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Response of harmonically and stochastically excited strongly nonlinear oscillators with delayed feedback bang-bang control*

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Abstract: We studied the response of harmonically and stochastically excited strongly nonlinear oscillators with delayed feedback bang-bang control using the stochastic averaging method. First, the time-delayed feedback bang-bang control force is expressed approximately in terms of the system state variables without time delay. Then the averaged Itô stochastic differential equations for the system are derived using the stochastic averaging method. Finally, the response of the system is obtained by solving the Fokker-Plank-Kolmogorov (FPK) equation associated with the averaged Itô equations. A Duffing oscillator with time-delayed feedback bang-bang control under combined harmonic and white noise excitations is taken as an example to illustrate the proposed method. The analytical results are confirmed by digital simulation. We found that the time delay in feedback bang-bang control will deteriorate the control effectiveness and cause bifurcation of stochastic jump of Duffing oscillator.

Key words: Nonlinear system, Delayed feedback bang-bang control, Combined harmonic and white noise excitation, Stochastic averaging, Stationary response, Stochastic jump

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INTRODUCTION

In the implementation of feedback control of a dynamical system, a time delay is usually unavoidable due to the time spent on measuring and estimating the system states, calculating and executing the control forces, etc. This delay causes unsynchronized application of the control forces, and this unsynchronization may not only deteriorate the control performance but also cause instability of the system. Thus, the time delay problem has drawn much attention of the control community.

Systems with time delay under deterministic excitation have been studied extensively (Malek-Zavarei and Jamshidi, 1987; Kuo, 1987; Stepan, 1989; Agrawal and Yang, 1997; Pu, 1998; Hu and Wang, 2002; Li *et al.*, 2006), and the study on those systems under stochastic excitation has attracted many researchers. Grigoriu (1997) studied the linearly controlled system with deterministic and random time delays excited by Gaussian white noise, and investigated the stability of such a system by means of Lyapunov exponent. Di Paola and Pirrotta (2001) studied the effects of time delay on the controlled linear systems under Gaussian random excitation using an approach based on the Taylor expansion of the control force and another approach finding exact stationary solution. Zhu and Liu (2007a; 2007b) studied the response of quasi-integrable Hamiltonian systems with delayed feedback control under Gaussian white noise excitation using the stochastic averaging method.

In all these studies, the excitation of systems is purely harmonic excitation or purely random noise. However, many mechanical and structural systems are subjected to both random and harmonic excita-

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tions. A typical example of such systems is helicopter rotor blade during forward flight in a turbulent atmosphere. Zhu and Wu (2005) proposed the bang-bang control strategy for minimizing the response of strongly nonlinear oscillators under combined harmonic and white noise excitations without time delay. In the present paper, we extended the stochastic averaging method (Huang *et al.*, 2000) to predict the response of strongly nonlinear oscillators with delayed feedback bang-bang control under combined harmonic and white noise excitations. The delayed feedback bang-bang control force is expressed equivalently in terms of feedback bang-bang control force without time delay, and the system is transformed into the Itô stochastic differential equations for the system with feedback bang-bang control without time delay, from which the averaged Itô equations are derived using the stochastic averaging method. The stationary probability densities of the amplitude and mean amplitude of the system are obtained by solving the averaged Fokker-Plank-Kolmogorov (FPK) equation associated with averaged Itô equations. A Duffing oscillator under external harmonic excitation and external and parametric white noise excitations is taken as an example to illustrate the proposed method. The effect of time delay in feedback bang-bang control on the response is analyzed. Based on the stationary probability of amplitude obtained by the proposed procedure, the effect of time delay in feedback bang-bang control on the stochastic jump of Duffing oscillator is also studied.

NONLINEAR SYSTEM WITH DELAYED FEEDBACK BANG-BANG CONTROL

For a strongly nonlinear oscillator with delayed feedback bang-bang control forces under external and/or parametric excitations of harmonic function and white noises, the motion equation of the system is of the form:

$$\begin{aligned}\ddot{X}+g(X)&=\varepsilon f(X,\dot{X},\Omega t)+\varepsilon u(\dot{X}_\tau)\\&\quad+\varepsilon^{1/2}h_k(X,\dot{X})\xi_k(t),\ \ k=1,\ 2,\ \cdots,m,\end{aligned}\tag{1}$$

