Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Free vibration of pre-tensioned nanobeams based on nonlocal stress theory*

C. W. LIM[†1], Cheng LI[1,2], Ji-lin YU[2]
([1]*Department of Building and Construction, City University of Hong Kong, Kowloon, Hong Kong SAR, China*)
([2]*Department of Modern Mechanics, University of Science and Technology of China, Hefei 230026, China*)
[†]E-mail: bccwlim@cityu.edu.hk
Received Jan. 17, 2009; Revision accepted Mar. 31, 2009; Crosschecked Sept. 7, 2009

Abstract: The transverse free vibration of nanobeams subjected to an initial axial tension based on nonlocal stress theory is presented. It considers the effects of nonlocal stress field on the natural frequencies and vibration modes. The effects of a small scale parameter at molecular level unavailable in classical macro-beams are investigated for three different types of boundary conditions: simple supports, clamped supports and elastically-constrained supports. Analytical solutions for transverse deformation and vibration modes are derived. Through numerical examples, effects of the dimensionless nanoscale parameter and pre-tension on natural frequencies are presented and discussed.

Key words: Nanobeam, Natural frequency, Nonlocal stress, Pre-tensioned, Vibration mode
doi:10.1631/jzus.A0900048 **Document code:** A **CLC number:** O34

1 Introduction

Recently, research on dynamic behavior of nano-structures has become a hot field because of the application prospects of nano-electromechanical systems (NEMSs) or nano-machine components. Although nano-structures, such as nanobeams and nanobelts, have been proposed to have practical applications, analysis in this field has been lacking in particular the dynamics of pre-tensioned nano-structures.

Vibration of axially moving macro-beams has been a subject of much concern recently. Mote (1965) constructed mathematical model of axially moving beams firstly based on the Hamilton principle and also determined the first three natural frequencies and modes. His results were confirmed by experiment (Mote and Naguleswaran, 1966). Simpson (1973) researched the natural frequency and mode function of axially moving beams without pre-tension and clamped at both ends. Oz *et al*. (2001) introduced axially moving beams with time-dependent velocity through multiple scale analysis. Liu and Zhang (2007) presented the nonlinear vibration of viscoelastic belts. The bifurcation of transverse vibration for axially accelerating moving strings was investigated by Chen and Wu (2005). Yang and Chen (2005) addressed dynamic stability problem of axially moving viscoelastic beams.

Most classical continuum theories are based on elastic constitutive relation, which assumes that the stress at a point is a function of strain at only that point. On the other hand, the nonlocal continuum mechanics assume that the stress at a point is a function of strains at all points in the domain. Such theories contain information about the forces between atoms, and the internal length scale is introduced into the constitutive equations as a material parameter. The nonlocal elasticity model was initiated by Eringen (1972; 1983; 2002) and Eringen and Edelen

* Project supported by the Collaboration Scheme from University of Science and Technology of China and City University of Hong Kong Joint Advanced Research Institute, and City University of Hong Kong (No. 7002357 (BC))

(1972). The nonlocal elasticity theory was applied in nanomechanics including lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, static deflection, fracture mechanics, and surface tension fluids, etc. (Reddy and Wang, 1998; Peddieson *et al*., 2003; Zhang *et al*., 2004; 2005; Wang, 2005; Lu *et al*., 2006; Wang and Varadan, 2006; Wang *et al*., 2006; 2008; Xu, 2006; Lim and Wang, 2007; Benzair *et al*., 2008; Kumar *et al*., 2008; Wang and Duan, 2008). The recent work by Tounsi *et al*. (2008) concluded that the scale coefficient was radius dependent.

Vibration behavior of beams has been developed for a long time. However, very few papers consider nanobeams with nonlocal effects. The nanomechanical vibration of a nanobeam is very different from the classical continuum mechanics theory which deals with the macroscopic scale of a beam. In this paper, we attempt to consider the nonlocal effects of a pre-tensioned nanobeam without axial motion and subsequently study the transverse vibration of such a nanobeam. The model is described by partial differential equations in dimensionless quantities such that the analysis is more general and distinctive to describe the difference between nanomechanics and classical mechanics. It is found that pre-tension and nonlocal stress play significant roles in the vibration behavior of a nanobeam. Their effects are analyzed and discussed in detail in a few numerical examples.

2 Problem definition and modeling

Consider a pre-tensioned nanobeam with the length L, initial axial tension P at the ends as illustrated in Fig. 1. The end boundary conditions are arbitrary and will be specified in various cases of study. The force equilibrium for an element of the nanobeam is shown in Fig. 2.

For vibration of a nanobeam, the bending rotation angle with respect to x-axis is denoted as θ. Because only small deformation is considered for linear vibration, we have

$$\cos\theta = 1, \quad \sin\theta = \frac{\partial w}{\partial x}, \tag{1}$$

where w is the transverse deformation.

