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# A co-rotational updated Lagrangian formulation for a 2D beam element with consideration of the deformed curvature

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**Abstract:** A tensor-based updated Lagrangian (UL) formulation for the geometrically nonlinear analysis of 2D beam-column structures is developed by using curvilinear coordinates, which has considered the effects of the deformed curvature. Between the known configuration $C_1$ and the desired configuration $C_2$, a configuration $C_2^*$ derived by rigid-body motion of $C_1$ is introduced to eliminate the element-end transverse displacements between $C_2^*$ and $C_2$. A stiffness matrix is obtained in $C_2^*$; and then by a transformation defined by the element-end displacements, the stiffness matrix in $C_2^*$ is transformed into that in $C_1$. Comparing the stiffness matrix with that in the conventional UL formulation for a 2D beam element, the initial displacement stiffness matrix emerges, which results from the deformed curvature within the element. Numerical examples have verified the accuracy and efficiency of the present formulation, and the results show that the deformed curvatures have significant effects when deformations are large.

**Key words:** Deformed curvature, Beam element, Updated Lagrangian (UL) formulation, Geometrical no-linearity, Finite element
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## INTRODUCTION

The geometrically nonlinear analysis of frames has been widely studied (Bathe and Bolourchi, 1979; Hsiao and Hou, 1987; Meek and Hoon Swee, 1984; Meek and Xue, 1996; Stolarski and Belytschko, 1982; Yang *et al.*, 2007). The commonly used formulations are total Lagrangian (TL) formulation, updated Lagrangian (UL) formulation and co-rotational (CR) formulation (Crisfield, 1991; Hsiao *et al.*, 1999; Izzuddin, 1996; 2001).

If the effect of axial forces is not considered, the deflection of a beam is found to be cubic polynomials, so cubic polynomials are naturally used as shape functions in the finite element analysis of frames. These shape functions are the Hermitian polynomials (HPs). Since the beam elements interpolated by HPs have a consistent linear stiffness matrix with the classical beam theory, they have attracted many scholars' attention. Bathe and Bolourchi (1979) and Yang and McGuire (1986) developed UL beam elements just based on the HPs. However, numerical tests indicate that available UL beam elements interpolated by HPs are almost the same as those interpolated by the linear polynomials, although they seem to be more accurate than the linear interpolations. For the benchmark problem of Williams' toggle (Williams, 1964), in order to get a good correlation with Williams' analysis, the UL beam elements interpolated by HPs must be eight elements per member (Teh and Clarke, 1998) or ten elements per member (Yang *et al.*, 2007), showing the poor efficiency of the elements. Many scholars (Saleeb *et al.*, 1992; Bathe, 1996) discarded the HPs and turned to the low order linear interpolation.

The conventional UL formulations for beam analysis did not consider the effect of deformed curvature within the element on the displacement and geometric stiffness matrices, i.e., it is assumed tacitly that each element is "straight" before each load

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increment. Because of this tacit simplification, more elements are required to achieve adequate accuracy for a nonlinear problem, as indicated by some literature (Teh and Clarke, 1998). In the investigation of nonlinear buckling and postbuckling of elastic arches, Pi and Trahair (1998) found that the effects of prebuckling deformations on the buckling loads are significant for shallow arches. According to the literature available to the authors, there is little research on the effect of deformed curvature in the geometrically nonlinear analysis. This paper will develop a new beam element based on the UL formulation in which the effect of deformed curvature on element stiffness matrices is included. Numerical examples show that the new element can achieve a similar convergence rate under much fewer elements used.

## CONFIGURATIONS AND COORDINATE SYSTEMS

For each element, the following three configurations are adopted (Fig.1):

(1) Configuration 1 ($C_1$)—a known balanced state. In the UL formulation, the deformations and strains are referred to this configuration, and the tangent stiffness matrix is obtained in this configuration.

(2) Configuration 2 ($C_2$)—a desired deformed state.

(3) CR configuration ($C_2^*$)—a configuration obtained through an imaginary rigid-body motion of $C_1$. The line joining the centroids of both end sections in $C_2^*$ is aligned with that in $C_2$.

Three coordinate systems below are adopted (Fig.1):

(1) Element coordinates *XY*—in $C_1$, the origin is located at the end cross-sectional centroid of Node $A_1$, and the *X* axis passes through both end cross-sectional centroids.

(2) Element CR coordinates *xy*—in $C_2^*$, the base vectors corresponding to *x* and *y* are $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$, respectively; the origin is located at the end cross-sectional centroid of Node A; the *x* axis passes through both end cross-sectional centroids. In this paper, it is in this coordinate system that the equation of virtual work is established, then by transformation defined by the element-end displacements, the displacements and rotations are transformed to those in the frame of *XOY*, and then the equation of virtual work in $C_1$ is obtained.

(3) Element cross-sectional coordinates $\zeta$, $\xi$—the orthogonal base vectors are $\boldsymbol{t}_0$ and $\boldsymbol{n}_0$ in $C_2^*$ or $\boldsymbol{t}$ and $\boldsymbol{n}$ in $C_2$; they are moving coordinates; the origin is located at the cross-sectional centroids, $\boldsymbol{t}$ and $\boldsymbol{t}_0$ coincide with the cross-sectional normal vectors, and $\boldsymbol{n}$ and $\boldsymbol{n}_0$ are the cross-sectional principal axial unit vectors.