where $g(X)$ represents strongly nonlinear restoring force; ε is a small positive parameter; $\varepsilon f(X,\dot{X},\Omega t)$ denotes light damping and/or parametric harmonic excitation with frequency Ω; $\varepsilon^{1/2}h_k(X,\dot{X})\xi_k(t)$ represent weak external and/or parametric excitations; $\xi_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $E[\xi_k(t)\xi_l(t+\tau)]=2D_{kl}\delta(\tau)$; $\varepsilon u(\dot{X}_\tau)$ with $\dot{X}_\tau=\dot{X}(t-\tau)$ denotes delayed feedback bang-bang control force:

$$\varepsilon u(\dot{X}_\tau)=-\varepsilon b\,\mathrm{sgn}(\dot{X}(t-\tau)),\tag{2}$$

where τ is the time delay. $u(\dot{X}_\tau)$ has constant magnitude b in the opposite direction of $\dot{X}$ and changes direction at $\dot{X}=0$.

The Hamiltonian (total energy) of system Eq.(1) is:

$$H=\frac{1}{2}\dot{X}^2+U(X),\tag{3}$$

where

$$U(X)=\int_0^X g(u)\mathrm{d}u\tag{4}$$

is the potential energy.

When ε=0, the Hamiltonian system with Hamilton H associated with Eq.(1) has a trivial solution $(X=B,\dot{X}=0)$ and a family of periodic solutions around the trivial solution. When ε is small, system Eq.(1) has random periodic solutions around the trivial solution. The sample solution of system Eq.(1) can be assumed as (Huang *et al.*, 2000):

$$\begin{cases}X(t)=A\cos\Phi(t)+B,\\ \dot{X}(t)=-A\nu(A,\Phi)\sin\Phi(t),\end{cases}\tag{5}$$

where $\cos\Phi(t)$ and $\sin\Phi(t)$ are called the generalized harmonic functions:

$$\begin{cases}\Phi(t)=\Psi(t)+\Theta(t),\\ \nu(A,\Phi)=\dfrac{\mathrm{d}\Psi}{\mathrm{d}t}=\sqrt{\dfrac{2[U(A+B)-U(A\cos\Phi+B)]}{A^2\sin^2\Phi}},\end{cases}\tag{6}$$

and A, B, Φ, Ψ, and Θ are all random processes. Expanding $\nu^{-1}(A,\Phi)$ in Eq.(5) into Fourier series:

$$\nu^{-1}(A,\Phi)=C_0(A)+\sum_{n=1}^{\infty}C_n(A)\cos(n\Phi), \tag{7}$$

and then integrating Eq.(7) with respect to Φ from 0 to 2π leads to the average period:

$$T(A)=2\pi C_0(A), \tag{8}$$

and average frequency:

$$\omega(A)=\frac{1}{C_0(A)}. \tag{9}$$

Thus, in average:

$$\Phi(t)\approx\omega(A)t+\Theta(t). \tag{10}$$

So, for a small time delay, in the sense of averaging we have the following approximate expression:

$$\begin{aligned}\dot{X}(t-\tau)&=-A(t-\tau)\frac{\mathrm{d}\Psi(t-\tau)}{\mathrm{d}t}\sin[\Phi(t-\tau)]\\&\approx-A(t)\omega\sin[\omega(t-\tau)+\Theta(t)].\end{aligned} \tag{11}$$

In the feedback bang-bang control of strongly nonlinear systems (Zhu and Wu, 2005), the optimal control force is in the form of

$$u=u(\dot{X})=-b\,\mathrm{sgn}(\dot{X}). \tag{12}$$

Using the approximate relation between $\dot{X}(t)$ and $\dot{X}(t-\tau)$ in Eq.(11) and Eq.(5), the delayed control force $u(\dot{X}_\tau)$ can be equivalently written as (Zhu and Liu, 2007a):

$$\begin{aligned}u(\dot{X}_\tau)&=u(\dot{X}(t-\tau))=u(\dot{X}(t))\cos(\omega\tau)\\&=-b\cos(\omega\tau)\mathrm{sgn}(\dot{X}(t))=b\cos(\omega\tau)\mathrm{sgn}(\sin\Phi).\end{aligned} \tag{13}$$

