Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Computing the topology of an arrangement of implicitly defined real algebraic plane curves[§*]

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Received July 16, 2008; revision accepted Aug. 20, 2008; CrossCheck deposited Oct. 27, 2008

Abstract: We introduce a new algebraic approach dealing with the problem of computing the topology of an arrangement of a finite set of real algebraic plane curves presented implicitly. The main achievement of the presented method is a complete avoidance of irrational numbers that appear when using the sweeping method in the classical way for solving the problem at hand. Therefore, it is worth mentioning that the efficiency of the proposed method is only assured for low-degree curves.

Key words: Topology computation, Real plane curves, Sweeping method
doi:10.1631/jzus.A08GMP01 **Document code:** A **CLC number:** TP391.72

INTRODUCTION

In the study of the topology of arrangements of plane curves presented implicitly, efficient algorithms are known, while to a high degree they require symbolic methods. The case of a single planar curve has been studied in (Sakkalis and Farouki, 1990; Sakkalis, 1991; Hong, 1996; Gonzalez-Vega and El Kahoui, 1996; Gonzalez-Vega and Necula, 2002; Seidel and Wolpert, 2005; Eigenwillig *et al.*, 2007; Liang *et al.*, 2007; Cheng *et al.*, 2008). Efficient algorithms for arrangements of straight segments can be found in (Mehlhorn and Noher, 1999; Flato *et al.*, 2000; Seel, 2001) and for conics in (Berberich *et al.*, 2002; Wein, 2002).

This paper is inspired by the ideas in (Gonzalez-Vega and Necula, 2002; Eigenwillig *et al.*, 2006), and will introduce several tools allowing one to perform an exact topological analysis for arrangements of curves of a general degree presented implicitly. The method in (Eigenwillig *et al.*, 2006) for n cubics $f_1, f_2, ..., f_n$ finds the topology of each curve f_i, then computes the topology of each pair f_i, f_j and, finally, puts all the information together providing the topology of the considered arrangement. Since the last step before mentioned does not depend on the degree of the considered curves, for completeness it will just be sketched at the end of Section 3.

Here we give a general method which is valid for every degree. Unfortunately, it needs an efficient implementation of Descartes' rule of signs to be efficient for a degree higher than 5.

To analyze a single curve, we consider the real roots $x_1<x_2<...<x_n$ (that we do not determine) of the discriminant of f_i with respect to y. It is well known that $f_i=0$ in the region $(x_j, x_{j+1})\times\mathbb{R}$ consists of (topologically) a finite number of disjoint segments. The number of segments is exactly the number of real roots of $f_i(r_{j+1}, y)$ (at most 4 due to degree limitations) for any $r_{j+1}\in(x_j, x_{j+1})$. Now we have to find out what happens over the x_j. Since x_j is a root of the discriminant of f, we know (or in other cases we will be warned beforehand) that $f_i(x_i, y)$ has one multiple real root (representing what we call an event point of the curve) and up to two single real roots (representing what we will call uninvolved arcs). We need to sort

[§] Presented as a Keynote at Geometric Modeling and Processing 2008 (GMP 2008), April 23-25, 2008, Hangzhou, China
[*] Project (No. MTM2005-08690-C02-02) partially supported by the Spanish Ministry of Science and Innovation Grant

these real roots, i.e., to find if the event is above, below or between the uninvolved arcs. With this information we can construct a graph with the same topology of our curve.

For two curves f and g, we consider as events both one-curve events and intersections, so we sort their x-coordinates $x_1<x_2<...<x_n$. The idea is the same. First we find what is happening over (x_j, x_{j+1}) with the help of a rational r_j in this interval. Now we have to sort the roots of $f(r_j, y)$ and $g(r_j, y)$ in order to find the relative positions of the arcs over the interval. For the events, we want over each x_j exactly one number y_j satisfying one (or more) of the following:

(1) y_j is a multiple root of $f(x_j, y)$;

(2) y_j is a multiple root of $g(x_j, y)$;

(3) y_j is a common root of $f(x_j, y)$ and $g(x_j, y)$.