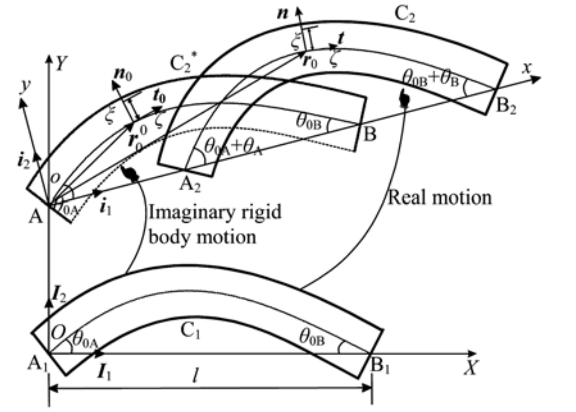


**Fig.1 Configurations and coordinate systems of a 2D beam element**

## KINEMATICS

### Geometry

The following assumptions are made in the derivation of the beam element: (1) The beam is prismatic and slender, and the Euler-Bernoulli hypothesis is valid; (2) The cross-section of the beam is bi-symmetric; (3) The material is isotropic and linear elastic.

The $C_2^*$ of the beam is described by a family of cross-sections, the centroids of which are connected by a centroidal axis, and the centroidal axis is defined as

$$\boldsymbol{r}_0^0 = x\boldsymbol{i}_1 + u_{y0}(x)\boldsymbol{i}_2 \ (0 \le x \le l), \tag{1}$$

where *l* is the length between both end cross-sectional centroids in $C_1$ or $C_2^*$. The tangent vector of the centroidal axis is

$$\boldsymbol{v}_0 = \boldsymbol{i}_1 + u'_{y0}\boldsymbol{i}_2, \tag{2}$$

where the superscript "′" denotes the differentiation with respect to $x$, sic passim.

The unit vector is

$$\boldsymbol{t}_0=\frac{1}{\sqrt{1+u_{y0}'^2}}\boldsymbol{i}_1+\frac{u_{y0}'}{\sqrt{1+u_{y0}'^2}}\boldsymbol{i}_2=\cos\theta_0\boldsymbol{i}_1+\sin\theta_0\boldsymbol{i}_2, \tag{3}$$

where $\theta_0$ is the angle between $\boldsymbol{t}_0$ and the $x$ axis.

An arbitrary point on the cross-section is described by the unit vector $\boldsymbol{n}_0$, orthogonal to $\boldsymbol{t}_0$ and the coordinate $\xi$. Denote the arbitrary point in $C_2^*$ by $\boldsymbol{r}^0$,

$$\boldsymbol{r}^0=\boldsymbol{r}_0^0+\boldsymbol{n}_0\xi=x\boldsymbol{i}_1+u_{y0}\boldsymbol{i}_2+\boldsymbol{n}_0\xi, \tag{4}$$

in which

$$\boldsymbol{n}_0=-\sin\theta_0\boldsymbol{i}_1+\cos\theta_0\boldsymbol{i}_2. \tag{5}$$

After deformation, the position vector $\boldsymbol{r}_0^0$ in $C_2^*$ moves to $\boldsymbol{r}_0$ in $C_2$, and $\boldsymbol{r}_0$ can be represented as

$$\boldsymbol{r}_0=(x+u_x)\boldsymbol{i}_1+(u_{y0}+u_y)\boldsymbol{i}_2, \tag{6}$$

where $u_x$ and $u_y$ are incremental displacements from $C_2^*$ to $C_2$, which are functions of $x$.

In $C_2$, the tangent vector of the centroidal axis is

$$\boldsymbol{v}=(1+u_x')\boldsymbol{i}_1+(u_{y0}'+u_y')\boldsymbol{i}_2. \tag{7}$$

The normalized vector $\boldsymbol{v}$ is denoted by $\boldsymbol{t}$,

$$\begin{aligned}\boldsymbol{t}&=\frac{(1+u_x')\boldsymbol{i}_1+(u_{y0}'+u_y')\boldsymbol{i}_2}{\sqrt{(1+u_x')^2+(u_{y0}'+u_y')^2}}\\&=\cos(\theta_0+\theta)\boldsymbol{i}_1+\sin(\theta_0+\theta)\boldsymbol{i}_2,\end{aligned} \tag{8}$$

where $\theta_0+\theta$ is the angle between $\boldsymbol{t}$ and the $x$ axis, and $\theta$ is the incremental angle from $C_2^*$ to $C_2$.

In Eq.(6), after the incremental deformation, the position vector of the point $(x, \xi)$ at $C_2^*$ becomes

$$\boldsymbol{r}=\boldsymbol{r}_0+\boldsymbol{n}\xi=(x+u_x)\boldsymbol{i}_1+(u_{y0}+u_y)\boldsymbol{i}_2+\boldsymbol{n}\xi, \tag{9}$$

where

$$\boldsymbol{n}=-\sin(\theta_0+\theta)\boldsymbol{i}_1+\cos(\theta_0+\theta)\boldsymbol{i}_2. \tag{10}$$

