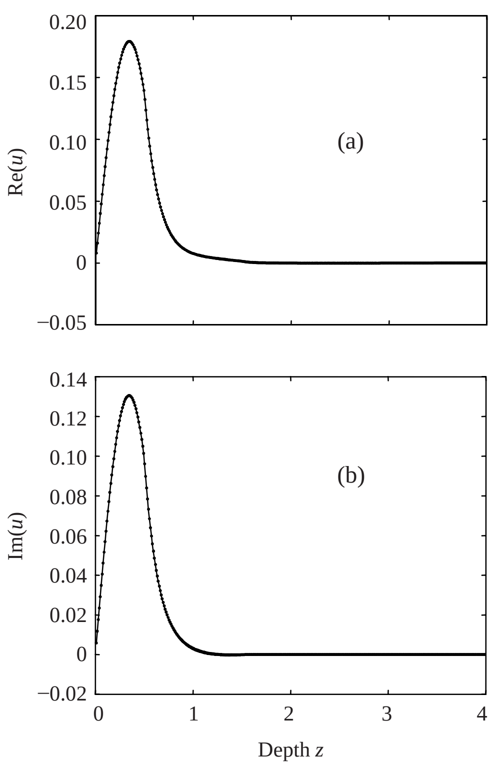

Fig.2 Comparison with the real part (a) and the imaginary part (b) of $u(L, z)$ for τ=1/8 (bold points) and τ=1/128 (solid line) in Example 1

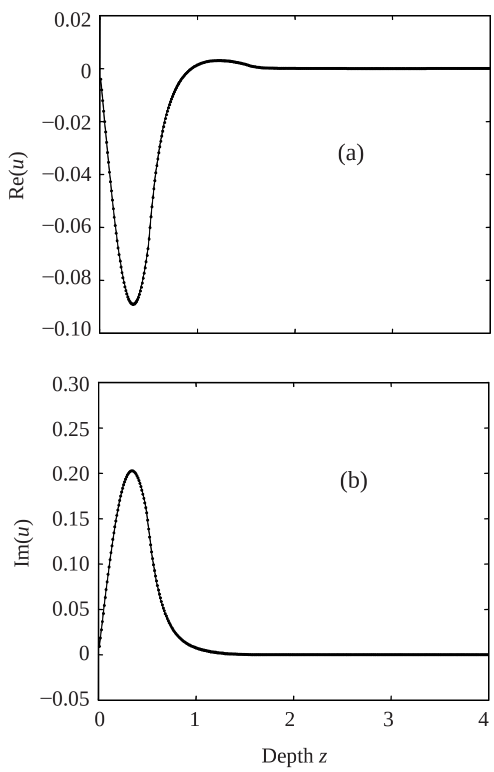

Fig.3 Comparison with the real part (a) and the imaginary part (b) of $u(L, z)$ for τ=1/8 (bold points) and τ=1/128 (solid line) in Example 2

Example 3 Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x),\\ 0.7\times16, & h_1(x)<z<h_2(x),\\ 0.22\times16, & h_2(x)<z<h_3(x),\\ 0.18\times16, & h_3(x)<z<D,\end{cases}$$

with L=10, n=30, H_1=0.5, D_1=1.0, H_2=1.5, D_2=2.0, H_3=2.5, D_3=3.0, D=4, N=400, ρ_1=1, ρ_2=1.7, ρ_3=2.7, ρ_4=2.8; $h_i(x)=H_i-\varepsilon_i\exp(-\sigma_i(x/L-\theta_i)^2)$ (i=1, 2, 3), $\varepsilon_1=\varepsilon_2=\varepsilon_3$=0.1, $\sigma_1=\sigma_2=\sigma_3$=10, and θ_1=0.35, θ_2=0.50, θ_3=0.65. Here $V(0,\hat{z})$ is given by the eigenfunction whose corresponding eigenvalue is the largest one at x=0.