If this is satisfied, we sort the other roots of $f(x_j, y)$ and $g(x_j, y)$ and y_j, which represent the event now. When this is done, we have to join the arcs to complete the graph.

For the analysis of a whole arrangement, there is no additional modification to include, so we refer to (Eigenwillig *et al.*, 2006) with the obvious adaptations to our case. The main advance with respect to other methods is that we do not need to compute any of the x_i, so we will work just with rational points.

The paper is laid out as follows. After some preliminaries, we sketch the algorithm without paying attention to details. These details will be analyzed in Section 4 (coverticality tests) and Section 5 (event analysis). We finish with an example illustrating the proposed method.

PRELIMINARIES

Here we will introduce the main tools and notations we will use in this paper.

Definition 1 Let

$$P=\sum_{i=0}^{m} a_{m-i}x^i$$

and

$$Q=\sum_{i=0}^{n} b_{n-i}x^i$$

be two polynomials in y with coefficients in a field K. We define the jth subresultant polynomial of P and Q with respect to the variable x in the following way (Li, 2006):

$$Sres_j(P,Q;x)=(-1)^{j(m-j+1)}\cdot\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_m & & & \\ & \ddots & \ddots & \ddots & & & \ddots & & \\ & & a_0 & a_1 & a_2 & \cdots & \cdots & a_m & \\ & & & & & 1 & -x & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 1 & -x \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & b_n & & \\ & \ddots & \ddots & \ddots & & & & \ddots & \\ & & b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & b_n \end{vmatrix}\begin{matrix} \left.\right\} n-j \\ \left.\right\} j \\ \left.\right\} m-j \end{matrix}$$

and we define the kth subresultant coefficient of P and Q with respect to x, $sres_k(P, Q; x)$, as the coefficient of x^k in $Sres_k(P, Q; x)$. Finally, the resultant of P and Q with respect to x is

$$res(P, Q; x)=Sres_0(P, Q; x)=sres_0(P, Q; x).$$

There are many ways of computing efficiently subresultants different from their definition: see, for example, Basu *et al.*(2006) and the references contained therein. Next we introduce the main property of subresultants that will be used in what follows. A proof can be found, for example, in (Basu *et al.*, 2006).

Theorem 1 Let f and g be two polynomials in y with coefficients in a field K. Then the following assertions are equivalent:

(1) $sres_i(f, g; y)=0$ for all $i\leq k$ and $sres_{k+1}(f, g; y)\neq 0$;

(2) The greatest common divisor of f and g has degree $k+1$ and it is $Sres_k(f, g; y)$ (up to multiplication by nonzero elements of K).

We now define some interesting points of a real algebraic plane curve implicitly defined which will guide the computation of the topological information we want to determine.

Definition 2 Let f and g be polynomials in two variables x and y with real coefficients. We write f_x and f_y for the partial derivatives of f and

$$V(f)=\{(\alpha, \beta)\in\mathbb{R}^2: f(\alpha, \beta)=0\},$$
$$V_{\mathbb{C}}(f)=\{(\alpha, \beta)\in\mathbb{C}^2: f(\alpha, \beta)=0\}.$$

We use analogous notation for g. Then,

(1) An event or event point is a point $P=(\alpha, \beta)$ which is either an intersection point of $V(f)$ and $V(g)$ or satisfy that β is a multiple root of $f(\alpha, y)$ or $g(\alpha, y)$;

(2) Given an event point $P=(\alpha, \beta)$, we say that an arc of $V(f)$ or $V(g)$ intersecting the line $V(x-\alpha)$ is uninvolved if the intersection point between the arc and the line is different from P.

Throughout this paper and when possible, we will use notation identifying event points with their x-coordinates or, when fixed, with their y-coordinates (in those cases, there will be only one event point with such x- or y-coordinate). The algorithm we describe in this paper is of the so-called 'sweeping' type. This kind of algorithm is based on the following well-known fact:

Theorem 2 Consider a square-free polynomial $f\in\mathbb{Q}[x, y]$ and a point $P=(x_0, y_0)\in V(f)\subset\mathbb{R}^2$. The implicit function theorem provides that, if $f_y(P)\neq 0$, then the intersection of $V(f)$ but with a sufficiently small (analytic) neighborhood of P has the shape of the graph of a function $y=F(x)$, where F is analytic. This means that at all points of $V(f)$ a finite number (i.e., $V(f)\cap V(f_y)$), $V(f)$ is locally homeomorphic to a horizontal segment. This means that, if the projection of the finite set $V(f)\cap V(f_y)$ with respect to the first coordinate consists of the points $x_1<x_2<\ldots<x_n$, the intersection of $V(f)$ with $(x_i, x_{i+1})\times\mathbb{R}$ is homeomorphic to a finite set of horizontal segments.