**Deformations and strains**

From Eq.(4), the covariant base vectors in $C_2^*$ can be obtained as

$$\boldsymbol{g}_1=\frac{\partial\boldsymbol{r}^0}{\partial\zeta}=(\boldsymbol{v}_0+\boldsymbol{n}_0'\xi)\cos\theta_0=(1-\xi\theta_0'\cos\theta_0)\boldsymbol{t}_0, \tag{11}$$

$$\boldsymbol{g}_2=\frac{\partial\boldsymbol{r}^0}{\partial\xi}=\boldsymbol{n}_0=-\sin\theta_0\boldsymbol{i}_1+\cos\theta_0\boldsymbol{i}_2, \tag{12}$$

where

$$\boldsymbol{n}_0'=-\theta_0'\boldsymbol{t}_0. \tag{13}$$

The covariant metric tensor $g_{\alpha\beta}$ in $C_2^*$ is

$$\left[g_{\alpha\beta}\right]=\begin{bmatrix}\boldsymbol{g}_1\boldsymbol{g}_1 & \boldsymbol{g}_1\boldsymbol{g}_2\\ \boldsymbol{g}_2\boldsymbol{g}_1 & \boldsymbol{g}_2\boldsymbol{g}_2\end{bmatrix}, \tag{14}$$

where $\boldsymbol{g}_1\boldsymbol{g}_1=(1-\xi\theta'_0\cos\theta_0)^2$, $\boldsymbol{g}_2\boldsymbol{g}_2=1$, and $\boldsymbol{g}_1\boldsymbol{g}_2=0$.

From Eq.(9), the covariant base vectors in $C_2$ can be obtained as

$$\begin{cases}\boldsymbol{G}_1=\dfrac{\partial\boldsymbol{r}}{\partial\zeta}=(\boldsymbol{v}+\boldsymbol{n}'\xi)\cos\theta_0\\ \quad=\cos\theta_0\left[\dfrac{1+u_x'}{\cos(\theta+\theta_0)}-\xi(\theta_0'+\theta')\right]\boldsymbol{t},\\ \boldsymbol{G}_2=\dfrac{\partial\boldsymbol{r}}{\partial\xi}=\boldsymbol{n}=-\sin(\theta_0+\theta)\boldsymbol{i}_1+\cos(\theta_0+\theta)\boldsymbol{i}_2,\end{cases} \tag{15}$$

where

$$\boldsymbol{n}'=-(\theta_0'+\theta')\boldsymbol{t}. \tag{16}$$

The covariant metric tensor $G_{\alpha\beta}$ in $C_2$ is

$$\left[G_{\alpha\beta}\right]=\begin{bmatrix}\boldsymbol{G}_1\boldsymbol{G}_1 & \boldsymbol{G}_1\boldsymbol{G}_2\\ \boldsymbol{G}_2\boldsymbol{G}_1 & \boldsymbol{G}_2\boldsymbol{G}_2\end{bmatrix}, \tag{17}$$

where

$$\begin{cases}\boldsymbol{G}_1\boldsymbol{G}_1=\cos^2\theta_0\left[\dfrac{1+u_x'}{\cos(\theta+\theta_0)}-\xi(\theta_0'+\theta')\right]^2,\\ \boldsymbol{G}_2\boldsymbol{G}_2=1,\quad \boldsymbol{G}_1\boldsymbol{G}_2=0.\end{cases} \tag{18}$$

In the curvilinear coordinate, the components of the stains can be defined as (Washizu, 1982)

$$f_{\alpha\beta}=\frac{1}{2}(G_{\alpha\beta}-g_{\alpha\beta}). \tag{19}$$

Neglecting the third order and the higher order terms, we have the following strain expressions,

$$\begin{aligned}f_{11}=u_x'-\theta'\xi+\frac{1}{2}(u_x'^2+\theta^2+\theta'^2\xi^2)\\ -u_x'(\theta'+\theta_0')\xi+\theta_0\theta+\theta_0'\theta'\xi^2,\end{aligned} \tag{20}$$

$$f_{12}=f_{21}=0,\ \ f_{22}=0. \tag{21}$$

The determinant of $[g_{\alpha\beta}]$ is

$$g=\left|g_{\alpha\beta}\right|=\left(1-\xi\theta_0'\cos\theta_0\right)^2\approx 1-2\theta_0'\xi. \tag{22}$$

The contravariant metric tensor is

$$\begin{aligned}\left[g^{\alpha\beta}\right]=\left[g_{\alpha\beta}\right]^{-1}=\begin{bmatrix}(1-\xi\theta_0'\cos\theta_0)^{-2} & 0\\ 0 & 1\end{bmatrix}\\ \approx\begin{bmatrix}1+2\theta_0'\xi & 0\\ 0 & 1\end{bmatrix}.\end{aligned} \tag{23}$$

From Eqs.(11) and (23), the contravariant base vectors can be obtained as

$$\begin{cases}\boldsymbol{g}^1=g^{1\mu}\boldsymbol{g}_\mu=g^{11}\boldsymbol{g}_1+g^{12}\boldsymbol{g}_2\\ \quad =(1-\xi\theta_0'\cos\theta_0)^{-1}(\cos\theta_0\boldsymbol{i}_1+\sin\theta_0\boldsymbol{i}_2)\\ \quad \approx(1+\theta_0'\xi)\boldsymbol{i}_1+\theta_0\boldsymbol{i}_2,\\ \boldsymbol{g}^2=g^{2\mu}\boldsymbol{g}_\mu=g^{21}\boldsymbol{g}_1+g^{22}\boldsymbol{g}_2=\boldsymbol{g}_2=\boldsymbol{n}_0\\ \quad =-\sin\theta_0\boldsymbol{i}_1+\cos\theta_0\boldsymbol{i}_2\approx-\theta_0\boldsymbol{i}_1+\boldsymbol{i}_2.\end{cases} \tag{24}$$