Substituting Eq.(13) into Eq.(1), and treating Eq.(5) as generalized van der Pol transformation from X, $\dot{X}$ to A, Θ, the following equations for A and Θ can be obtained:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t}&=\varepsilon F_1(A,\Phi,\Omega t)+\varepsilon F_1^u(A,\Phi,\tau)\\&\quad+\varepsilon^{1/2}H_{1k}(A,\Phi)\xi_k(t),\\\frac{\mathrm{d}\Theta}{\mathrm{d}t}&=\varepsilon F_2(A,\Phi,\Omega t)+\varepsilon F_2^u(A,\Phi,\tau)\\&\quad+\varepsilon^{1/2}H_{2k}(A,\Phi)\xi_k(t),\\&\quad k=1,2,\cdots,m,\end{aligned} \tag{14}$$

where

$$\begin{cases}F_1=\dfrac{-A}{g(A+B)(1+h)}f(A\cos\Phi+B,\\\quad -A\nu(A,\Phi)\sin\Phi,\Omega t)\nu(A,\Phi)\sin\Phi,\\F_1^u=\dfrac{-A}{g(A+B)(1+h)}b|\sin\Phi|\nu(A,\Phi)\cos(\omega\tau),\\F_2=\dfrac{-1}{g(A+B)(1+h)}f(A\cos\Phi+B,\\\quad -A\nu(A,\Phi)\sin\Phi,\Omega t)\nu(A,\Phi)(\cos\Phi+h),\\F_2^u=\dfrac{-1}{g(A+B)(1+h)}b\,\mathrm{sgn}(\sin\Phi)\\\quad\cdot\cos(\omega\tau)\nu(A,\Phi)(\cos\Phi+h),\\H_{1k}=\dfrac{-A}{g(A+B)(1+h)}h_k(A\cos\Phi+B,\\\quad -A\nu(A,\Phi)\sin\Phi)\nu(A,\Phi)\sin\Phi,\\H_{2k}=\dfrac{-1}{g(A+B)(1+h)}h_k(A\cos\Phi+B,\\\quad -A\nu(A,\Phi)\sin\Phi)\nu(A,\Phi)(\cos\Phi+h),\end{cases} \tag{15}$$

with X, $\dot{X}$ replaced by A, Θ according to transformations Eq.(5), and

$$h=\frac{g(-A+B)+g(A+B)}{g(-A+B)-g(A+B)}. \tag{16}$$

Eq.(14) can be modeled as Stratonovich stochastic differential equations and then transformed into the Itô stochastic differential equation by adding Wong-Zakai correction terms. The result is:

$$\begin{aligned}\mathrm{d}A&=\varepsilon[m_1(A,\Phi,\Omega t)+F_1^u(A,\Phi,\tau)]\mathrm{d}t\\&\quad+\varepsilon^{1/2}\sigma_{1r}(A,\Phi)\mathrm{d}B_r(t),\\\mathrm{d}\Theta&=\varepsilon[m_2(A,\Phi,\Omega t)+F_2^u(A,\Phi,\tau)]\mathrm{d}t\\&\quad+\varepsilon^{1/2}\sigma_{2r}(A,\Phi)\mathrm{d}B_r(t),\\&\quad r=1,2,\cdots,m,\end{aligned} \tag{17}$$

where $B_r(t)$ are independent unit Wiener processes:

$$m_i = F_i + D_{kl}\frac{\partial H_{ik}}{\partial A}H_{1l} + D_{kl}\frac{\partial H_{ik}}{\partial \Phi}H_{2l},$$
$$b_{ij} = \sigma_{ir}\sigma_{jr} = 2D_{kl}H_{ik}H_{jl},$$
$$i,j=1,2,\quad k,l,r=1,2,\cdots,m. \tag{18}$$

AVERAGED EQUATIONS AND STATIONARY SOLUTIONS

System Eq.(1) has the harmonic excitation and so two cases can be identified: resonant case and non-resonant case. In the non-resonant case, the harmonic excitation has no effect on the first approximation of the response. So, only the resonant case is considered in the following. Assume that we are interested in the resonant case, i.e.,

$$\frac{\Omega}{\omega(A)} = \frac{q}{p} + \varepsilon\sigma, \tag{19}$$

where p and q are relatively prime positive small integers and $\varepsilon\sigma$ is the detuning parameter. In this case, multiplying Eq.(19) by t and utilizing the approximate relation Eq.(10) yield

$$\Omega t = \frac{q}{p}\Phi + \varepsilon\sigma\Psi - \frac{q}{p}\Theta. \tag{20}$$