In the case of several curves the previous theorem can be generalized easily in the following way:

Theorem 3 In the case of several curves $f_1, f_2, ..., f_r$, we consider the projection of the finite set

$$\left[\bigcup_{1\le i<j\le r}(V(f_i)\cap V(f_j))\right]\cup\left[\bigcup_{i=1}^{n}(V(f_i)\cap V(f_{i,y}))\right]$$

with respect to the first coordinate. If such projection consists of the numbers $x_1<x_2<\ldots<x_n$, then the intersections of $V(f_1)$, $V(f_2)$, ..., $V(f_r)$ with $(x_i, x_{i+1})\times\mathbb{R}$ are homeomorphic to a finite set of horizontal segments.

ALGORITHM OVERVIEW

Throughout this section we briefly describe the algorithm without specifying concrete methods for the events analysis (which will be discussed in Section 5). We begin with the general explanation of the analysis of the topology of a single curve. Then we proceed to describe the analysis of a two-curve arrangement. Finally, we explain what to do in the general case.

One curve

We ask any curve f in our arrangement to satisfy the following properties:

(1) y-regularity (i.e., the coefficient of y^4 is not zero),

(2) square-freeness, and

(3) no two points of $V_{\mathbb{C}}(f)\cap V_{\mathbb{C}}(f_y)$ being covertical (i.e., sharing the first coordinate).

The first condition is trivial to check and can be avoided just by performing a change of coordinates. The second condition is easy to check and can be avoided just by computing and (if different from 1) dividing by the greatest common divisor of f and f_y. The third condition is checked by using subresultants as explained in Section 4.

The analysis of the topology of f begins with the computation of its discriminant with respect to y, $discrim(f; y)=sres_0(f, f_y; y)$, and the next subresultants $sres_i(f, f_y; y)$ until

$$\gcd(discrim(f; y), sres_1(f, f_y; y), sres_2(f, f_y; y), ..., sres_k(f, f_y; y))=1.$$

We denote the square-free part of the discriminant by R_f, and sort and isolate its real roots using Descartes' rule of signs. Therefore, without knowing these roots $x_1, x_2, ..., x_n$, we can choose rational numbers $r_1, r_2, ..., r_{n+1}$ such that

$$r_1<x_1<r_2<\ldots<x_n<r_{n+1}.$$

For all $i=1, 2, ..., n$ we call m_i to the least j such that $sres_j(f, f_y; y)(x_i)=0$ and for all $i=1, 2, \ldots, n+1$ we define k_i as the number of real roots of $f(r_i, y)$ (computed by Descartes' rule of signs, since $f(r_i, y)$ is square free).

Later, we proceed as explained in Section 5 to know how many uninvolved arcs lay over the event in the line $x=x_i$ and how many below. In this way we can obtain a scheme of the following shape (called Scheme (1)):

		a		
	e	a	a	
a	a	a	e	a
r_1	x_1	r_2	x_2	r_3

(1)

where a denotes a smooth arc with non-vertical tangent line and e denotes an event point. Any a must be joined exactly with one symbol to the right and one to the left. After assuring this, the remaining e's are joined with the a's immediately to left or right which are not linked at that side. In this way we can obtain the graph in Fig.1.

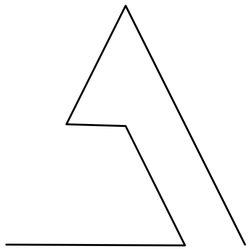

Fig.1 Topology graph corresponding to Scheme (1)

Two curves

In the case of two curves $f, g \in \mathbb{Q}[x, y]$, we need some more conditions:

(1) f and g are co-prime;

(2) No two points of

$$[V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(f_y)] \cup [V_{\mathbb{C}}(g) \cap V_{\mathbb{C}}(g_y)] \cup [V(f) \cap V(g)]$$

are covertical.