## STRESSES, GEOMETRICAL PROPERTIES AND STRESS-STRAIN RELATIONS

### Stresses in the cross-section and the resultant

In $C_2^*$, the cross-sectional area is

$$\mathrm{d}\boldsymbol{A}=\mathrm{d}\xi\boldsymbol{g}_2\times\boldsymbol{g}_3=\mathrm{d}A_1\boldsymbol{g}^1+\mathrm{d}A_2\boldsymbol{g}^2+\mathrm{d}A_3\boldsymbol{g}^3. \tag{25}$$

In Eq.(25), it is assumed that the width of the cross-section is unit length; $\boldsymbol{g}_3$ and $\boldsymbol{g}^3$ are covariant and contravariant base vectors corresponding to the direction of the width, and they are unit vectors

$$\mathrm{d}A_1=\mathrm{d}\xi\sqrt{g},\quad \mathrm{d}A_2=0,\quad \mathrm{d}A_3=0. \tag{26}$$

In Eq.(26), $g$ is defined by Eq.(22).

The relation between the resultants, stresses and the area is as follows:

$$\mathrm{d}\boldsymbol{P}=\tau^{11}\mathrm{d}A_1\boldsymbol{g}_1+\tau^{12}\mathrm{d}A_1\boldsymbol{g}_2. \tag{27}$$

Let

$$\mathrm{d}\boldsymbol{P}=\mathrm{d}F_x\boldsymbol{i}_1+\mathrm{d}F_y\boldsymbol{i}_2. \tag{28}$$

Multiplying Eq.(27) by $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ respectively, and considering Eq.(28), we obtain

$$\begin{cases}\tau^{11}\mathrm{d}A_1=(1+\theta_0'\xi)\mathrm{d}F_x+\theta_0\mathrm{d}F_y,\\ \tau^{12}\mathrm{d}A_1=-\theta_0\mathrm{d}F_x+\mathrm{d}F_y,\end{cases} \tag{29}$$

where the directions of $\mathrm{d}F_x$ and $\mathrm{d}F_y$ are defined by Eq.(28).

The resultants of the cross-section can be defined by

$$F_x=\int\mathrm{d}F_x,\quad F_y=\int\mathrm{d}F_y,\quad M=-\int\xi\mathrm{d}F_x. \tag{30}$$

### Geometrical properties

The parameters of cross-sections are defined as follows:

$$A=\int\mathrm{d}\xi,\quad I=\int\xi^2\mathrm{d}\xi. \tag{31}$$

In Eq.(31), it is assumed that the cross-sectional width is the unit length. Eq.(31) defines the cross-sectional area $A$ and the second moment $I$ of cross-sectional area.

### Stress-strain relation

For homogenous, isotropic, linear and elastic material, the incremental stress-strain relation is as follows (Washizu, 1982):

$$\begin{aligned}\Delta\tau^{ij}=\frac{E\mu}{(1+\mu)(1-2\mu)}g^{ij}g^{mn}f_{mn}\\ +G(g^{im}g^{jn}+g^{in}g^{jm})f_{mn}.\end{aligned} \tag{32}$$

Based on the convenient beam-column theory (Washizu, 1964), here yields

$$\Delta\tau^{22}=0,\quad \Delta\tau^{33}=0,\quad \Delta\tau^{23}=0. \tag{33}$$

From Eq.(32), six equations can be obtained. Considered Eq.(33), the system of equations is solved. Then Eq.(23) is substituted into the solution, and the following can be obtained

$$\Delta\tau^{11}=E(g^{11})^2 f_{11}\approx E(1+4\xi\theta_0')f_{11}. \tag{34}$$

## PRINCIPLE OF VIRTUAL WORK IN $C_2^*$

In curvilinear coordinate, Washizu (1982) presented the expression of principle of virtual work:

$$\iiint_V {}^2\tau^{11}\delta f_{11}\mathrm{d}V-\iint_s \boldsymbol{p}\cdot\delta\boldsymbol{r}\mathrm{d}S=0, \tag{35}$$

where ${}^2\tau^{11}$ is stress in $C_2$, and measured in $C_2^*$; $f_{11}$ is strain in $C_2$, and measured in $C_2^*$; $\boldsymbol{p}$ is the traction on the end cross-section; $\delta\boldsymbol{r}$ is virtual displacement of a point on the end cross-section; $V$ is a volume of the beam in $C_2^*$; $S$ is the union of both end cross-sections at Nodes A and B in $C_2$.

In Eq.(35), the gravity is ignored, and ${}^2\tau^{11}$ can be broken into

$${}^2\tau^{11}=\Delta\tau^{11}+{}^1\tau^{11}, \tag{36}$$

where $\Delta\tau^{11}$ is the incremental stress from $C_2^*$ to $C_2$, and ${}^1\tau^{11}$ is the real stress in $C_2^*$.