Introduce a new variable Γ such that

$$\Gamma = \varepsilon\sigma\Psi - \frac{q}{p}\Theta \tag{21}$$

is a measure of the phase difference between the response and the harmonic excitation. Then, Eq.(20) can be rewritten as

$$\Omega t = \frac{q}{p}\Phi + \Gamma. \tag{22}$$

With the transformation from Θ to Γ as defined by Eq.(21), Eq.(17) can be rewritten as

$$\mathrm{d}A = \varepsilon(m_1(A,\Phi,\Gamma) + F_1^u(A,\Phi,\Gamma))\mathrm{d}t$$
$$+\varepsilon^{1/2}\sigma_{1r}(A,\Phi)\mathrm{d}B_r(t),$$

$$\mathrm{d}\Gamma = \{\varepsilon[m_2(A,\Phi,\Gamma) + F_2^u(A,\Phi,\Gamma)](-q/p)$$
$$+(\Omega/\omega(A) - q/p)\nu(A,\Phi)\}\mathrm{d}t \tag{23}$$
$$-\varepsilon^{1/2}\frac{q}{p}\sigma_{2r}(A,\Phi)\mathrm{d}B_r(t),$$

where A and Γ are slowly varying processes while Φ is a rapidly varying process. Averaging the drift and diffusion coefficients with respect to Φ yields the following averaged Itô equations:

$$\begin{cases}\mathrm{d}A = \varepsilon\bar{m}_1(A,\Gamma,\tau)\mathrm{d}t + \varepsilon^{1/2}\bar{\sigma}_{1r}(A)\mathrm{d}B_r(t),\\ \mathrm{d}\Gamma = \varepsilon\bar{m}_2(A,\Gamma,\tau)\mathrm{d}t + \varepsilon^{1/2}\bar{\sigma}_{2r}(A)\mathrm{d}B_r(t),\end{cases} \tag{24}$$

where

$$\bar{m}_1 = \langle m_1(A,\Phi,\Gamma)\rangle_\Phi + \langle F_1^u(A,\Phi,\tau)\rangle_\Phi,$$
$$\bar{m}_2 = \left\langle m_2(A,\Phi,\Gamma)\left(-\frac{q}{p}\right) + \left(\frac{\Omega}{\omega(A)} - \frac{q}{p}\right)\frac{\nu(A,\Phi)}{\varepsilon}\right\rangle_\Phi$$
$$-\frac{q}{p}\langle F_2^u(A,\Phi,\tau)\rangle_\Phi,$$
$$\bar{b}_{ij} = \bar{\sigma}_{ir}\bar{\sigma}_{jr} = -\frac{q}{p}\langle\sigma_{1r}\sigma_{2r}\rangle_\Phi, \tag{25}$$

and $\langle\ \rangle_\Phi$ represents the averaging with respect to Φ from 0 to 2π.

The averaged FPK equation associated with averaged Itô Eq.(24) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial a}(\bar{m}_1 p) - \frac{\partial}{\partial\gamma}(\bar{m}_2 p) + \frac{1}{2}\frac{\partial^2}{\partial a^2}(\bar{b}_{11}p)$$
$$+\frac{\partial^2}{\partial a\partial\gamma}(\bar{b}_{12}p) + \frac{1}{2}\frac{\partial^2}{\partial\gamma^2}(\bar{b}_{22}p), \tag{26}$$

where $p(a,\gamma,t\,|\,a_0,\gamma_0)$ is the transition probability density of amplitude A and phase Γ. The initial condition of FPK Eq.(26) is

$$p(a,\gamma,0\,|\,a_0,\gamma_0) = \delta(a-a_0)\delta(\gamma-\gamma_0),\ t=0. \tag{27}$$

Since $p(a,\gamma,t\,|\,a_0,\gamma_0)$ is a periodic function of γ, the boundary condition of FPK Eq.(26) with respect to γ is

$$p(a,\gamma+2n\pi,t\,|\,a_0,\gamma_0) = p(a,\gamma,t\,|\,a_0,\gamma_0). \tag{28}$$

As for the boundary conditions of Eq.(26) with respect to a, one is

$$p = \text{finite at } a = 0, \tag{29}$$

which means that a=0 is a reflecting boundary. The other boundary conditions are

$$p \to 0, \partial p / \partial a \to 0 \text{ as } a \to \infty. \tag{30}$$

The solution of FPK Eq.(26) under initial and boundary conditions Eqs.(27)~(30) can be obtained using the finite difference method.