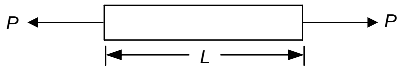

Fig. 1 Free-body diagram of a nanobeam

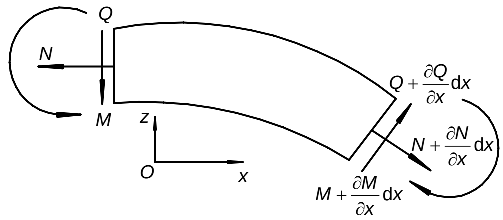

Fig. 2 Force equilibrium for an element of the nanobeam
M: bending moment; N: internal axial force; Q: shear force; x: axial coordinate; z: transverse coordinate

The equilibrium equation of an element with respect to the z-axis as shown in Fig. 2 can be obtained based on the D'Alembert Principle (Fung, 1965) as

$$\frac{\partial Q}{\partial x}\mathrm{d}x - \left(N\frac{\partial^2 w}{\partial x^2} + \frac{\partial N}{\partial x}\frac{\partial w}{\partial x}\right)\mathrm{d}x - \rho\frac{\partial^2 w}{\partial t^2}\mathrm{d}x = 0, \tag{2}$$

where ρ is the line density and t is time. The moment balance condition yields

$$\frac{\partial M}{\partial x} - Q = 0. \tag{3}$$

Substituting Eq. (3) into Eq. (2) and dividing dx throughout, we have (Yang and Lim, 2008)

$$\frac{\partial^2 M}{\partial x^2} - N\frac{\partial^2 w}{\partial x^2} - \frac{\partial N}{\partial x}\frac{\partial w}{\partial x} - \rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{4}$$

which can also be expressed as

$$\frac{\partial^2 M}{\partial x^2} - \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right) - \rho\frac{\partial^2 w}{\partial t^2} = 0. \tag{5}$$

For such a nanobeam with a constant, external axial tension P at the ends, we have

$$\frac{\partial^2 M}{\partial x^2} - P\frac{\partial^2 w}{\partial x^2} - \rho\frac{\partial^2 w}{\partial t^2} = 0. \tag{6}$$

According to Eringen (1983), the nonlocal stress in a 2D domain can be approximately governed by a second-order differential equation

$$\left[1-(e_0a)^2\nabla^2\right]\sigma_{ij}=\sigma'_{ij}, \tag{7}$$

where σ_{ij} (i, j=1, 2) are the nonlocal stresses, σ_{ij}' (i, j=1, 2) the classical local stresses, e_0 a constant dependent on material, and a an internal characteristic length, e.g., for lattice parameter, C-C bond length. For a nanobeam, the governing equation above with respect to the neutral axis can be reduced to an ordinary differential equation as

$$\left[1-(e_0a)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\sigma=\sigma', \tag{8}$$

where σ indicates the nonlocal normal stresses while σ' the classical local normal stresses along the x-axis. From Eq. (8), the nonlocal normal stresses can be solved and expressed in an infinite series as (Lim and Wang, 2007)

$$\sigma=-Ez\frac{\partial^2 w}{\partial x^2}-Ez(e_0a)^2\frac{\partial^4 w}{\partial x^4}+\cdots, \tag{9}$$

where E is the Young's modulus and z is the transverse coordinate defined in Fig. 2.

Integrating Eq. (8) above with respect to the distance from the neutral axis and over the cross-sectional area, the nonlocal bending moment is governed by

$$M-(e_0a)^2\frac{\partial^2 M}{\partial x^2}=-EI\frac{\partial^2 w}{\partial x^2}, \tag{10}$$

where EI is the flexural stiffness. From Eqs. (6) and (10), the following governing differential equation of motion for a nanobeam subjected to an initial axial tension P can be derived as

$$\rho\frac{\partial^2 w}{\partial t^2}+P\frac{\partial^2 w}{\partial x^2}-(e_0a)^2\left(\rho\frac{\partial^4 w}{\partial x^2\partial t^2}+P\frac{\partial^4 w}{\partial x^4}\right)=-EI\frac{\partial^4 w}{\partial x^4}. \tag{11}$$

For generality, dimensionless formulation is adopted using the following parameters $\bar{x}=x/L$, $\bar{w}=w/L$, and $\bar{t}=t\sqrt{EI/(\rho L^4)}$. In dimensionless quantities, Eq. (11) then becomes

$$\frac{\partial^2\bar{w}}{\partial\bar{t}^2}+\bar{P}\frac{\partial^2\bar{w}}{\partial\bar{x}^2}-\tau^2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}+\left(1-\bar{P}\tau^2\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4}=0, \tag{12}$$

where $\tau=e_0a/L$ and $\bar{P}=PL^2/EI$. From Eqs. (6) and (10), the nonlocal bending moment can be expressed as

$$M=(e_0a)^2\left(\rho\frac{\partial^2 w}{\partial t^2}+P\frac{\partial^2 w}{\partial x^2}\right)-EI\frac{\partial^2 w}{\partial x^2}. \tag{13}$$

Similarly, the non-dimensional form of the equation above is

$$\bar{M}=\tau^2\frac{\partial^2\bar{w}}{\partial\bar{t}^2}+\left(\bar{P}\tau^2-1\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}, \tag{14}$$

where $\bar{M}=ML/EI$.