Substituting Eq.(36) into Eq.(35), we get the following equation

$$\iiint_V {}^2\tau^{11}\delta f_{11}\mathrm{d}V=\iiint_V {}^1\tau^{11}\delta f_{11}\mathrm{d}V+\iiint_V \Delta\tau^{11}\delta f_{11}\mathrm{d}V. \tag{37}$$

Substituting Eqs.(20), (21) and (29) into the first term of the right-hand side of Eq.(37), neglecting the second and higher order terms about the deformation, we obtain

$$\begin{aligned}
&\iiint_V {}^1\tau^{11}\delta f_{11}\mathrm{d}V=\iiint_V {}^1\tau^{11}\delta f_{11}\mathrm{d}A_1\mathrm{d}x=\\
&=\underbrace{\int (F_x+M\theta_0')\delta(u_x')\mathrm{d}x+\int\left(M+\frac{F_xI}{A}\theta_0'\right)\delta(\theta')\mathrm{d}x+\int F_x\theta_0\delta(\theta)\mathrm{d}x}_{\text{equivalent element-end forces in } C_2^*}\\
&\quad\underbrace{+\int M\delta(u_x'\theta')\mathrm{d}x+\frac{1}{2}\int F_x\left[\delta(u_x'^2)+\delta(\theta^2)+\frac{I}{A}\delta(\theta'^2)\right]\mathrm{d}x}_{\text{to form the geometric stiffness matrix in } C_2^*}
\end{aligned} \tag{38}$$

Substituting Eqs.(20), (21) and (34) into the second term of the right-hand side of Eq.(37), neglecting the second and higher order terms related to the deformation, we have

$$\begin{aligned}
&\iiint_V \Delta\tau^{11}\delta f_{11}\mathrm{d}V\\
&=\iiint_V \Delta\tau^{11}\delta f_{11}\sqrt{g}\mathrm{d}\xi\mathrm{d}x\\
&=\underbrace{\frac{1}{2}EA\int\delta(u_x'^2)\mathrm{d}x+\frac{1}{2}EI\int\delta(\theta'^2)\mathrm{d}x}_{\text{to form the linear stiffness matrix in } C_2^*}\\
&\quad+\underbrace{EA\int\theta_0\delta(u_x'\theta)\mathrm{d}x-EI\int\theta_0'\delta(u_x'\theta')\mathrm{d}x.}_{\text{to form the initial displacement stiffness matrix in } C_2^*}
\end{aligned} \tag{39}$$

Substituting Eqs.(38) and (39) into Eq.(37), and then substituting Eq.(37) into Eq.(35), we get the equation of virtual work expressed by displacement.

In Eq.(38), it is assumed that $F_x$ is uniform within the element, and that $M$ is linearly distributed, namely

$$F_x=F_{x\mathrm{B}},\ M=-M_{\mathrm{A}}n_1+M_{\mathrm{B}}n_2, \tag{40}$$

where

$$n_1=1-t,\ n_2=t, \tag{41}$$

where $t=x/l$, $F_{x\mathrm{B}}$ is the axial force at Node B, $M_{\mathrm{A}}$ and $M_{\mathrm{B}}$ are the moments at Node A and Node B, respectively.

Note that in Eqs.(38) and (39) incremental displacements are only dependent upon the axial displacements and rotations, so it is just necessary to consider the interpolation of axial displacements and rotations. The interpolations of $u_x$ and $\theta$ are as follows:

$$u_x = \begin{bmatrix} n_1 & 0 & n_2 & 0 \end{bmatrix}\boldsymbol{u},\ \theta = \begin{bmatrix} 0 & n_3 & 0 & n_4 \end{bmatrix}\boldsymbol{u}, \quad (42)$$

where

$$\boldsymbol{u} = \begin{bmatrix} u_{xA} & \theta_A & u_{xB} & \theta_B \end{bmatrix}^T, \quad (43)$$

$$n_3 = 1 - 4t + 3t^2, \quad n_4 = 3t^2 - 2t. \quad (44)$$

In Eq.(42), $u_{xA}$ and $u_{xB}$ are the $x$-direction incremental displacements at Node A and Node B in $C_2^*$, $\theta_A$ and $\theta_B$ are the incremental rotations at Node A and Node B, respectively.

The interpolation for the deformation is

$$\theta_0 = \theta_{0A} n_3 + \theta_{0B} n_4, \quad (45)$$

where $\theta_{0A}$ and $\theta_{0B}$ are the deformed rotations at Node A and Node B (Fig.1), $n_3$ and $n_4$ are the first differentiation of Hermitian function with respect to $x$.

Substituting Eqs.(40) and (42)~(45) into Eqs.(38) and (39), by virtue of Eqs.(37) and (35), we have the equation of virtual work in $C_2^*$.

$$\delta(\boldsymbol{u}^T)(\boldsymbol{k}_e + \boldsymbol{k}_g + \boldsymbol{k}_d)\boldsymbol{u} = \delta(\boldsymbol{u}^T)\,{}^2\boldsymbol{Q} - \delta(\boldsymbol{u}^T)\,{}^1\boldsymbol{Q}, \quad (46)$$

where

$$\begin{cases} {}^2\boldsymbol{Q} = \begin{bmatrix} {}^2F_{xA} & {}^2M_A & {}^2F_{xB} & {}^2M_B \end{bmatrix}^T, \\ {}^1\boldsymbol{Q} = \begin{bmatrix} {}^1F_{xA} & {}^1M_A & {}^1F_{xB} & {}^1M_B \end{bmatrix}^T. \end{cases} \quad (47)$$

## STIFFNESS MATRIX IN $C_1$

Eq.(46) is the equation of virtual work in $C_2^*$ in finite element formulation, from which the tangent stiffness matrix can be derived. Since the configuration of $C_2^*$ is unknown, we cannot adopt the stiffness matrix in this configuration. It is essential to obtain relations of displacements and rotation between $C_1$ and $C_2^*$.