The first condition is checked after computing the resultant of f and g with respect to y (if this resultant vanishes, then there is a common component) and solved after removing the common component h from f and g and performing an analysis of the topology of the arrangement consisting of f/h, g/h and h (see the next subsection). The second condition can be checked with subresultants as explained in Section 4.

The analysis of the topology of the pair of curves begins with the analysis of each curve separately. Then we compute the subresultants $sres_i(f, g; y)$ until

$$\gcd(res(f, g; y), sres_1(f, g; y), sres_2(f, g; y), ..., sres_k(f, g; y))=1.$$

We denote by R_{fg} the square-free greatest divisor of the resultant of f and g with respect to y. We now sort and isolate the real roots of R_f, R_g and R_{fg} using Descartes' rule of signs. So without knowing these roots $x_1, x_2, ..., x_n$ we can choose rational numbers $r_1, r_2, ..., r_{n+1}$ such that (Eigenwillig *et al.*, 2006)

$$r_1<x_1<r_2<...<x_n<r_{n+1}.$$

We denote by m_i^f, m_i^g and m_i^{fg} the smallest numbers j, k, l such that

$$sres_j(f, f_y; y)(x_i) \neq 0,$$
$$sres_k(g, g_y; y)(x_i) \neq 0,$$
$$sres_l(f, g; y)(x_i) \neq 0,$$

for all i=1, 2, …, n. We use Descartes' rule to sort the real roots of $f(r_i, y)$ and $g(r_i, y)$ for all i=1, 2, ..., n+1.

We now proceed as explained in Section 5 to know how many uninvolved arcs lay over the event in the line $x=x_i$ and how many below. With all this information, we can obtain a scheme of the following shape (called Scheme (2)):

		f						g		
	f	f	e				g	f	g	
f	f	g	g	g		g	f	g	f	g
f	g	f	f	f	g	f	e	g	g	f
g	e	f	f	f	f	g	g	g	e	g
g	g	g	g	g	e	f	f	f	f	f
r_1	x_1	r_2	x_2	r_3	x_3	r_4	x_4	r_5	x_5	r_6

(2)

where f or g represents a regular arc of f or g, and we denote an event by e. Now we join the arcs in the same way as we explained for Scheme (1), and obtain the topology of the arrangement of curves f and g as shown in Fig.2.

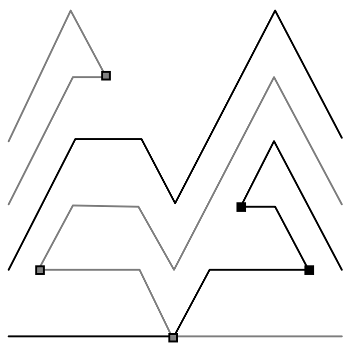

Fig.2 Topology of the arrangement *f*=0 (gray) and *g*=0 (black). Solid boxes represent the event points of the arrangement

General case

In the case of an arrangement of three or more curves $f_1, f_2, ..., f_r$, we do not need to impose further coverticality conditions. In fact, after we sort the real roots of all R_{f_i} and all R_{f_i,f_j} (as defined in previous sections) we have again rational numbers between the real roots:

$$r_1<x_1<r_2<...<x_n<r_{n+1}.$$

For each r_i, we know (from the two-curve analysis) that the relative position of the arcs through the vertical line $x=r_i$ is already known for each pair of curves. Then there is no need to perform further analysis to know the ordering of all real roots of $f_1(r_i, y), f_2(r_i, y), ..., f_r(r_i, y)$.

The same thing happens over each x_i: the two-curve analysis provides the knowledge of the number of arcs over and below each event and the amount of arcs involved on it. Therefore, even if there are two covertical events (but involving two different pairs of curves), we can put clearly all data together to build a diagram like Scheme (1) or Scheme (2) and then obtain the topology (see e.g. Eigenwillig *et al.* (2006) for further details).