For linear free vibration of a nanobeam, the modes are harmonic in time. Hence the time-dependent transverse deformation of the nanobeam can be represented by

$$\bar{w}(\bar{x},\bar{t})=\bar{W}_n(\bar{x})\mathrm{e}^{i\omega_n\bar{t}}, \tag{15}$$

where $\bar{W}_n(\bar{x})$ is the dimensionless amplitude of vibration and n=1, 2, … denotes the vibration mode number. Substituting Eq. (15) into Eq. (12), the governing equation is transformed into the frequency domain as

$$-\omega_n^2\bar{W}_n+(\tau^2\omega_n^2+\bar{P})\frac{\mathrm{d}^2\bar{W}_n}{\mathrm{d}\bar{x}^2}+(1-\bar{P}\tau^2)\frac{\mathrm{d}^4\bar{W}_n}{\mathrm{d}\bar{x}^4}=0. \tag{16}$$

Substituting $\bar{W}_n(\bar{x})=C_n\mathrm{e}^{i\lambda_{jn}\bar{x}}$, with C_n as an arbitrary nonzero constant, into the equation above, we obtain a dispersion relation

$$\omega_n^2+(\tau^2\omega_n^2+\bar{P})\lambda_{jn}^2-(1-\bar{P}\tau^2)\lambda_{jn}^4=0. \tag{17}$$

Since Eq. (17) is a fourth-order polynomial in terms of λ_n, the four roots are denoted by λ_{jn} (j=1, 2, 3, 4), respectively. Because only linear free vibration is concerned, the superposition of the four solutions with respect to each root λ_{jn} is also a solution of Eq. (16). Hence

$$\overline{W}_n(\overline{x}) = C_{1n}(\mathrm{e}^{i\lambda_{1n}\overline{x}} + C_{2n}\mathrm{e}^{i\lambda_{2n}\overline{x}} + C_{3n}\mathrm{e}^{i\lambda_{3n}\overline{x}} + C_{4n}\mathrm{e}^{i\lambda_{4n}\overline{x}}), \quad (18)$$

where C_{jn} (j=1, 2, 3, 4) are four arbitrary constants of integration associated with Eq. (16) which is a fourth-order ordinary differential equation.

3 Examples and discussion

To illustrate the effects of nonlocal stress and initial axial tension on the free vibration frequency of a nanobeam, the following examples for various boundary conditions are presented and discussed.

3.1 Simply supported nanobeams

For a nanobeam simply supported at both ends, the boundary conditions for the bending moments and displacements are

$$\overline{M}(0,\overline{t}) = 0,\ \overline{M}(1,\overline{t}) = 0,\ \overline{w}(0,\overline{t}) = 0,\ \overline{w}(1,\overline{t}) = 0. \quad (19)$$

Substitute Eqs. (14) and (15) into Eq. (19) and simplify the results. Further substituting Eq. (18) into the results obtained above yields

$$\begin{aligned}
&(\overline{P}\tau^2 - 1)C_{1n}(\lambda_{1n}^2 + \lambda_{2n}^2 C_{2n} + \lambda_{3n}^2 C_{3n} + \lambda_{4n}^2 C_{4n}) = 0,\\
&(\overline{P}\tau^2 - 1)C_{1n}(\lambda_{1n}^2 \mathrm{e}^{i\lambda_{1n}} + \lambda_{2n}^2 C_{2n}\mathrm{e}^{i\lambda_{2n}}\\
&\quad + \lambda_{3n}^2 C_{3n}\mathrm{e}^{i\lambda_{3n}} + \lambda_{4n}^2 C_{4n}\mathrm{e}^{i\lambda_{4n}}) = 0, \qquad (20)\\
&C_{1n}(1 + C_{2n} + C_{3n} + C_{4n}) = 0,\\
&C_{1n}(\mathrm{e}^{i\lambda_{1n}} + C_{2n}\mathrm{e}^{i\lambda_{2n}} + C_{3n}\mathrm{e}^{i\lambda_{3n}} + C_{4n}\mathrm{e}^{i\lambda_{4n}}) = 0.
\end{aligned}$$

For $\overline{P}\tau^2 - 1 = 0$ or equivalently $P = EI/(e_0 a)^2$, the amplitude solution $\overline{W}_n(\overline{x})$ is an arbitrary function and this is apparently not a solution of interest. On the other hand, for $\overline{P}\tau^2 - 1 \neq 0$, Eq. (20) can be expressed in a matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1\\ \mathrm{e}^{i\lambda_{1n}} & \mathrm{e}^{i\lambda_{2n}} & \mathrm{e}^{i\lambda_{3n}} & \mathrm{e}^{i\lambda_{4n}}\\ \lambda_{1n}^2 & \lambda_{2n}^2 & \lambda_{3n}^2 & \lambda_{4n}^2\\ \lambda_{1n}^2\mathrm{e}^{i\lambda_{1n}} & \lambda_{2n}^2\mathrm{e}^{i\lambda_{2n}} & \lambda_{3n}^2\mathrm{e}^{i\lambda_{3n}} & \lambda_{4n}^2\mathrm{e}^{i\lambda_{4n}} \end{pmatrix} \begin{pmatrix} 1\\ C_{2n}\\ C_{3n}\\ C_{4n}\end{pmatrix} C_{1n} = 0. \quad (21)$$

For an arbitrary $C_{1n} \neq 0$, the coefficients in Eq. (20) can be obtained as

$$\begin{aligned}
C_{2n} &= \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{3n}^2)}{(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{4n}^2 - \lambda_{3n}^2)} + \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{4n}^2)}{(\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{3n}})(\lambda_{4n}^2 - \lambda_{3n}^2)} - 1,\\
C_{3n} &= \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{4n}^2)}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{4n}^2 - \lambda_{3n}^2)},\\
C_{4n} &= \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{3n}^2)}{(\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{4n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}. \qquad (22)
\end{aligned}$$