### Relations of displacements and rotations at both ends between $C_1$ and $C_2^*$

In Eq.(46), the expression is only dependent upon the element-end displacements. Since the movement from $C_1$ to $C_2^*$ is a rigid-body motion, the displacements in the coordinates $xoy$ and $XOY$ satisfy the following relation:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \end{bmatrix}, \quad (48)$$

where

$$\omega = \arcsin\frac{u_{YB} - u_{YA}}{l + u_{xB} - u_{xA}} \approx \frac{u_{YB} - u_{YA}}{l}\left(1 - \frac{u_{XB} - u_{XA}}{l}\right). \quad (49)$$

In Eq.(49), $u_{XA}$, $u_{XB}$ and $u_{YA}$, $u_{YB}$ are the $X$-direction and the $Y$-direction incremental displacements at Node A and Node B from $C_1$ to $C_2$, respectively.

In Eq.(46), $\boldsymbol{u}$ is only dependent upon the element-end displacements of $u_x$, and independent of those of $u_y$, so it is just necessary to consider the relation of $u_x$ with $u_X$ and $u_Y$. Taking account of the first equation of Eq.(48), we have

$$u_{x\alpha} = u_{X\alpha}\cos\omega + u_{Y\alpha}\sin\omega \approx u_{X\alpha}\left(\frac{u_{YB} - u_{YA}}{l}\right)^2 + u_{Y\alpha}\frac{u_{YB} - u_{YA}}{l}\left(1 - \frac{u_{XB} - u_{XA}}{l}\right) \ (\alpha = A, B). \quad (50)$$

The incremental rotations satisfy the following relation at both ends

$$\theta_\alpha = \vartheta_\alpha - \omega \approx \vartheta_\alpha - \frac{u_{YB} - u_{YA}}{l} + \frac{u_{YB} - u_{YA}}{l}\frac{u_{XB} - u_{XA}}{l} \quad (\alpha = A, B), \quad (51)$$

where $\vartheta_\alpha$ ($\alpha$=A, B) is the incremental rotation at ends A or B from $C_1$ to $C_2$.

### Linearization of the incremental displacements and rotations from $C_1$ to $C_2^*$

Since it is a rigid-body motion from $C_1$ to $C_2^*$, there are no stresses to be produced during the movement. It is just necessary to transfer the element-end displacements from the coordinate frame $xoy$ to the coordinate frame $XOY$. By this, the virtual work in $C_1$ is obtained. If the nonlinear relations in Eqs.(50) and (51) are directly substituted into Eq.(46), a system of nonlinear equations will appear. So it is necessary to linearize the relations in Eqs.(50) and (51). Considering the linear terms in Eqs.(50) and (51), we have

$$u_{xA} = u_{XA},\ u_{xB} = u_{XB},$$
$$\theta_A = \vartheta_A - \frac{u_{YB} - u_{YA}}{l}, \quad \theta_B = \vartheta_B - \frac{u_{YB} - u_{YA}}{l}. \quad (52)$$

Eq.(52) can be expressed in a matrix form as

$$\boldsymbol{u}=\boldsymbol{L}\boldsymbol{U}, \tag{53}$$

where $\boldsymbol{u}$ is referred to Eq.(43), and

$$\boldsymbol{L}=\frac{1}{l}\begin{bmatrix} l & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & l & 0 & -1 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & l \end{bmatrix}, \tag{54}$$

$$\boldsymbol{U}=\begin{bmatrix} u_{XA} & u_{YA} & \vartheta_{A} & u_{XB} & u_{YB} & \vartheta_{B} \end{bmatrix}^{T}. \tag{55}$$

The relation of element-end forces between *xoy* and *XOY* is

$$\boldsymbol{F}=\boldsymbol{L}^{T}\boldsymbol{Q}, \tag{56}$$

where $\boldsymbol{Q}$ is the element-end forces in $C_2^*$ and referred to Eq.(47), $\boldsymbol{F}$ is the element-end forces in $C_1$, and expressed as

$$\boldsymbol{F}=\begin{bmatrix} F_{XA} & F_{YA} & M_{A} & F_{XB} & F_{YB} & M_{B} \end{bmatrix}^{T}. \tag{57}$$

Substituting Eq.(53) and its transpose into Eq.(46), we get the incremental system of equations in $C_1$

$$\boldsymbol{L}^{T}(\boldsymbol{k}_{e}+\boldsymbol{k}_{g}+\boldsymbol{k}_{d})\boldsymbol{L}\boldsymbol{U}=\boldsymbol{L}^{T}\,{}^{2}\boldsymbol{Q}-\boldsymbol{L}^{T}\,{}^{1}\boldsymbol{Q}. \tag{58}$$

Rewrite it as

$$(\boldsymbol{k}_{E}+\boldsymbol{k}_{G}+\boldsymbol{k}_{D})\boldsymbol{U}={}^{2}\boldsymbol{F}-{}^{1}\boldsymbol{F}, \tag{59}$$

where

$$\boldsymbol{k}_{E}=\boldsymbol{L}^{T}\boldsymbol{k}_{e}\boldsymbol{L},\ \boldsymbol{k}_{G}=\boldsymbol{L}^{T}\boldsymbol{k}_{g}\boldsymbol{L},\ \boldsymbol{k}_{D}=\boldsymbol{L}^{T}\boldsymbol{k}_{d}\boldsymbol{L}, \tag{60}$$

$${}^{2}\boldsymbol{F}=\boldsymbol{L}^{T}\,{}^{2}\boldsymbol{Q},\ {}^{1}\boldsymbol{F}=\boldsymbol{L}^{T}\,{}^{1}\boldsymbol{Q}. \tag{61}$$