The peaks of the interfaces in Examples 1 and 2 are in the same position. In Example 3 we compute wave propagation in a waveguide with peaks of the interfaces in different positions. The corresponding solution is shown in Fig.4.

Example 4 Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x),\\ 0.7\times16, & h_1(x)<z<h_2(x),\\ 0.22\times16, & h_2(x)<z<h_3(x),\\ 0.18\times16, & h_3(x)<z<D,\end{cases}$$

with L=10, n=30, H_1=0.5, D_1=1.0, H_2=1.5, D_2=2.0, H_3=2.5, D_3=3.0, D=4, N=400, ρ_1=1, ρ_2=1.7, ρ_3=2.7, ρ_4=2.6; $h_i(x)=H_i-\varepsilon_i\exp(-\sigma_i(x/L-0.5)^2)$ (i=1, 2, 3), $\varepsilon_1=\varepsilon_2=\varepsilon_3$=0.1 and $\sigma_1=\sigma_2=\sigma_3$=20. Here $V(0,\hat{z})$ is given by the eigenfunction whose corresponding eigenvalue is the sixth one in descending order at x=0.

In this example another special situation, in which the density in the bottom layer is smaller than that in its upper adjoining layer, is considered. We take the sixth eigenfunction at x=0 as $V(0,\hat{z})$ which is highly oscillating. The corresponding solution is shown in Fig.5.

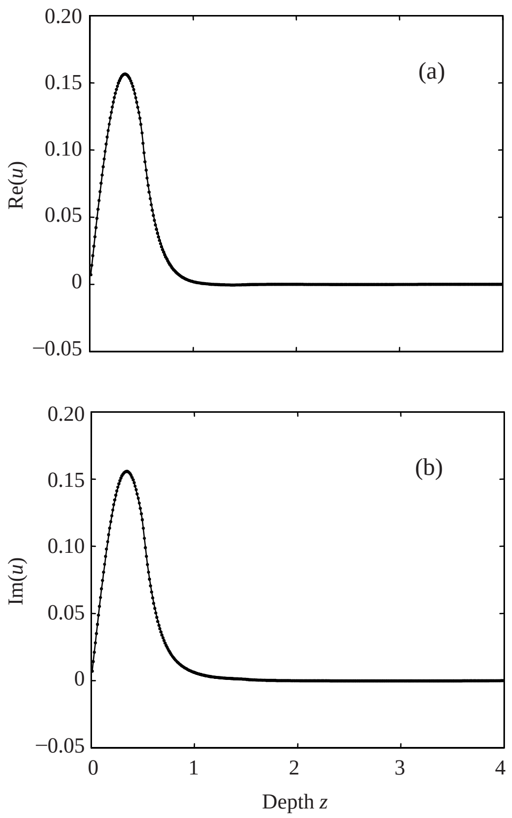

Fig.4 Comparison with the real part (a) and the imaginary part (b) of $u(L, z)$ for τ=1/8 (bold points) and τ=1/128 (solid line) in Example 3