Therefore, the n-mode amplitude of vibration from Eqs. (18) and (22) is

$$\begin{aligned}
\overline{W}_n(\overline{x}) = C_{1n}\Bigg\{ &\mathrm{e}^{i\lambda_{1n}\overline{x}} + \Bigg[\frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{3n}^2)}{(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}\\
&+ \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{4n}^2)}{(\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{3n}})(\lambda_{4n}^2 - \lambda_{3n}^2)} - 1\Bigg]\mathrm{e}^{i\lambda_{2n}\overline{x}}\\
&+ \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{4n}^2)}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}\mathrm{e}^{i\lambda_{3n}\overline{x}}\\
&+ \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{3n}^2)}{(\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{4n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}\mathrm{e}^{i\lambda_{4n}\overline{x}}\Bigg\}, \qquad (23)
\end{aligned}$$

and the corresponding time-dependent displacement from Eq. (15) is

$$\begin{aligned}
\overline{w}(\overline{x},\overline{t}) = C_{1n}\Bigg\{ &\mathrm{e}^{i\lambda_{1n}\overline{x}} + \Bigg[\frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{3n}^2)}{(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}\\
&+ \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{4n}^2)}{(\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{3n}})(\lambda_{4n}^2 - \lambda_{3n}^2)} - 1\Bigg]\mathrm{e}^{i\lambda_{2n}\overline{x}}\\
&+ \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{4n}^2)}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}\mathrm{e}^{i\lambda_{3n}\overline{x}}\\
&+ \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\lambda_{1n}^2 - \lambda_{3n}^2)}{(\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{4n}})(\lambda_{4n}^2 - \lambda_{3n}^2)}\mathrm{e}^{i\lambda_{4n}\overline{x}}\Bigg\}\mathrm{e}^{i\omega_n\overline{t}}, \qquad (24)
\end{aligned}$$

where $C_{1n} \neq 0$ is an arbitrary constant. For nontrivial solution of Eq. (21), the determinant of the coefficient matrix must be zero

$$\begin{vmatrix} 1 & 1 & 1 & 1\\ \mathrm{e}^{i\lambda_{1n}} & \mathrm{e}^{i\lambda_{2n}} & \mathrm{e}^{i\lambda_{3n}} & \mathrm{e}^{i\lambda_{4n}}\\ \lambda_{1n}^2 & \lambda_{2n}^2 & \lambda_{3n}^2 & \lambda_{4n}^2\\ \lambda_{1n}^2\mathrm{e}^{i\lambda_{1n}} & \lambda_{2n}^2\mathrm{e}^{i\lambda_{2n}} & \lambda_{3n}^2\mathrm{e}^{i\lambda_{3n}} & \lambda_{4n}^2\mathrm{e}^{i\lambda_{4n}} \end{vmatrix} = 0, \quad (25)$$

which yields a characteristic equation as

$$\begin{aligned}&\lambda_{3n}^2\lambda_{4n}^2(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{1n}})\\&+\lambda_{2n}^2\lambda_{4n}^2(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{3n}})\\&+\lambda_{1n}^2\lambda_{4n}^2(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})\\&+\lambda_{2n}^2\lambda_{3n}^2(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})\\&+\lambda_{1n}^2\lambda_{3n}^2(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{1n}})(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{4n}})\\&+\lambda_{1n}^2\lambda_{2n}^2(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{1n}})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})=0.\end{aligned} \tag{26}$$

By combining Eqs. (17) and (26), the five unknown quantities λ_{jn} (j=1, 2, 3, 4) and ω_n can be solved. Subsequently, substituting the results into Eqs. (23) and (24), the n-mode vibration mode and transverse deformation can be solved to the extent of an arbitrary constant $C_{1n}\neq 0$.

The analysis above can be described clearly through numerical examples. For instance, taking τ=0.3 and $\bar{P}=2$, the roots for ω_n (n=1, 2, …) satisfying Eqs. (17) and (26) can be obtained as the intercepts of the horizontal axis in Fig. 3 where the determinant Eq. (25) vanishes.

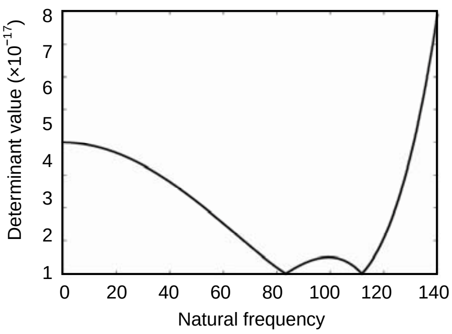

Fig. 3 The relationship between determinant value and natural frequency

It is obvious that there are infinite modes of frequency which make the determinant zero. The first intercept with the horizontal axis is the fundamental frequency; the second intercept is the second mode frequency, and so on. Following the numerical procedure above, the relationship between ω_1, ω_2 and the dimensionless nanoscale parameter τ can be obtained as shown in Fig. 4.

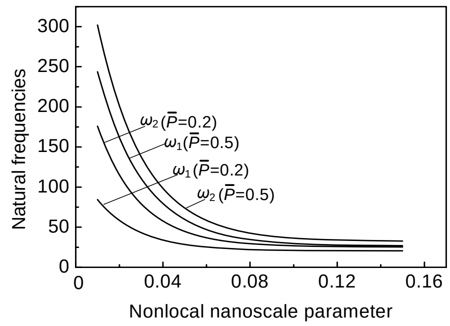

Fig. 4 Nanoscle effects on the first two mode frequencies for simply supported nanobeams

We can find that the fundamental and the second mode frequencies reduce with the increasing τ. Hence, the natural frequencies reduce when the stronger nonlocal stress effect is present. It is also obvious that the frequencies increase with the dimensionless pre-tension $\bar{P}$. Obviously, τ and $\bar{P}$ affect very much the natural vibration frequencies.