EXAMPLE

As an example, consider a Duffing oscillator subject to external harmonic excitation and external and parametric white noise excitations with delayed feedback bang-bang control. The motion equation of the system is of the form:

$$\begin{aligned}\ddot{X} + \omega_0^2 X + \alpha X^3 = -\beta \dot{X} + E\cos(\Omega t) \\ + \xi_1(t) + X\xi_2(t) + u_\tau,\end{aligned} \tag{31}$$

where ω_0, α, β, E, Ω are positive constants denoting the natural frequency of degenerated linear oscillator, intensity of nonlinearity, damping coefficient, amplitude and frequency of the harmonic excitation, respectively; $\xi_k(t)$ (k=1,2) are independent Gaussian white noises in the sense of Stratonovich with intensities $2D_k$. It is assumed that β, E, D_k are of the same order of ε. $u_\tau = -b\,\mathrm{sgn}(\dot{X}(t-\tau))$ is the delayed feedback bang-bang control force.

For this oscillator:

$$\begin{cases} g(x) = \omega_0^2 x + \alpha x^3, \\ U(x) = \omega_0^2 x^2/2 + \alpha x^4/4, \\ b = h = 0, \end{cases} \tag{32}$$

and

$$\begin{cases} \nu(a,\varphi) = [(\omega_0^2 + 3\alpha a^2/4)(1 + \lambda\cos(2\varphi))]^{1/2}, \\ \lambda = \alpha a^2 / [4(\omega_0^2 + 3\alpha a^2/4)]. \end{cases} \tag{33}$$

$\nu(a,\varphi)$ can be approximated by the following finite sum with a relative error less than 0.03%:

$$\begin{aligned}\nu(a,\varphi) = \nu_0(a) + \nu_2(a)\cos(2\varphi) + \nu_4(a)\cos(4\varphi) \\ + \nu_6(a)\cos(6\varphi),\end{aligned} \tag{34}$$

where

$$\begin{cases} \nu_0(a) = (\omega_0^2 + 3\alpha a^2/4)^{1/2}(1 - \lambda^2/16), \\ \nu_2(a) = (\omega_0^2 + 3\alpha a^2/4)^{1/2}(\lambda/2 + 3\lambda^3/64), \\ \nu_4(a) = (\omega_0^2 + 3\alpha a^2/4)^{1/2}(-\lambda^2/16), \\ \nu_6(a) = (\omega_0^2 + 3\alpha a^2/4)^{1/2}(\lambda^3/64). \end{cases} \tag{35}$$

So, the averaged frequency $\omega(a)=\nu_0(a)$. Following Eq.(13), the delayed bang-bang control force u_τ can be approximately expressed as

$$u_\tau = b\cos(\omega\tau)\,\mathrm{sgn}(\sin\varphi). \tag{36}$$

By using the generalized van der Pol transformations Eq.(5) with B=0, Eq.(31) is converted into

$$\begin{cases} \dfrac{\mathrm{d}A}{\mathrm{d}t} = F_1^{(1)}(A,\Phi,\Omega t) + F_1^{(2)}(A,\Phi,\tau) \\ \qquad + h_{11}(A,\Phi)\xi_1(t) + h_{12}(A,\Phi)\xi_2(t), \\ \dfrac{\mathrm{d}\Theta}{\mathrm{d}t} = F_2^{(1)}(A,\Phi,\Omega t) + F_2^{(2)}(A,\Phi,\tau) \\ \qquad + h_{21}(A,\Phi)\xi_1(t) + h_{22}(A,\Phi)\xi_2(t), \end{cases} \tag{37}$$