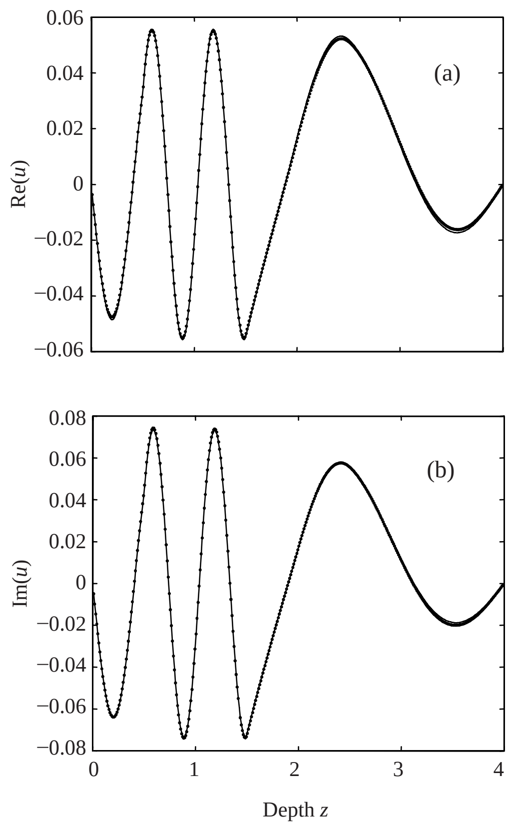

Fig.5 Comparison with the real part (a) and the imaginary part (b) of $u(L, z)$ for τ=1/8 (bold points) and τ=1/128 (solid line) in Example 4

The solutions obtained with τ=1/128 and represented by solid lines in Figs.2~5 act as the 'exact' solutions. We computed the relative errors of $u(L, z)$. The relative errors of Examples 1~4 are 0.0051, 0.0073, 0.0075 and 0.0179, respectively. The numerical examples demonstrate that good approximate solutions can be obtained by quite large steps.

CONCLUSION

The result of this work provides a theoretical foundation for developing a practical numerical scheme for a class of acoustic waveguides with n curved interfaces. By constructing a local orthogonal coordinate transformation and an equation transformation, the original problem is changed into a new problem that may be solved by some marching methods. This extends our work (Zhu and Li, 2007) which treats waveguides with two curved interfaces. This method is particularly useful for solving long range wave propagation problems in slowly varying waveguides with a multilayered medium structure. Numerical examples demonstrate that our treatment is feasible for solving the Helmholtz equation with n layered media by a large range step.

However, the case for supposing that there is a straight line between adjoint interfaces does not always hold and there is one possible way to solve the problem in this theoretical frame, by dividing the waveguide properly into several parts along the range direction and letting the supposition of a straight line between two curved interfaces hold in each part. Furthermore, in the case where the density is varied, the waveguide can be divided into many small parts along the depth. The density in each part is an approximately regarded constant. Then our treatment can be applied. Further research will appear soon.

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APPENDIX A

The formulas for coefficients α_j, β_j, γ_j (j=1, 2), α_3, β_3, $\gamma_{3,i}$ are given as follows.

If $h_i'(x)\neq 0$, then we have Eq.(A1) (see the next page), where

$$\frac{\partial \breve{x}_{i-1}}{\partial x}=\frac{h_i'(\breve{x}_{i-1})(h_i(x)-D_i)}{h_i'(x)(h_i(\breve{x}_{i-1})-D_i)},$$

and

$$\frac{\partial \breve{x}_{i-1}}{\partial z}=(z-D_i)\frac{h_i'(\breve{x}_{i-1})}{h_i(\breve{x}_{i-1})-D_i}.$$

If $h_i'(\bar{x}_{i-1})=0$ and $h_i''(\bar{x}_{i-1})\neq 0$, then we have Eq.(A2), with

$$\lim_{\breve{x}_{i-1}\to\bar{x}_{i-1}}\frac{h_i'(x)}{h_i'(\breve{x}_{i-1})}=\exp\left(-\frac{(z-D_i)^2}{2}\frac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1})-D_i}\right)$$

and $\bar{x}_{i-1}=\breve{x}_{i-1}$.

If $h_i'(\bar{x}_{i-1})=0$ and $h_i''(\bar{x}_{i-1})=0$, then

$$\begin{aligned}&\alpha_1(\hat{x},\hat{z})=(h_i(\bar{x}_{i-1})-D_i)^{-2},\\&\beta_1(\hat{x},\hat{z})=0,\\&\gamma_1(\hat{x},\hat{z})=\kappa^2(\bar{x}_{i-1},z)-\frac{(z-D_i)^2}{4}\frac{h_i^{(4)}(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1})-D_i}.\end{aligned}$$

If $h_i'(x)\neq 0$, then we have Eq.(A3).