3.2 Clamped nanobeams

The problem of a clamped, pre-tensioned nanobeam is presented in the following example. The clamped boundary conditions are

$$\bar{w}(0,\bar{t})=0,\ \bar{w}(1,\bar{t})=0,\ \frac{\partial\bar{w}}{\partial\bar{x}}(0,\bar{t})=0,\ \frac{\partial\bar{w}}{\partial\bar{x}}(1,\bar{t})=0. \tag{27}$$

From the above equation, the result can be deduced by Eqs. (15) and (18)

$$\begin{pmatrix}1&1&1&1\\ \mathrm{e}^{i\lambda_{1n}}&\mathrm{e}^{i\lambda_{2n}}&\mathrm{e}^{i\lambda_{3n}}&\mathrm{e}^{i\lambda_{4n}}\\ \lambda_{1n}&\lambda_{2n}&\lambda_{3n}&\lambda_{4n}\\ \lambda_{1n}\mathrm{e}^{i\lambda_{1n}}&\lambda_{2n}\mathrm{e}^{i\lambda_{2n}}&\lambda_{3n}\mathrm{e}^{i\lambda_{3n}}&\lambda_{4n}\mathrm{e}^{i\lambda_{4n}}\end{pmatrix}\begin{pmatrix}1\\C_{2n}\\C_{3n}\\C_{4n}\end{pmatrix}C_{1n}=0, \tag{28}$$

which yields

$$C_{2n}=\frac{(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})+(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{3n}})+(\lambda_{2n}-\lambda_{1n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}, \tag{29}$$

$$C_{3n}=\frac{(\lambda_{2n}-\lambda_{4n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{2n}-\lambda_{1n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})},\ C_{4n}=\frac{(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{1n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}.$$

Hence, the n-mode amplitude of vibration is

$$\begin{aligned}\bar{W}_n(\bar{x}) = C_{1n}\Bigg\{ & \mathrm{e}^{i\lambda_{1n}\bar{x}} + \left[\frac{(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})+(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{3n}})+(\lambda_{2n}-\lambda_{1n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}\right]\mathrm{e}^{i\lambda_{2n}\bar{x}} \\ & + \frac{(\lambda_{2n}-\lambda_{4n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{2n}-\lambda_{1n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}\mathrm{e}^{i\lambda_{3n}\bar{x}} \\ & + \frac{(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{1n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}\mathrm{e}^{i\lambda_{4n}\bar{x}}\Bigg\},\end{aligned} \tag{30}$$

and the corresponding time-dependent displacement is

$$\begin{aligned}\bar{w}(\bar{x},\bar{t}) = C_{1n}\Bigg\{ & \mathrm{e}^{i\lambda_{1n}\bar{x}} + \left[\frac{(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})+(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{3n}})+(\lambda_{2n}-\lambda_{1n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}\right]\mathrm{e}^{i\lambda_{2n}\bar{x}} \\ & + \frac{(\lambda_{2n}-\lambda_{4n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{2n}-\lambda_{1n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}\mathrm{e}^{i\lambda_{3n}\bar{x}} \\ & + \frac{(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{1n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})}{(\lambda_{4n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})-(\lambda_{3n}-\lambda_{2n})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})}\mathrm{e}^{i\lambda_{4n}\bar{x}}\Bigg\}\mathrm{e}^{i\omega_n\bar{t}}.\end{aligned} \tag{31}$$

For nontrivial solution of matrix Eq. (28), the determinant of the coefficient matrix must be zero, or

$$\begin{aligned}&\lambda_{3n}\lambda_{4n}(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{1n}})+\lambda_{2n}\lambda_{4n}(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{3n}}) \\ &\quad+\lambda_{1n}\lambda_{4n}(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})+\lambda_{2n}\lambda_{3n}(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}}) \\ &\quad+\lambda_{1n}\lambda_{3n}(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{1n}})(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{4n}})+\lambda_{1n}\lambda_{2n}(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{1n}})(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})=0.\end{aligned} \tag{32}$$

Analogously, from Eqs. (17) and (32), we can solve the unknown quantities in Eqs. (30) and (31). To solve the problem numerically, the relationship between ω_1, ω_2 and τ is presented in Fig. 5 for two values of $\bar{P}$. Again it is obvious that frequency decreases with an increase in τ while it increases with an increase in $\bar{P}$.