where

$$\begin{cases} F_1^{(1)} = -\dfrac{A}{g(A)}[\beta A\nu(A,\Phi)\sin\Phi \\ \qquad + E\cos(\Omega t)]\nu(A,\Phi)\sin\Phi, \\ F_1^{(2)} = -\dfrac{A}{g(A)}b|\sin\Phi|\nu(A,\Phi)\cos(\omega\tau), \\ F_2^{(1)} = -\dfrac{1}{g(A)}[\beta A\nu(A,\Phi)\sin\Phi \\ \qquad + E\cos(\Omega t)]\nu(A,\Phi)\cos\Phi, \\ F_2^{(2)} = -\dfrac{1}{g(A)}b\,\mathrm{sgn}(\sin\Phi)\cos(\omega\tau)\nu(A,\Phi)\cos\Phi, \\ h_{11} = -\dfrac{A}{g(A)}\nu(A,\Phi)\sin\Phi, \\ h_{12} = -\dfrac{A^2}{g(A)}\nu(A,\Phi)\sin\Phi\cos\Phi, \\ h_{21} = -\dfrac{1}{g(A)}\nu(A,\Phi)\cos\Phi, \\ h_{22} = -\dfrac{A}{g(A)}\nu(A,\Phi)\cos^2\Phi. \end{cases} \tag{38}$$

Eq.(38) can be modeled as the following Itô stochastic differential equations by adding Wong-Zakai correction terms

$$\begin{cases} \mathrm{d}A=[m_1^{(1)}(A,\Phi,\Omega t)+F_1^{(2)}(A,\Phi,\tau)]\mathrm{d}t \\ \qquad +\sigma_{1r}(A,\Phi)\mathrm{d}B_r(t), \\ \mathrm{d}\Theta=[m_2^{(1)}(A,\Phi,\Omega t)+F_2^{(2)}(A,\Phi,\tau)]\mathrm{d}t \\ \qquad +\sigma_{2r}(A,\Phi)\mathrm{d}B_r(t), \end{cases} \quad r=1,2, \tag{39}$$

where

$$\begin{cases} m_i^{(1)}=F_i^{(1)}+D_r\left(h_{1r}\dfrac{\partial h_{ir}}{\partial A}+h_{2r}\dfrac{\partial h_{ir}}{\partial \Phi}\right), \\ b_{ij}=\sigma_{ir}\sigma_{jr}=2D_r h_{ir}h_{jr}. \end{cases} \quad i,j,r=1,2. \tag{40}$$

Consider the case of primary resonance:

$$\frac{\Omega}{\omega(a)}=1+\sigma, \tag{41}$$

where σ is of the same order of ε. Introducing the new variable

$$\Gamma=\sigma\Psi-\Theta. \tag{42}$$

Eq.(39) is transformed into

$$\begin{cases} \mathrm{d}A=[m_1^{(1)}(A,\Phi,\Gamma)+F_1^{(2)}(A,\Phi,\tau)]\mathrm{d}t \\ \qquad +\sigma_{1r}(A,\Phi)\mathrm{d}B_r(t), \\ \mathrm{d}\Gamma=\{-[m_2^{(1)}(A,\Phi,\Gamma)+F_2^{(2)}(A,\Phi,\tau)] \\ \qquad +(\Omega/\omega(A)-1)\nu(A,\Phi)\}\mathrm{d}t \\ \qquad -\sigma_{2r}(A,\Phi)\mathrm{d}B_r(t). \end{cases} \tag{43}$$

Averaging the drift and diffusion coefficients in Itô stochastic differential Eq.(43) with respect to Φ leads to

$$\begin{aligned} \mathrm{d}A&=\overline{m}_1(A,\Gamma,\tau)\mathrm{d}t+\overline{\sigma}_{1r}(A)\mathrm{d}B_r(t), \\ \mathrm{d}\Gamma&=\overline{m}_2(A,\Gamma,\tau)\mathrm{d}t+\overline{\sigma}_{2r}(A)\mathrm{d}B_r(t), \end{aligned} \tag{44}$$