$$
\begin{cases}
\alpha_1(\hat{x},\hat{z}) = \dfrac{(H_i - D_i)^2 (h_i'(x))^2 (h_i(\breve{x}_{i-1}) - D_i)^2}{(h_i(x) - D_i)^4 (h_i'(\breve{x}_{i-1}))^2}, \\[2ex]
\beta_1(\hat{x},\hat{z}) = \dfrac{2(z - D_i)(H_i - D_i)[2(h_i'(x))^2 - (h_i(x) - D_i)h_i''(x)](h_i'(x))^2 (h_i(\breve{x}_{i-1}) - D_i)^2}{(h_i(x) - D_i)^3 (h_i'(\breve{x}_{i-1}))^2 [(h_i(x) - D_i)^2 + (z - D_i)^2 (h_i'(x))^2]}, \\[2ex]
\gamma_1(\hat{x},\hat{z}) = \left\{ 2\dfrac{(h_i'(x))^2}{(h_i(x) - D_i)^2} - 2\dfrac{h_i''(x)}{h_i(x) - D_i} + \dfrac{(z - D_i)^2}{4}\left[\dfrac{(h_i''(\breve{x}_{i-1}))^2 - 2h_i'(\breve{x}_{i-1})h_i'''(\breve{x}_{i-1})}{(h_i(\breve{x}_{i-1}) - D_i)^2} + \dfrac{4h_i'(\breve{x}_{i-1})h_i''(\breve{x}_{i-1})}{(h_i(\breve{x}_{i-1}) - D_i)^3} \right.\right. \\[2ex]
\quad \left. - \dfrac{3(h_i'(\breve{x}_{i-1}))^4}{(h_i(\breve{x}_{i-1}) - D_i)^4} \right] + \dfrac{(h_i(x) - D_i)^2}{4(h_i'(x))^2}\left[\dfrac{2h_i'(x)h_i'''(x) - (h_i''(x))^2}{(h_i(x) - D_i)^2} - \dfrac{2h_i'(\breve{x}_{i-1})h_i'''(\breve{x}_{i-1}) - (h_i''(\breve{x}_{i-1}))^2}{(h_i(\breve{x}_{i-1}) - D_i)^2} \right] \\[2ex]
\quad \left. + \dfrac{(h_i'(\breve{x}_{i-1}))^2}{(h_i'(x))^2}\left[\dfrac{(h_i(x) - D_i)^2 h_i''(\breve{x}_{i-1})}{(h_i(\breve{x}_{i-1}) - D_i)^3} - \dfrac{3}{4}\dfrac{(h_i(x) - D_i)^2 (h_i'(\breve{x}_{i-1}))^2}{(h_i(\breve{x}_{i-1}) - D_i)^4} \right] + \kappa^2(x,z) \right\} \Bigg/ \left[\left(\dfrac{\partial \breve{x}_{i-1}}{\partial x}\right)^2 + \left(\dfrac{\partial \breve{x}_{i-1}}{\partial z}\right)^2 \right].
\end{cases} \tag{A1}
$$

$$
\begin{cases}
\alpha_1(\hat{x},\hat{z}) = \dfrac{(H_i - D_i)^2}{(h_i(\bar{x}_{i-1}) - D_i)^2} \exp\left(-(z - D_i)^2 \dfrac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1}) - D_i} \right), \\[2ex]
\beta_1(\hat{x},\hat{z}) = -2(z - D_i)(H_i - D_i)\dfrac{h_i''(\bar{x}_{i-1})}{(h_i(\bar{x}_{i-1}) - D_i)^2} \exp\left(-(z - D_i)^2 \dfrac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1}) - D_i} \right), \\[2ex]
\gamma_1(\hat{x},\hat{z}) = \left\{ \dfrac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1}) - D_i}\left[\exp\left((z - D_i)^2 \dfrac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1}) - D_i} \right) - 2 \right] + \dfrac{(h_i''(\bar{x}_{i-1}))^2 + (h_i(\bar{x}_{i-1}) - D_i)h_i^{(4)}(\bar{x}_{i-1})}{4(h_i(\bar{x}_{i-1}) - D_i)h_i''(\bar{x}_{i-1})} \right. \\[2ex]
\quad \left. \cdot \left[1 - \exp\left((z - D_i)^2 \dfrac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1}) - D_i} \right) \right] + \dfrac{(z - D_i)^2}{4}\dfrac{(h_i''(\bar{x}_{i-1}))^2}{(h_i(\bar{x}_{i-1}) - D_i)^2} + \kappa^2(\bar{x}_{i-1}, z) \right\} \exp\left(-(z - D_i)^2 \dfrac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1}) - D_i} \right).
\end{cases} \tag{A2}
$$