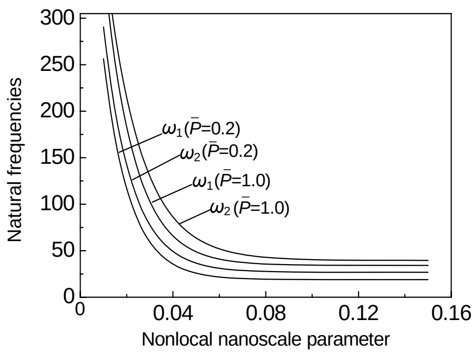

Fig. 5 Nanoscale effects on the first two mode frequencies for clamped nanobeams

3.3 Nanobeams with elastically constrained ends

In this example, we consider a special supporting condition for nanobeams with elastically constrained ends (Xie, 2007). The support conditions may be formulated with the following boundary conditions

$$\begin{aligned}&\bar{w}(0,\bar{t})=0,\ \bar{w}(1,\bar{t})=0, \\ &\frac{\partial^2\bar{w}(0,\bar{t})}{\partial\bar{x}^2}-\bar{k}\,\frac{\partial\bar{w}(0,\bar{t})}{\partial\bar{x}}=0, \\ &\frac{\partial^2\bar{w}(1,\bar{t})}{\partial\bar{x}^2}+\bar{k}\,\frac{\partial\bar{w}(1,\bar{t})}{\partial\bar{x}}=0,\end{aligned} \tag{33}$$

where $\bar{k}=k/(PL)$ is the dimensionless stiffness of the elastically constrained ends in which k is the physical stiffness of the elastic constraint. If $\bar{k}$ approaches 0, these ends degenerate to simply supports as discussed in subsection 3.1, while if $\bar{k}$ approaches infinity, they degenerate to clamped ones in subsection 3.2.

Substituting Eqs. (15) and (18) into Eq. (33) yields

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathrm{e}^{i\lambda_{1n}} & \mathrm{e}^{i\lambda_{2n}} & \mathrm{e}^{i\lambda_{3n}} & \mathrm{e}^{i\lambda_{4n}} \\ \lambda_{1n}^2 + i\bar{k}\lambda_{1n} & \lambda_{2n}^2 + i\bar{k}\lambda_{2n} & \lambda_{3n}^2 + i\bar{k}\lambda_{3n} & \lambda_{4n}^2 + i\bar{k}\lambda_{4n} \\ (\lambda_{1n}^2 - i\bar{k}\lambda_{1n})\mathrm{e}^{i\lambda_{1n}} & (\lambda_{2n}^2 - i\bar{k}\lambda_{2n})\mathrm{e}^{i\lambda_{2n}} & (\lambda_{3n}^2 - i\bar{k}\lambda_{3n})\mathrm{e}^{i\lambda_{3n}} & (\lambda_{4n}^2 - i\bar{k}\lambda_{4n})\mathrm{e}^{i\lambda_{4n}} \end{pmatrix} \begin{pmatrix} 1 \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{pmatrix} C_{1n} = 0. \tag{34}$$

For $C_{1n} \neq 0$, the following solutions of coefficients are obtained by solving Eq. (34):

$$\begin{aligned} C_{2n} &= \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{3n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right] + (\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{4n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})^2\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} + \frac{\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{3n}}}{\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}}}, \\ C_{3n} &= \frac{(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})^2\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} + \frac{\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{1n}}}{\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}}}, \\ C_{4n} &= \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]}. \end{aligned} \tag{35}$$

Then the *n*-mode amplitude of vibration can be obtained as

$$\begin{aligned} \bar{W}_n(\bar{x}) = C_{1n} \Bigg\{ & \mathrm{e}^{i\lambda_{1n}\bar{x}} \\ & + \left[\frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{3n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right] + (\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{4n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})^2\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} + \frac{\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{3n}}}{\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}}} \right] \mathrm{e}^{i\lambda_{2n}\bar{x}} \\ & + \left[\frac{(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})^2\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} + \frac{\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{1n}}}{\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}}} \right] \mathrm{e}^{i\lambda_{3n}\bar{x}} \\ & + \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} \mathrm{e}^{i\lambda_{4n}\bar{x}} \Bigg\}, \end{aligned} \tag{36}$$

and the corresponding time-dependent displacement is shown in Eq. (37):

$$\begin{aligned} \bar{w}(\bar{x}, \bar{t}) = C_{1n} \Bigg\{ & \mathrm{e}^{i\lambda_{1n}\bar{x}} \\ & + \left[\frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{3n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right] + (\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{4n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})^2\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} + \frac{\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{3n}}}{\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}}} \right] \mathrm{e}^{i\lambda_{2n}\bar{x}} \\ & + \left[\frac{(\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})^2\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} + \frac{\mathrm{e}^{i\lambda_{2n}} - \mathrm{e}^{i\lambda_{1n}}}{\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}}} \right] \mathrm{e}^{i\lambda_{3n}\bar{x}} \\ & + \frac{(\mathrm{e}^{i\lambda_{1n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{1n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{1n} - \lambda_{2n})\right]}{(\mathrm{e}^{i\lambda_{3n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{4n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{4n} - \lambda_{2n})\right] - (\mathrm{e}^{i\lambda_{4n}} - \mathrm{e}^{i\lambda_{2n}})\left[\lambda_{3n}^2 - \lambda_{2n}^2 + i\bar{k}(\lambda_{3n} - \lambda_{2n})\right]} \mathrm{e}^{i\lambda_{4n}\bar{x}} \Bigg\} \mathrm{e}^{i\omega_n\bar{t}}. \end{aligned} \tag{37}$$