CHECKING COVERTICALITY CONDITIONS

Throughout this section we introduce the way we propose to check coverticality conditions. The main advantage of this criterion is that it checks the coverticality globally for the whole curve, instead of once for each event. We avoid the notation of Section 3 for the convenience of the reader who prefers to refer to this section from the very moment it is referred to.

One curve

Let $f\in\mathbb{Q}[x, y]$ be the polynomial defining the curve to be studied and let α be a real root of $res(f, f_y; y)$. Then there exists at least one common root of $f(\alpha, y)$ and $f_y(\alpha, y)$. Coverticality over α is equivalent to $f(\alpha, y)$ and $f_y(\alpha, y)$ having two (or more) different common real roots.

Let i_α be the first natural number such that $sres_{i_\alpha}(f, f_y; y)\neq 0$, and define

$$S_{i_\alpha}(x, y) = Sres_{i_\alpha}(f, f_y; y),$$

then the coverticality over α is equivalent to $S_{i_\alpha}(\alpha, y)\neq k(y-\beta)^{i_\alpha-1}$ for certain $k, \beta\in\mathbb{R}$. This is characterized by the vanishing of the polynomials in $x=\alpha$:

$$P_{i_\alpha, j}(x) = \binom{i_\alpha}{j}\binom{i_\alpha}{j+2}\left(coeff(S_{i_\alpha}, y, j+1)\right)^2 - \binom{i_\alpha}{j+1}^2 coeff(S_{i_\alpha}, y, j)coeff(S_{i_\alpha}, y, j+2).$$

Therefore, to check the existence of coverticalities we have to check that, for all $i_\alpha=1, 2, ..., \deg(f)$ all square-free factors of

$$\gcd(discrim(f; y), sres_1(f, f_y), sres_2(f, f_y), ..., sres_{i_\alpha}(f, f_y))$$

divide

$$\gcd(P_{i_\alpha,0}, P_{i_\alpha,1}, ..., P_{i_\alpha,i_\alpha-2}).$$

Remark 1 All the subresultant coefficients that are used here must also be used in Section 3. In fact, the fast methods for computing subresultant coefficients obtain them through the subresultant polynomials, so no further subresultants computation is needed.

Remark 2 If $\deg(f)=3$, there is nothing to check (coverticality is impossible, see e.g. Eigenwillig *et al.* (2006)), and if $\deg(f)=4$, then the computations are quite reduced, as shown in (Caravantes and Gonzalez-Vega, 2007a).

Two curves

For the two curves $f, g\in\mathbb{Q}[x, y]$ analysis, we have to check three different coverticalities, depending on the type of events to compare:

(1) events of f with events of g,

(2) events of f with intersections (also with g, but analogous), and

(3) intersections with intersections.

If α is a common real root of the discriminants of f and g, then there is an event of f and an event of g over $x=\alpha$. We want them to be the same point. After checking coverticalities from the one-curve analysis, we know that the y-coordinates of these events are the only (maybe multiple) roots of $S^f_{i^f_\alpha}(\alpha, y)$ and

$S^g_{i^g_\alpha}(\alpha, y)$. We want them to share the only (perhaps multiple) root, and this is equivalent to the vanishing in $x=\alpha$ of the polynomial

$$Q_{i^f_\alpha, i^g_\alpha}(x) = i^f_\alpha coeff(S^f_{i^f_\alpha}, y, i^f_\alpha) coeff(S^g_{i^g_\alpha}, y, i^g_\alpha - 1) - i^g_\alpha coeff(S^f_{i^f_\alpha}, y, i^f_\alpha - 1) coeff(S^g_{i^g_\alpha}, y, i^g_\alpha).$$

So we avoid the first kind of coverticality by checking that the polynomial

$$\gcd(discrim(f; y), sres_1(f, f_y; y), ..., sres_{i^f_\alpha}(f, f_y; y), discrim(g; y), sres_1(g, g_y; y), ..., sres_{i^g_\alpha}(g, g_y; y))$$

divides $Q_{i^f_\alpha, i^g_\alpha}(x)$.

The third condition is checked as in the one-curve analysis but with g doing the job of f_y.