$$
\begin{cases}
\alpha_2(\hat{x},\hat{z}) = (D_{i+1} - H_i)^2 \dfrac{(D_{i+1} - h_i(x_i^*))^2 (h_i'(x))^2 (h_i(\breve{x}_{i-1}) - D_i)^2}{(D_{i+1} - h_i(x))^4 (h_i'(\breve{x}_{i-1}))^2 (h_i(x_i^*) - D_i)^2}, \\[2ex]
\beta_2(\hat{x},\hat{z}) = 2(D_{i+1} - H_i)(z - D)\dfrac{2(h_i'(x))^2 + (D_{i+1} - D_i)h_i''(x) - h_i''(x)h_i'(x)}{(D_{i+1} - h_i(x))^2 + (h_i'(x))^2 (D_{i+1} - z)^2}\left(\dfrac{h_i'(x)}{h_i'(\breve{x}_{i-1})} \right)^2 \left(\dfrac{h_i(\breve{x}_{i-1}) - D_i}{h_i(x_i^*)} \right)^2 \dfrac{(D_{i+1} - h_i(x_i^*))^2}{(D_{i+1} - h_i(x))^3}, \\[2ex]
\gamma_2(\hat{x},\hat{z}) = \left\{ \dfrac{\left((D_{i+1} - h_i(x))/h_i'(x) \right)^2 + (D_{i+1} - z)^2}{(D_{i+1} - h_i(x_i^*))^2}\left[\dfrac{(h_i(x_i^*) - D_i)^2}{(h_i(\breve{x}_{i-1}) - D_i)^2}\left(-\dfrac{3}{4}\dfrac{(h_i'(\breve{x}_{i-1}))^4}{(h_i(\breve{x}_{i-1}) - D_i)^2} + \dfrac{(h_i'(\breve{x}_{i-1}))^4 h_i''(\breve{x}_{i-1})}{h_i(\breve{x}_{i-1}) - D_i} \right.\right.\right. \\[2ex]
\quad \left. + \dfrac{1}{4}(h_i''(\breve{x}_{i-1}))^2 - \dfrac{1}{2}h_i'(\breve{x}_{i-1})h_i'''(\breve{x}_{i-1}) \right) + \dfrac{(D_{i+1} - D_i)(h_i'(x_i^*))^2}{(h_i(x_i^*) - D_i)(D_{i+1} - h_i(x_i^*))(1 + (h_i'(x_i^*))^2)^2} \\[2ex]
\quad \left. \cdot \left(\dfrac{3}{4}\dfrac{(h_i'(x_i^*))^2 (D_{i+1} - 2h_i(x_i^*) + D_i)}{(h_i(x_i^*) - D_i)(D_{i+1} - h_i(x_i^*))} - \dfrac{h_i''(x_i^*)}{1 + (h_i'(x_i^*))^2} \right) \right] + \dfrac{2(h_i'(x))^2}{(D_{i+1} - h_i(x))^2} + \dfrac{2h_i''(x)}{D_{i+1} - h_i(x)} + \dfrac{h_i'''(x)}{2h_i'(x)} \\[2ex]
\quad \left. - \dfrac{1}{4}\dfrac{(h_i''(x))^2}{(h_i'(x))^2} + \kappa^2(x,z) \right\} \dfrac{(h_i(\breve{x}_{i-1}) - D_i)^2 (D_{i+1} - h_i(x_i^*))^2 (h_i'(x))^2}{(h_i(x_i^*) - D_i)^2 (h_i'(\breve{x}_{i-1}))^2 [(D_{i+1} - h_i(x))^2 + (h_i'(x))^2 (D_{i+1} - z)^2]}.
\end{cases} \tag{A3}
$$