Eq.(60) has presented the tangent stiffness matrix in $C_1$, and it is obvious that the stiffness matrix in Eq.(59) is rigid-body-qualified (Yang *et al.*, 2003). In Eq.(60), $\boldsymbol{k}_E$ is the elastic stiffness matrix and is identical with the literature (Przemieniecki, 1968), $\boldsymbol{k}_G$ is the geometric stiffness matrix, and $\boldsymbol{k}_D$ is the initial displacement stiffness matrix. $\boldsymbol{k}_G$ and $\boldsymbol{k}_D$ are as follows:

$$\boldsymbol{k}_{G}=\begin{bmatrix}
\frac{F_{XB}}{l} & -\frac{M_A+M_B}{l^2} & -\frac{M_A}{l} & -\frac{F_{XB}}{l} & \frac{M_A+M_B}{l^2} & -\frac{M_B}{l} \\
-\frac{M_A+M_B}{l^2} & \frac{F_{XB}}{5l}+\frac{12F_{XB}I}{Al^3} & \frac{F_{XB}}{10}+\frac{6F_{XB}I}{Al^2} & \frac{M_A+M_B}{l^2} & -\frac{F_{XB}}{5l}-\frac{12F_{XB}I}{Al^3} & \frac{F_{XB}}{10}+\frac{6F_{XB}I}{Al^2} \\
-\frac{M_A}{l} & \frac{F_{XB}}{10}+\frac{6F_{XB}I}{Al^2} & \frac{2F_{XB}l}{15}+\frac{4F_{XB}I}{Al} & \frac{M_A}{l} & -\frac{F_{XB}}{10}-\frac{6F_{XB}I}{Al^2} & -\frac{F_{XB}l}{30}+\frac{2F_{XB}I}{Al} \\
-\frac{F_{XB}}{l} & \frac{M_A+M_B}{l^2} & \frac{M_A}{l} & \frac{F_{XB}}{l} & -\frac{M_A+M_B}{l^2} & \frac{M_B}{l} \\
\frac{M_A+M_B}{l^2} & -\frac{F_{XB}}{5l}-\frac{12F_{XB}I}{Al^3} & -\frac{F_{XB}}{10}-\frac{6F_{XB}I}{Al^2} & -\frac{M_A+M_B}{l^2} & \frac{F_{XB}}{5l}+\frac{12F_{XB}I}{Al^3} & -\frac{F_{XB}}{10}-\frac{6F_{XB}I}{Al^2} \\
-\frac{M_B}{l} & \frac{F_{XB}}{10}+\frac{6F_{XB}I}{Al^2} & -\frac{F_{XB}l}{30}+\frac{2F_{XB}I}{Al} & \frac{M_B}{l} & -\frac{F_{XB}}{10}-\frac{6F_{XB}I}{Al^2} & \frac{2F_{XB}l}{15}+\frac{4F_{XB}I}{Al}
\end{bmatrix}, \tag{62}$$

$$\boldsymbol{k}_{D}=\frac{2EI}{l^3}\begin{bmatrix}
0 & 3a_{11} & la_{21} & 0 & -3a_{11} & la_{12} \\
3a_{11} & 0 & 0 & -3a_{11} & 0 & 0 \\
la_{21} & 0 & 0 & -la_{21} & 0 & 0 \\
0 & -3a_{11} & -la_{21} & 0 & 3a_{11} & -la_{12} \\
-3a_{11} & 0 & 0 & 3a_{11} & 0 & 0 \\
la_{12} & 0 & 0 & -la_{12} & 0 & 0
\end{bmatrix}-\frac{EA}{30l}\begin{bmatrix}
0 & 3a_{11} & la_{41} & 0 & -3a_{11} & la_{14} \\
3a_{11} & 0 & 0 & -3a_{11} & 0 & 0 \\
la_{41} & 0 & 0 & -la_{41} & 0 & 0 \\
0 & -3a_{11} & -la_{41} & 0 & 3a_{11} & -la_{14} \\
-3a_{11} & 0 & 0 & 3a_{11} & 0 & 0 \\
la_{14} & 0 & 0 & -la_{14} & 0 & 0
\end{bmatrix}, \tag{63}$$

where $a_{11}=\theta_{0A}+\theta_{0B}$, $a_{21}=2\theta_{0A}+\theta_{0B}$, $a_{12}=\theta_{0A}+2\theta_{0B}$, $a_{41}=4\theta_{0A}-\theta_{0B}$, $a_{14}=4\theta_{0B}-\theta_{0A}$.

Comparing the tangent stiffness matrix in this paper with the literature (Yang *et al.*, 2007), we can find that in the present UL formulation $\boldsymbol{k}_D$ emerges in the tangent stiffness matrix and is dependent on the existed deformation.

## NUMERICAL EXAMPLES

The most satisfactory method of solving the nonlinear problem is to combine incremental methods with an iteration technique. The iteration strategy involved in the analysis is the Newton-Raphson method. All the displacements are controlled by the arc-length method suggested by Crisfield (1991). Presented below are the calculations with their results performed on some classical examples.