We now study the case of α being a common real root of $discrim(f; y)$ and $res(f, g; y)$. Let i^f_α be as above and let i^{fg}_α be the smallest i such that $sres_i(f, g; y)\neq 0$ and define

$$S^{fg}_{i^{fg}_\alpha}(x, y) = Sres_{i_\alpha}(f, g; y).$$

Due to the previous coverticality analysis, we know that both $S^f_{i^f_\alpha}(\alpha, y)$ and $S^{fg}_{i^{fg}_\alpha}(\alpha, y)$ (each one) have at most one (perhaps multiple) root. We want them to be the same, which is equivalent to the vanishing in $x=\alpha$ of the polynomial

$$Q'_{i^f_\alpha, i^{fg}_\alpha}(x) = i^f_\alpha coeff(S^f_{i^f_\alpha}, y, i^f_\alpha) coeff(S^{fg}_{i^{fg}_\alpha}, y, i^{fg}_\alpha - 1) - y^{fg}_\alpha coeff(S^f_{i^f_\alpha}, y, i^f_\alpha - 1) coeff(S^{fg}_{i^{fg}_\alpha}, y, i^{fg}_\alpha).$$

So we avoid the second kind of coverticality by checking that the polynomial

$$\gcd(discrim(f; y), sres_1(f, f_y; y), ..., sres_{i^f_\alpha}(f, f_y; y), res(f, g; y), sres_1(f, g; y), ..., sres_{i^{fg}_\alpha}(f, g; y))$$

divides

$$Q'_{i^f_\alpha, i^{fg}_\alpha}(x).$$

DEALING WITH EVENTS

Throughout this section, we describe a general method to check the situation through a line $x=x_i$ which contains an event, first for the one-curve analysis and later for the two-curve one. The main method is strongly inspired by the one introduced in (Gonzalez-Vega and Necula, 2002), but with the introduction of some ideas of (Eigenwillig *et al.*, 2006) to avoid irrational numbers and floating point numbers so we can certify the topology provided by the algorithm.

Locatable and non-locatable events

The x-coordinate of an event is always the root of a subresultant. Due to the fact that these roots are isolated by means of Descartes' rule of signs, it is advisable to perform a square-free decomposition in order to work with lower degree polynomials.

In fact, if one of these square-free factors is linear, its root x_i can be immediately computed since it is a rational number. In fact, after the coverticality tests, we know that there is exactly one event through the line $x=x_i$, so the multiple/common (depending on the event type) root y_i of the polynomials after the substitution $x=x_i$ is a rational number and can be computed by finding a square-free factor or a greatest common divisor and solving a linear equation.

We call these rational events that can be explicitly computed 'locatable'. After the determination of x_i and y_i one can use Descartes' algorithm to isolate the real roots of $f(x_i, y)$ and/or $g(x_i, y)$ and compare them with y_i. The next two subsections deal with the generic way to treat those events that are not locatable.

One-curve analysis

Let $x_i=\alpha$ be the real x-coordinate of a non-locatable event of the curve f. Let i_α be defined, as in Section 4, as the smallest i such that $sres_i(f, f_y; y)\neq 0$. Then, as stated before,

$$S(\alpha, y) = Sres_{i_\alpha}(f(\alpha, y), f_y(\alpha, y); y) = k(y-\beta)^{i_\alpha}.$$

This means that (since no coverticality has been allowed) β is a root of $f(\alpha, y)$ of multiplicity $i_\alpha+1$. Now we define

$$\eta(x) := -\frac{coeff(S, y, i_\alpha - 1)}{i_\alpha coeff(S, y, i_\alpha)}.$$

Note that $\eta(\alpha)=\beta$. Then one can apply Horner's rule (i_α+1 times and considering y as the variable) to compute the quotient $\tilde{f}$ of the division of f by $(y-\eta(x))^{i_\alpha+1}$ (which, for $x=\alpha$ is exact).