If $h_i'(\bar{x}_{i-1}) = 0$ and $h_i''(\bar{x}_{i-1}) \neq 0$, then we have Eq.(A4) with

$$
\lim_{\breve{x}_{i-1} \to \bar{x}_{i-1}} \frac{h_i'(x)}{h_i'(\breve{x}_{i-1})} = \exp\left(\frac{1}{2}\frac{h_i''(\bar{x}_{i-1})}{D_{i+1} - h_i(\bar{x}_{i-1})}\Big((z - D_i)^2 - \right.
$$

$$
\left. -2(D_{i+1} - D_i)(z - D_i) + (D_{i+1} - D_i)(h_i(\bar{x}_{i-1}) - D_i) \Big) \right)
$$

and $\bar{x}_{i-1} = x_{i-1}^*$.

If $h_i'(\bar{x}_{i-1}) = 0$ and $h_i''(\bar{x}_{i-1}) = 0$, then

$$
\alpha_2(\hat{x},\hat{z}) = \frac{(D_{i+1} - H_i)^2}{(D_{i+1} - h_i(\breve{x}_{i-1}))^2},
$$

$$
\beta_2(\hat{x},\hat{z}) = 0,
$$

$$\gamma_2(\hat{x},\hat{z})=\kappa^2(\bar{x}_{i-1},z)+\Big((z-D_i)^2-2(D_{i+1}-D_i)(z-D_i)$$
$$+(D_{i+1}-D_i)(h_i(\bar{x}_{i-1})-D_i)h_i^{(4)}(\breve{x}_{i-1})\Big)\Big/[4(D_{i+1}-h_i(\bar{x}_{i-1}))].$$

If $h_i'(x)\neq 0$, then we have Eq.(A5), where $D_{n+1}=D$. If $h_i'(\bar{x}_{i-1})=0$ and $h_i''(\bar{x}_{i-1})\neq 0$, then we have Eq.(A6), where $\bar{x}_{i-1}=\breve{x}_{i-1}$.

If $h_i'(\bar{x}_{i-1})=0$ and $h_i''(\bar{x}_{i-1})=0$, then

$$\alpha_3(\hat{x},\hat{z})=1,$$
$$\beta_3(\hat{x},\hat{z})=0,$$
$$\gamma_{3,i}(\hat{x},\hat{z})=\kappa^2(\bar{x}_{i-1},z)-\frac{1}{4}h_i^{(4)}(\bar{x}_{i-1})(D_{i+1}-D_i).$$