For nontrivial solution of Eq. (34), the determinant of the coefficient matrix must be zero, or Eq. (38)

$$
\begin{aligned}
&(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{1n}})\Big[(\lambda_{3n}^2+i\bar{k}\lambda_{3n})(\lambda_{4n}^2-i\bar{k}\lambda_{4n})\mathrm{e}^{i\lambda_{4n}}-(\lambda_{4n}^2+i\bar{k}\lambda_{4n})(\lambda_{3n}^2-i\bar{k}\lambda_{3n})\mathrm{e}^{i\lambda_{3n}}\Big]\\
&+(\mathrm{e}^{i\lambda_{1n}}-\mathrm{e}^{i\lambda_{3n}})\Big[(\lambda_{2n}^2+i\bar{k}\lambda_{2n})(\lambda_{4n}^2-i\bar{k}\lambda_{4n})\mathrm{e}^{i\lambda_{4n}}-(\lambda_{4n}^2+i\bar{k}\lambda_{4n})(\lambda_{2n}^2-i\bar{k}\lambda_{2n})\mathrm{e}^{i\lambda_{2n}}\Big]\\
&+(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{1n}})\Big[(\lambda_{2n}^2+i\bar{k}\lambda_{2n})(\lambda_{3n}^2-i\bar{k}\lambda_{3n})\mathrm{e}^{i\lambda_{3n}}-(\lambda_{3n}^2+i\bar{k}\lambda_{3n})(\lambda_{2n}^2-i\bar{k}\lambda_{2n})\mathrm{e}^{i\lambda_{2n}}\Big]\\
&+(\mathrm{e}^{i\lambda_{3n}}-\mathrm{e}^{i\lambda_{2n}})\Big[(\lambda_{1n}^2+i\bar{k}\lambda_{1n})(\lambda_{4n}^2-i\bar{k}\lambda_{4n})\mathrm{e}^{i\lambda_{4n}}-(\lambda_{4n}^2+i\bar{k}\lambda_{4n})(\lambda_{1n}^2-i\bar{k}\lambda_{1n})\mathrm{e}^{i\lambda_{1n}}\Big]\\
&+(\mathrm{e}^{i\lambda_{2n}}-\mathrm{e}^{i\lambda_{4n}})\Big[(\lambda_{1n}^2+i\bar{k}\lambda_{1n})(\lambda_{3n}^2-i\bar{k}\lambda_{3n})\mathrm{e}^{i\lambda_{3n}}-(\lambda_{3n}^2+i\bar{k}\lambda_{3n})(\lambda_{1n}^2-i\bar{k}\lambda_{1n})\mathrm{e}^{i\lambda_{1n}}\Big]\\
&+(\mathrm{e}^{i\lambda_{4n}}-\mathrm{e}^{i\lambda_{3n}})\Big[(\lambda_{1n}^2+i\bar{k}\lambda_{1n})(\lambda_{2n}^2-i\bar{k}\lambda_{2n})\mathrm{e}^{i\lambda_{2n}}-(\lambda_{2n}^2+i\bar{k}\lambda_{2n})(\lambda_{1n}^2-i\bar{k}\lambda_{1n})\mathrm{e}^{i\lambda_{1n}}\Big]=0.
\end{aligned}
\tag{38}
$$

The relationship between the natural frequencies ω_1, ω_2 and nanoscale parameter τ is presented in Fig. 6 for $\bar{k}=0.2$. Again, we observe similar effects of τ and $\bar{P}$ where increases in τ and $\bar{P}$ cause the frequencies to decrease and increase, respectively.

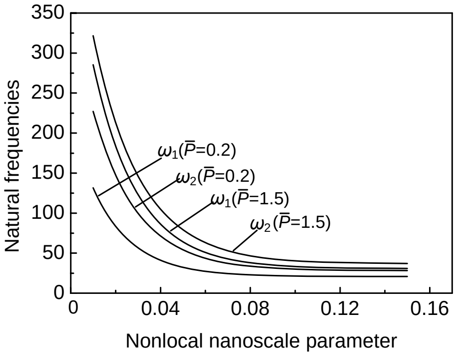

Fig. 6 Nanoscale effects on the first two mode frequencies for nanobeams with elastically constrained ends

4 Conclusion

In this paper, we concluded that the transverse free vibration of a nanobeam is significantly influenced by the existence of a pre-tension and the dimensionless nanoscale parameter. Three numerical examples are presented which include simply supported nanobeams, clamped nanobeams and nanobeams with elastically constrained ends. In the numerical examples, we find that the first two mode frequencies drop quickly with increasing dimensionless nanoscale parameter. On the contrary, the first two mode frequencies increase with increasing pre-tension. The effects are similar for the three examples investigated.

References

Benzair, A., Tounsi, A., Besseghier, A., Heireche, H., Moulay, N., Boumia, L., 2008. The thermal effect on vibration of single-walled carbon nanotubes using nonlocal Timoshenko beam theory. *Journal of Physics D: Applied Physics*, **41**(22):225404. [doi:10.1088/0022-3727/41/22/225404]

Chen, L.Q., Wu, J., 2005. Bifurcation in transverse vibration of axially accelerating viscoelastic strings. *Acta Mechanica Solida Sinica*, **26**(1):83-86.

Eringen, A.C., 1972. Nonlocal polar elastic continua. *International Journal of Engineering Science*, **10**(1):1-16. [doi:10.1016/0020-7225(72)90070-5]

Eringen, A.C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of Applied Physics*, **54**(9):4703-4710. [doi:10.1063/1.332803]

Eringen, A.C., 2002. Nonlocal Continuum Field Theories. Springer-Verlag, New York, USA, p.285-316.