where

$$\begin{cases} \overline{m}_1(A,\Gamma,\tau)= \\ \quad -5\alpha\beta A^3\big/[16(\alpha A^2+\omega_0^2)]-\beta\omega_0^2A\big/[2(\alpha A^2+\omega_0^2)] \\ \quad +(4\alpha^2D_1A^4+14\alpha^2D_2A^6+12\alpha\omega_0^2D_1A^2+34\alpha\omega_0^2D_2A^4 \\ \quad +32\omega_0^4D_1+24\omega_0^4D_2A^2)\big/[64A(\alpha A^2+\omega_0^2)^3] \\ \quad +(2\nu_0-\nu_2)E\sin\Gamma\big/[4(\alpha A^2+\omega_0^2)] \\ \quad -2bA\cos(\nu_0\tau)(\nu_0-\nu_2/3-\nu_4/15-\nu_6/35)\big/[\pi g(A)], \\ \overline{m}_2(A,\Gamma,\tau)=\Omega-\nu_0 \\ \quad +(2\nu_0+\nu_2)E\cos\Gamma\big/[4A(\alpha A^2+\omega_0^2)], \\ \overline{b}_{11}(A)=\sigma_{1k}\sigma_{1k}=[3\alpha D_2A^4+16\omega_0^2D_1 \\ \quad +2A^2(5\alpha D_1+2\omega_0^2D_2)]\big/[16(\alpha A^2+\omega_0^2)^2], \\ \overline{b}_{22}(A)=\sigma_{2k}\sigma_{2k}=[11\alpha D_2A^4+16\omega_0^2D_1 \\ \quad +2A^2(7\alpha D_1+6\omega_0^2D_2)]\big/[16A^2(\alpha A^2+\omega_0^2)^2], \\ \overline{b}_{12}(A)=\overline{b}_{21}(A)=0. \end{cases} \tag{45}$$

The averaged stationary FPK equation associated with Itô Eq.(44) is

$$-\frac{\partial}{\partial a}(\overline{m}_1p)-\frac{\partial}{\partial\gamma}(\overline{m}_2p)+\frac{1}{2}\frac{\partial^2}{\partial a^2}(\overline{b}_{11}p)+\frac{\partial^2}{\partial a\partial\gamma}(\overline{b}_{12}p)+\frac{1}{2}\frac{\partial^2}{\partial\gamma^2}(\overline{b}_{22}p)=0. \tag{46}$$

Eq.(46) is an elliptic partial differential equation and can be solved numerically by using the finite difference method to yield the stationary joint probability density $p(a,\gamma)$. Then the stationary probability density $p(a)$ and the stationary mean amplitude $E[A]$ can be obtained as follows:

$$p(a)=\int_0^{2\pi}p(a,\gamma)\mathrm{d}\gamma, \tag{47}$$

$$E[A]=\int_0^{\infty}\int_0^{2\pi}ap(a,\gamma)\mathrm{d}\gamma\mathrm{d}a. \tag{48}$$

Some numerical results for the stationary mean amplitude value $E[A]$ as a function of control force b are shown in Fig.1. It is seen that the results obtained by using the proposed method agree well with those from digital simulation. It is also seen that the time delay deteriorates the control effectiveness remarkably. When τ=0, the control force can reduce the response of the system effectively and the control effectiveness is weakened when τ=1. The response even increases as the control force increases when τ=2.

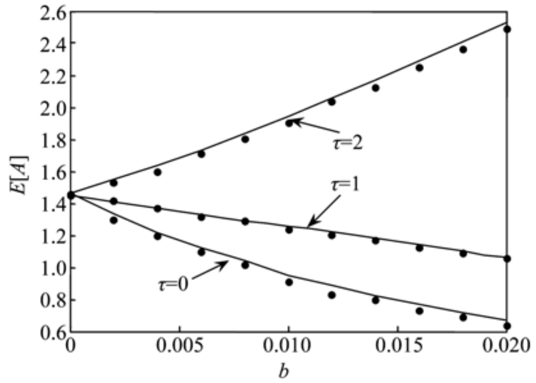

Fig.1 Stationary mean amplitude $E[A]$ of system Eq.(31) in primary external resonance using the proposed stochastic averaging method; Dots represent the results from digital simulation of the original system Eq.(31). ω_0=1.0, Ω=1.5, α=0.2, β=0.01, E=0.02, D_1=0.01, D_2=0.01

It is well known that a Duffing oscillator with hardening stiffness subject to harmonic excitation may exhibit the phenomenon of sharp jumps in amplitude (Den Hartog, 1956). The jump phenomenon may also occur when the Duffing oscillator is subjected to combined harmonic and white noise excitations, which has been studied by the stochastic averaging method (Huang *et al.*, 2000). The control of jump and its bifurcation of Duffing oscillator subject to combined harmonic and white noise excitations have been studied by Zhu and Wu (2005). Here, the effect of delayed feedback bang-bang control on the jump and its bifurcation of system Eq.(31) can also be seen in Fig.2. Fig.2a shows that without control force the probability density $p(a)$ is bimodal and so stochastic jump may occur in system Eq.(31). For τ=0 with appropriate control force, the probability density $p(a)$ is unimodal (Fig.2b) and no jump may occur in system Eq.(31). The probability density $p(a)$ becomes bimodal again when τ=1 (Fig.2c) and τ=2 (Fig.2d), thus the stochastic jump may occur. It is also seen from Fig.2 that the results obtained by using the proposed method agree well with those from digital simulation.