In this situation, $\tilde{f}(\alpha, y)$ vanishes in all single roots of $f(\alpha, y)$, and $\eta(\alpha)=\beta$, the multiple root. We choose $r\in\mathbb{Q}$ such that, between r and α, there does not exist a root of $sres_{i_\alpha}(f(\alpha, y), f_y(\alpha, y); y)$ (the denominator of η), the numerator of $\tilde{f}(x, \eta(x))$ or $discrim(\tilde{f}; y)$. Then the relative position of $\eta(r)$ with respect to the roots of $\tilde{f}(r, y)$ is the relative position of β with respect to the simple real roots of $f(\alpha, y)$.

Remark 3 Note that $\tilde{f}$ depends on i_α instead of α, so the subresultant computation does not need to be done once for each root.

Remark 4 In the cases of deg(f)=3 or 4, some specific methods have been developed in (Eigenwillig *et al.*, 2006; Caravantes and Gonzalez-Vega, 2007a; 2007b) to deal with certain types of events (in this paper, we are not distinguishing between them).

Two-curve analysis

Let $x_i=\alpha$ be a root of $discrim(f; y)$, $discrim(f; j)$ or $res(f, g; y)$. For the two-curve analysis, we have three cases to distinguish:

(1) Just one of the two discriminants of f and g vanishes in α, which is not a root of $res(f, g; y)$;

(2) All the three polynomials vanish in α;

(3) $res(f; y)(\alpha)=0$ and one of the discriminants does not vanish in $x=\alpha$.

The first case (we can say it is the discriminant of f who vanishes without loss of generality) can be dealt with easily if the event is not an isolated point (and this is checked during the one-curve analysis). In this case, the relative position of the uninvolved arcs with respect to the event is completely determined by the position of the arcs over r_i and r_{i+1}. This is because all changes among arcs between r_i and r_{i+1} occur over α, and therefore we can compare the position of the arcs (over r_i or r_{i+1}) connected to the event with the position of the arcs that are not connected to it. In the case of an isolated point, we recover the $\eta(x)$ we got from the one-curve analysis and compare $\eta(r)$ with the roots of $g(r, y)$ for a rational number r such that $g(x, \eta(x))$ does not vanish between r and α, so we get the relative position of the event with respect to the arcs of g. The relative position of the uninvolved arcs is determined by what we obtained over r_i and r_{i+1}.

The second case is probably the easiest one. Since the event is an event for both curves, we know (from each one-curve analysis) how many arcs of each curve lay above and below the event. The relative position of these arcs among themselves is given by the situation over $x=r_i$ and $x=r_{i+1}$.

For the third case, we know that $sres_1(f, g; y)(\alpha)$ $\neq 0$ (since at least one of $f(\alpha, y)$ and $g(\alpha, y)$ has only single roots). Let then $\eta(x)$ be

$$-\frac{coeff(S, y, 0)}{coeff(S, y, 1)},$$

where $S(x, y)=Sres_1(f, g; y)$. Then we divide (by Horner's rule and taking y as the variable) both f and g by $y=\eta(x)$ to get $\tilde{f}$ and $\tilde{g}$. We then choose a rational number r such that between α and r there is no root of $res(\tilde{f}, \tilde{g}; y)$, $res(f, g; y)$, and the numerators of $\tilde{f}(x, \eta(x))$ and $\tilde{g}(x, \eta(x))$. Now, the relative position of β (the y-coordinate of the event) and the uninvolved arcs is the same as the relative position of $\eta(r)$ with respect to the roots of $\tilde{f}(r, y)$ and $\tilde{g}(r, y)$.

AN EXAMPLE

For a somehow complete illustration, we consider three curves:

$$f := y^3-2xy^2-y^2+2xy-x^2,$$
$$g := y^4+6x^2y^2+x^4-16,$$
$$h := y^5-2y^3x^2+x^4y-7x^2y^2+xy-32.$$

After computing the suitable discriminants R_f, R_g and R_h and resultants R_{fg}, R_{fh} and R_{gh}, and after the coverticality tests, one obtains rational numbers that isolate the real roots of these polynomials:

$$-5<x_1<-7/2<x_2<-3<x_3<-7/3<x_4<-3/2<x_5<-1<x_6<$$
$$-1/2<x_7<1/64<x_8<1/2<x_9<1<x_{10}<3/2<x_{11}<3<x_{12}<4.$$

We also obtain the remaining data of Section 3 (note that the coverticality over x_7 is not forbidden):

m_i^f	0	0	0	0	0	0	1	0	0	0	0	0
m_i^g	0	0	1	0	0	0	0	0	0	0	1	0
m_i^h	0	1	0	0	0	0	0	0	0	0	0	1
m_i^{fg}	0	0	0	1	1	1	0	0	1	0	0	0
m_i^{fh}	1	0	0	0	0	0	0	0	0	1	0	0
m_i^{gh}	0	0	0	0	0	0	1	1	0	0	0	0
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}

We sketch the analysis of the events over x_7 (which is locatable and illustrates the management of allowed coverticalities) and over x_{11} (which is not).