$$\left\{\begin{aligned}
&\alpha_2(\hat{x},\hat{z})=\frac{(D_{i+1}-H_i)^2}{(D_{i+1}-h_i(\bar{x}_{i-1}))^2}\exp\left(\frac{h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}\Big((D_{i+1}-D_i)(h_i(\bar{x}_{i-1})-D_i)-2(D_{i+1}-D_i)(z-D_i)+(z-D_i)^2\Big)\right),\\
&\beta_2(\hat{x},\hat{z})=\frac{2(D_{i+1}-H_i)(z-D)h_i''(\bar{x}_{i-1})}{(D_{i+1}-h_i(\bar{x}_{i-1}))^3}\exp\left(\frac{h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}\Big((z-D_i)^2-2(D_{i+1}-D_i)(z-D_i)+(D_{i+1}-D_i)(h_i(\bar{x}_{i-1})-D_i)\Big)\right),\\
&\gamma_2(\hat{x},\hat{z})=\exp\left(\frac{h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}\Big((z-D_i)^2-2(D_{i+1}-D_i)(z-D_i)+(D_{i+1}-D_i)(h_i(\bar{x}_{i-1})-D_i)\Big)\right)\\
&\qquad\cdot\left\{\frac{h_i''(\bar{x}_{i-1})}{h_i(\bar{x}_{i-1})-D_i}\left[\frac{(h_i(\bar{x}_{i-1})-D_i)h_i''(\bar{x}_{i-1})(D_{i+1}-z)^2}{(D_{i+1}-h_i(\bar{x}_{i-1}))^2}-\frac{h_i(\bar{x}_{i-1})-D_i}{D_{i+1}-h_i(\bar{x}_{i-1})}+3\exp\left(-\frac{h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}\Big((z-D_i)^2\right.\right.\right.\\
&\qquad\left.\left.-2(D_{i+1}-D_i)(z-D_i)+(D_{i+1}-D_i)(h_i(\bar{x}_{i-1})-D_i)\Big)\right)\right]-\frac{3(D_{i+1}-D_i)h_i''(\bar{x}_{i-1})}{4(h_i(\bar{x}_{i-1})-D_i)(D_{i+1}-h_i(\bar{x}_{i-1}))}\exp\left(\frac{h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}\right.\\
&\qquad\left.\cdot(z-h_i(\bar{x}_{i-1}))(2D_{i+1}-z-h_i(\bar{x}_{i-1}))\right)+\frac{2h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}+\kappa^2(\bar{x}_{i-1},z)+\frac{h_i^{(4)}(\bar{x}_{i-1})}{4h_i''(\bar{x}_{i-1})}\left[1-\exp\left(-\frac{h_i''(\bar{x}_{i-1})}{D_{i+1}-h_i(\bar{x}_{i-1})}\right.\right.\\
&\qquad\left.\left.\left.\cdot\Big((z-D_i)^2-2(D_{i+1}-D_i)(z-D_i)+(D_{i+1}-D_i)(h_i(\bar{x}_{i-1})-D_i)\Big)\right)\right]\right\}.
\end{aligned}\right. \tag{A4}$$

$$\left\{\begin{aligned}
&\alpha_3(\hat{x},\hat{z})=\frac{(D-h_n(x_n^*))^2(h_n'(x))^2(h_n(\breve{x}_{n-1})-D_n)^2}{(D-h_n(x))^2(h_n'(\breve{x}_{n-1}))^2(h_n(x_n^*)-D_n)^2},\\
&\beta_3(\hat{x},\hat{z})=0,\\
&\gamma_{3,i}(\hat{x},\hat{z})=\frac{h_i''(\breve{x}_{i-1})}{h_i(\breve{x}_{i-1})-D_i}-\frac{3(h_i'(\breve{x}_{i-1}))^2}{4(h_i(\breve{x}_{i-1})-D_i)^2}+\frac{(h_i''(\breve{x}_{i-1}))^2}{4(h_i'(\breve{x}_{i-1}))^2}-\frac{h_i'''(\breve{x}_{i-1})}{2(h_i(\breve{x}_{i-1})-D_i)}+\frac{D_{i+1}-D_i}{(D_{i+1}-h_i(x_i^*))(h_i(x_i^*)-D_i)}\\
&\qquad\cdot\left(\frac{3(h_i'(x_i^*))^2(D_{i+1}-2h_i(x_i^*)+D_i)}{4(h_i(x_i^*)-D_i)(D_{i+1}-h_i(x_i^*))}-\frac{h_i''(x_i^*)}{1+(h_i'(x_i^*))^2}\right)\left(\frac{h_i'(x_i^*)(h_i(\breve{x}_{i-1})-D_i)}{(1+(h_i'(x_i^*))^2)h_i'(\breve{x}_{i-1})(h_i(x_i^*)-D_i)}\right)^2\\
&\qquad+\left(\kappa^2(x,z)+\frac{2h_i'(x)h_i'''(x)-(h_i''(x))^2}{4(h_i'(x))^2}+\frac{h_i''(x)}{2(D_{i+1}-h_i(x))}+\frac{2h_i''(x)(D_{i+1}-h_i(x))+3(h_i'(x))^2}{4(D_{i+1}-h_i(x))^2}\right)\\
&\qquad\cdot\left(\frac{h_i'(x)(h_i(\breve{x}_{i-1})-D_i)(D_{i+1}-h_i(x_i^*))}{h_i'(\breve{x}_{i-1})(h_i(x_i^*)-D_i)(D_{i+1}-h_i(x))}\right)^2.
\end{aligned}\right. \tag{A5}$$