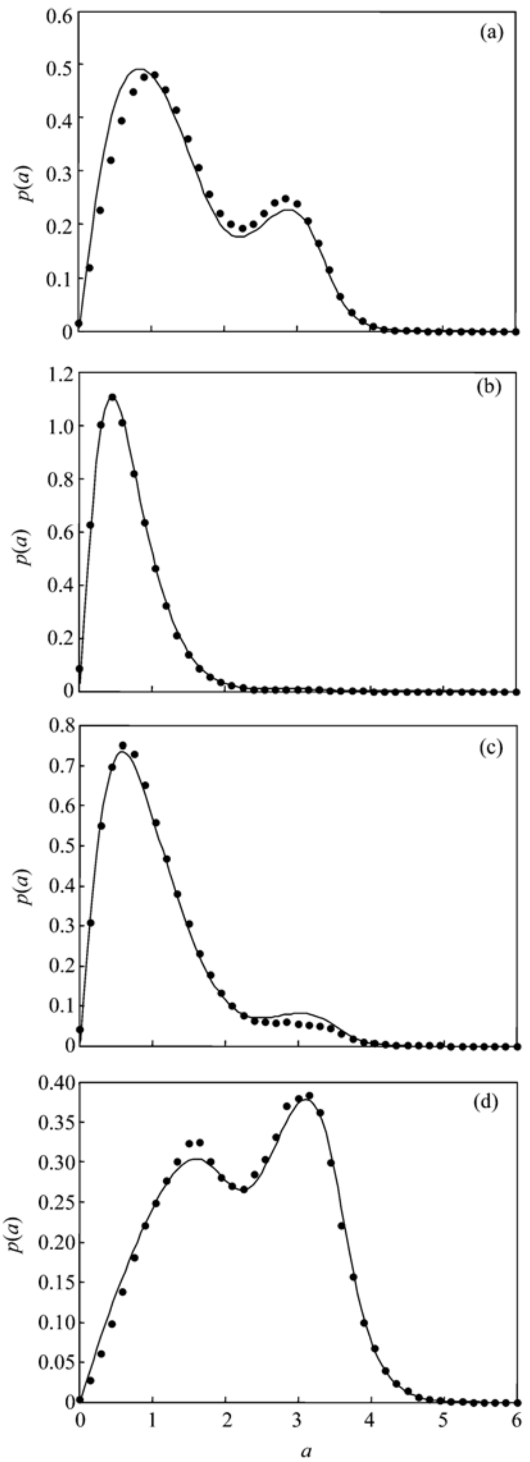

Fig.2 Stationary probability density $p(a)$ of amplitude in primary external resonance using the proposed stochastic averaging method; Dots represent the results from digital simulation of the original system Eq.(31). ω_0=1.0, Ω=1.5, α=0.2, β=0.01, E=0.2, D_1=0.008, D_2=0.006. (a) Uncontrolled, b=0; (b) Controlled, b=0.015, τ=0; (c) Controlled, b=0.015, τ=1.0; (d) Controlled, b=0.015, τ=2.0

CONCLUSION

We proposed a procedure for predicting the response of strongly nonlinear systems under combined harmonic and white noise excitations and delayed feedback bang-bang control. By replacing the delayed feedback bang-bang control force approximately with the equivalent non-delayed feedback bang-bang control force in an average sense, the stochastic averaging method for strongly nonlinear systems under combined harmonic and white noise excitations was extended to the case of strongly nonlinear systems with delayed feedback bang-bang control under combined harmonic and white noise excitations. A Duffing oscillator with hardening stiffness was taken as an example to illustrate the application of the extended stochastic averaging method. It showed that all the results obtained for the example using the proposed procedure agree well with those from the Monte Carlo simulation of the original system. It also showed that by using the proposed method the time delay in feedback bang-bang control deteriorates the control effectiveness and causes bifurcation of stochastic jump of the Duffing oscillator.

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