From R_f (in fact, its square-free factorization), we can find that $x_7=0$ is its only triple root. Moreover, after a square-free factorization, we see that $f(0, y)=y^2(y-1)$. Regarding g and h, we see that $\gcd(g(0, y), h(0, y))=y-2$. Moreover, from Descartes' rule of signs we find that there is an extra negative root of $g(0, y)$ and that $h(0, y)$ has no more real roots. Therefore, the situation over $x_7=0$, from below to above, is: an arc of g, an event of f, an arc of f, and an intersection between g and h.

For $3/2<x_{11}<3$, we compute

$$S_1^g=Sres_1(g, g_y; y)=(1\,536x^6+3\,072x^2)y.$$

Therefore, $\eta(x)=0$. We now divide twice g by $y-\eta(x)=y$ and obtain the quotient

$$\tilde{g}(x,y)=y^2+6x^2.$$

Since it consists of two complex lines, there are no roots to compare with $\eta(x_{11})=0$. On the other side, we can compare $\eta(r_{11})$ with the roots of $f(r_{11}, y)$ and $h(r_{11}, y)$. By Descartes' rule, we find that $f(r_{11}, y)-\eta(r_{11})$ has a positive root and so does $h(r_{11}, y)-\eta(r_{11})$. Since we know from before (Descartes' rule applied to the rational isolating points) that the real root of $f(r_{11}, y)$ is greater than the root of $h(r_{11}, y)$, therefore, the situation over x_{11} is, from below to above: the event of g, an arc of h, and an arc of f.

Remark 5 The comparison of $\eta(x_{11})$ with the arcs of f and h can be done just observing the arcs over r_{11}, since the event is not an isolated point. We chose this method in order to illustrate the management of $\eta(x)$.

After analyzing all the events, we obtain Scheme (3) (see below).

Now, there is just a way to join these points, but for x_7. But from the one-curve analysis we know that the lower event corresponds to the two lower arcs of f to the left and that the upper event corresponds to the highest arcs of f and g to both sides. Therefore, Scheme (3) corresponds to Fig.3.

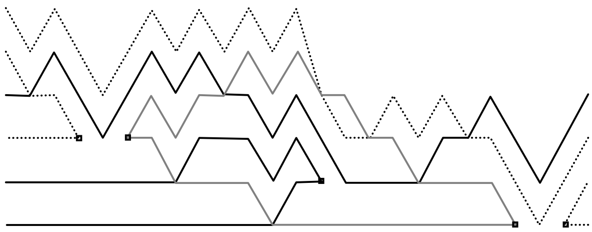

Fig.3 Topology of the arrangement f=0 (gray), g=0 (black) and h=0 (discontinuous). Solid boxes represent the event points of the arrangement

h		h				h		h		h		h											
h	h	f	h		h	f	h	f	h	g	h	g											
f	e	h	f	h	f	g	f	g	e	f	g	f	e	g		h		h		f			
h	h	h	e	f	e	g	g	f	f	f	f	f	f	h	e	g	h	f	e	h	f		f
f	f	f	f	f	f	f	e	g	g	g	f	f	e	f	f	f	e	g	g	g	h	f	h
f	f	f	f	f	f	f	f	f	f	f	e	g	g	g	g	g	g	g	g	g	e	h	e
−5	x_1	−7/2	x_2	−3	x_3	−7/3	x_4	−3/2	x_5	−1	x_6	−1/2	x_7	1/64	x_8	1/2	x_9	1	x_{10}	3/2	x_{11}	3	x_{12}

(3)

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