$$\left\{\begin{aligned}
&\alpha_3(\hat{x},\hat{z})=\exp\big(-(D-D_n)h_n''(\bar{x}_{n-1})\big),\\
&\beta_3(\hat{x},\hat{z})=0,\\
&\gamma_{3,i}(\hat{x},\hat{z})=\frac{3h_i''(\bar{x}_{i-1})}{4(h_i(\bar{x}_{i-1})-D_i)}+\left(\kappa^2(\bar{x}_{i-1},z)+\frac{3h_i''(\bar{x}_{i-1})}{4(D_{i+1}-h_i(\bar{x}_{i-1}))}\right)\exp\big(-(D_{i+1}-D_i)h_i''(\bar{x}_{i-1})\big)\\
&\quad+\frac{h_i''(\bar{x}_{i-1})}{(h_i(\bar{x}_{i-1})-D_i)(D_{i+1}-h_i(\bar{x}_{i-1}))}\left(\frac{h_i(\bar{x}_{i-1})-D_i}{4}-\frac{3}{4}(D_{i+1}-D_i)\right)+\frac{h_i^{(4)}(\bar{x}_{i-1})}{4h_i''(\bar{x}_{i-1})}\Big(\exp\big(-(D_{i+1}-D_i)h_i''(\bar{x}_{i-1})\big)-1\Big).
\end{aligned}\right. \tag{A6}$$

APPENDIX B

When $h_i'(\bar{x}_i)=0$, we define $P_i(\bar{x}_i)=\lim_{\breve{x}_i\to\bar{x}_i} P_i(\breve{x}_i)$, $M_i(\bar{x}_i)=\lim_{\breve{x}_i\to\bar{x}_i} M_i(\breve{x}_i)$. The formulas for $P_i(\breve{x}_i)$ and $M_i(\breve{x}_i)$ are given as follows:

$$P_i(\breve{x}_i)=\left(\frac{h_i'(\breve{x}_i)}{h_i'(\breve{x}_{i-1})}\right)^{1/2}\left(\frac{D_i-h_i(x_i^*)}{D_i-h_i(\breve{x}_i)}\right)^{1/2}\left(\frac{h_i(\breve{x}_{i-1})-D_{i-1}}{h_i(x_i^*)-D_{i-1}}\right)^{1/2}\cdot P_{i-1}(\breve{x}_{i-1}),$$

and

$$M_i(\breve{x}_i)=(P_i(\breve{x}_i))^4=\left(\frac{h_i'(\breve{x}_i)}{h_i'(\breve{x}_{i-1})}\right)^2\left(\frac{D_i-h_i(x_i^*)}{D_i-h_i(\breve{x}_i)}\right)^2\left(\frac{h_i(\breve{x}_{i-1})-D_{i-1}}{h_i(x_i^*)-D_{i-1}}\right)^2 M_{i-1}(\breve{x}_{i-1}),$$

with $P_0(\breve{x}_0)=1$.

If $h_i'(\bar{x}_i)=0$, then

$$P_i(\bar{x}_i)=\exp\left(-D_i h_i''(\bar{x}_i)/4\right)P_{i-1}(\bar{x}_{i-1}),$$
$$M_i(\bar{x}_i)=\exp\left(-D_i h_i''(\bar{x}_i)\right)M_{i-1}(\bar{x}_{i-1}).$$