### Williams' toggle

Williams (1964) solved analytically and experimentally the toggle shown in Fig.2. Yang *et al.*(2007) analyzed the toggle by employing the finite element approach. The present load-deflection curves are shown in Fig.3. With one element per member, the present results are in very close agreement with the analytical solution of Williams and the solutions of Yang *et al.*(2007). Fig.3 has also given the load-deflection curve without consideration of the deformed curvature. The example shows that the present beam element is efficient, and that the deformed curvature has a significant effect on the results.

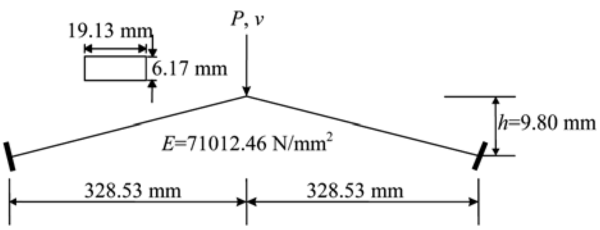


**Fig.2 Geometry and material of Williams' toggle**

### Shallow circular arch under concentrated load

Gjelsvik and Bodner (1962) solved analytically and experimentally the shallow circular arch shown in Fig.4, and Stolarski and Belytschko (1982) analyzed the arch by employing nonlinear finite element method. Bathe and Bolourchi (1979) solved the arch by the UL formulation with twelve elements for half of the arch. In present analysis, the arch is divided into four elements for half of the arch. Fig.5 shows the load-deflection curves with and without consideration of the deformed curvature.

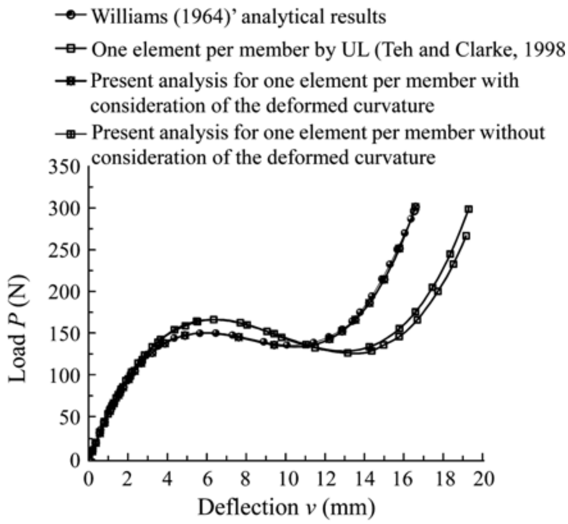


**Fig.3 Load-deflection curves of Williams' toggle with *h*=9.80 mm**

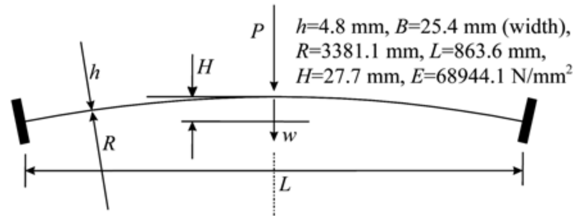


**Fig.4 Geometry of shallow circular arch under concentrated load**

## CONCLUSION

In this paper we have developed a tensor-based finite element model for the geometrically nonlinear analysis of 2D beam-column structures, which has considered the deformed curvature. Numerical examples presented herein clearly show the validity and accuracy of the present beam element.

The salient features of the present work are summarized as follows:

(1) The formulation has considered the effect of deformed curvature within the element that is ignored

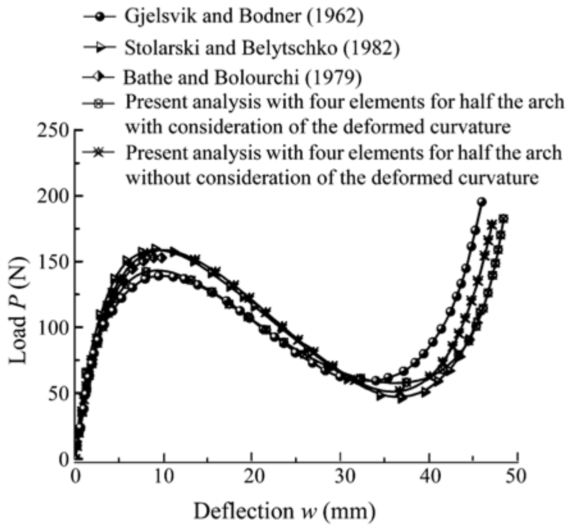


**Fig.5 Load-deflection curves of shallow circular arch under concentrated load**

in the conventional UL formulation. In the present beam element, the initial displacement stiffness matrix has been explicitly presented. If the deformation is large, the stiffness matrix has significant effects on the results. The present formulation has better computational efficiency, because for obtaining the results with similar accuracy the structural member can be subdivided into fewer elements than that without considering the effect of deformed curvature within the element.

(2) The formulation was referred to as a direct approach, since the element stiffness matrices were derived directly from the stress-strain relations of the incremental theory, which has the advantage of being physically interpretable compared with other established methods. The geometric stiffness matrix and the initial displacement stiffness matrix were demonstrated to be qualified for simulating the behaviors of initial element-end forces and moments undergoing rigid rotations.

(3) The formulation has adopted the rotation interpolation which is consistent with the HPs interpolations for transverse displacements, and the present beam element has a consistent linear stiffness matrix with the classical beam theory.

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