Eringen, A.C., Edelen, D.G.B., 1972. On nonlocal elasticity. *International Journal of Engineering Science*, **10**(3):233-248. [doi:10.1016/0020-7225(72)90039-0]

Fung, Y.C., 1965. Foundations of Solid Mechanics. Prentice-Hall, Englewood Cliffs, NJ, USA, p.305-359.

Kumar, D., Heinrich, C., Wass, A.M., 2008. Buckling analysis of carbon nanotubes modeled using nonlocal continuum theories. *Journal of Applied Physics*, **103**(7):073521. [doi:10.1063/1.2901201]

Lim, C.W., Wang, C.M., 2007. Exact variational nonlocal stress modeling with asymptotic higher-order strain gradients for nanobeams. *Journal of Applied Physics*, **101**(5):054312. [doi:10.1063/1.2435878]

Liu, Y.Q., Zhang, W., 2007. Transverse nonlinear dynamical characteristic of viscoelastic belt. *Journal of Beijing University of Technology*, **33**(11):10-13.

Lu, P., Lee, H.P., Lu, C., Zhang, P.Q., 2006. Dynamic properties of flexural beams using a nonlocal elasticity model. *Journal of Applied Physics*, **99**(7):073510. [doi:10.1063/1.2189213]

Mote, C.D.Jr., 1965. A study of band saw vibrations. *Journal of the Franklin Institute*, **279**(6):430-444. [doi:10.1016/0016-0032(65)90273-5]

Mote, C.D.Jr., Naguleswaran, S., 1966. Theoretical and experimental band saw vibrations. *ASME Journal of Engineering Industry*, **88**(2):151-156.

Oz, H.R., Pakdemirli, M., Boyaci, H., 2001. Non-linear vibrations and stability of an axially moving beam with time-dependent velocity. *International Journal of Non-Linear Mechanics*, **36**(1):107-115. [doi:10.1016/S0020-7462(99)00090-6]

Peddieson, J., Buchanan, G.R., McNitt, R.P., 2003. Application of nonlocal continuum models to nanotechnology. *International Journal of Engineering Science*, **41**(3-5): 305-312. [doi:10.1016/S0020-7225(02)00210-0]

Reddy, J.N., Wang, C.M., 1998. Deflection relationships between classical and third-order plate theories. *Acta Mechanica Sinica*, **130**(3):199-208.

Simpson, A., 1973. Transverse modes and frequencies of beams translating between fixed end supports. *Journal of Mechanical Engineering Science*, **15**(3):159-164. [doi:10.1243/JMES_JOUR_1973_015_031_02]

Tounsi, A., Heireche, H., Berrabah, H.M., Benzair, A., Boumia, L., 2008. Effect of small size on wave propagation in double-walled carbon nanotubes under temperature field. *Journal of Applied Physics*, **104**(10):104301. [doi:10.1063/1.3018330]

Wang, C.M., Duan, W.H., 2008. Free vibration of nanorings/arches based on nonlocal elasticity. *Journal of Applied Physics*, **104**(1):014303. [doi:10.1063/1.2951642]

Wang, C.M., Zhang, Y.Y., Ramesh, S.S., Kitipornchai, S., 2006. Buckling analysis of micro- and nano-rods/tubes based on nonlocal Timoshenko beam theory. *Journal of Applied Physics*, **39**(17):3904-3909. [doi:10.1088/0022-3727/39/17/029]

Wang, C.M., Kitipornchai, S., Lim, C.W., Eisenberger, M., 2008. Beam bending solutions based on nonlocal Timoshenko beam theory. *Journal of Engineering Mechanics, ASCE*, **134**(6):475-481. [doi:10.1061/(ASCE)0733-9399(2008)134:6(475)]

Wang, Q., 2005. Wave propagation in carbon nanotubes via nonlocal continuum mechanics. *Journal of Applied Physics*, **98**(12):124301. [doi:10.1063/ 1.2141648]

Wang, Q., Varadan, V.K., 2006. Vibration of carbon nanotubes studied using nonlocal continuum mechanics. *Smart Materials and Structures*, **15**(2):659-666. [doi:10.1088/0964-1726/15/2/050]

Xie, G.M., 2007. Vibration Mechanics. National Defense Industry Press, Beijing, China, p.198-203 (in Chinese).

Xu, M.T., 2006. Free transverse vibrations of nano-to-micron scale beams. *Proceedings of the Royal Society A: Mathematical Physical and Engineering Sciences*, **462**(2074):2977-2995. [doi:10.1098/rspa.2006.1712]

Yang, X.D., Chen, L.Q., 2005. Dynamic stability of axially moving viscoelastic beams with pulsating speed. *Applied Mathematics and Mechanics*, **26**(8):905-910.

Yang, X.D., Lim, C.W., 2008. Nonlinear Vibrations of Nano-beams Accounting for Nonlocal Effect. Fourth Jiangsu-Hong Kong Forum on Mechanics and Its Application, Suzhou, China. Jiangsu Society of Mechanics, Nanjing, p.16-17.

Zhang, Y.Q., Liu, G.R., Wang, J.S., 2004. Small-scale effects on buckling of multiwalled carbon nanotubes under axial compression. *Physical Review B*, **70**(20):205430. [doi:10.1103/PhysRevB.70.205430]

Zhang, Y.Q., Liu, G.R., Xie, X.Y., 2005. Free transverse vibrations of double-walled carbon nanotubes using a theory of nonlocal elasticity. *Physical Review B*, **71**(19): 195404. [doi:10.1103/PhysRevB.71.195404]