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Internal force and deformation matrixes and their applications in load path

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Abstract: This paper deals with the internal force and the deformation matrixes, both of which can be used to analyze the topological relationship of a structure. Based on the reciprocal theorem, the relationship between the two matrixes is established, which greatly simplifies the computation of the internal force matrix. According to the characteristics of the internal force matrix, the transfer law of the matrix itself (due to the removal of components) is established based on the principle of linear superposition. With the relation of the two matrixes, the transfer law of the deformation matrix is also obtained. The transfer law illuminates the change regularity of internal force or deformation of the remnant structure when certain members are cut off one after another. The results of numerical examples show that the proposed methods are correct, reliable and effective.

Key words: Truss structure, Redundancy, Robustness, Topological relationship of structure, Removal of members, Transfer law
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1 Introduction

The methods of structural reliability analysis enable structures to perform well under identified conventional loads. For statically indeterminate structures, however, the load effects will be caused not only by the direct loads, but also by the indirect loads, such as errors in manufacture, settlement of supports, and shrinkage or creep of material. To calculate the indirect actions, a matrix, which can be called deformation matrix, has been put forward (Kou *et al*., 2008).

However, damages to one or more components do occur when extraordinary loads appear. The extraordinary loads can be fires or explosions, human errors in design or construction, undetected degradation of components, and other factors which are difficult to predict. The consequent effect on the whole structure, after the damage, depends both on the conventional loads and on the location of damaged components, and the way they are connected (Agarwal *et al*., 2003). Medium and high levels of protection categories require the use of the alternate path (AP) method to investigate the capability of the structural system to transfer loads safely from a notionally removed column to the remaining structural elements (Fu, 2009; Mohamen, 2009). This is robustness analysis, the purpose of which is to avoid possible progressive failure or the disproportionately large consequences in comparison with that damage.

Pertinent developments in robust structural design almost always rely on probabilistic uncertainty models (Allen and Maute, 2005; Zang *et al*., 2005; Jensen, 2006; Papadopoulos and Lagaros, 2009). Thus, the uncertainty models must be specified according to the underlying nature, to obtain reliable results (Beer and Liebscher, 2008). Anyway, the required precise probabilistic information on the extraordinary loads is in fact often difficult to obtain, especially because the number of samples and observation data is limited (Qiu *et al*., 2008; Guo *et al*., 2009). As a result, the available information is frequently imprecise, fluctuating, incomplete, fragmentary, vague, ambiguous or of subjective background

(Möller and Beer, 2008). In addition, the quantitative measures of structural robustness, which considered the possibility of failure (Lind, 1995; Pinto *et al.*, 2002; Baker *et al.*, 2008), are not very practical for complex systems.

Therefore, non-probabilistic modeling (Ni and Qiu, 2010; Qiu and Wang, 2010) is used when the probabilistic information is limited. England *et al.* (2008) examined the potential for damage to propagate through a structure using a new measure of hazard potential and vertical pushover analysis. A method to calculate the importance indices of structural components was provided by Liu and Liu (2005). This method considers the effect of components stiffness, structural redundancy, and external loads actions. By applying unit axial balance force to a member, the inner force of that member is regarded as the component importance index. If all the inner forces in the truss structure are calculated when all the components are subjected to unit axial balance forces one by one, a matrix may be formed with the inner forces listed by the sequence numbers of unit axial balance force. This matrix can be called the internal force matrix.

Both the internal force matrix and the deformation matrix integrate the topological relationship of a structure and the action effects. However, either the identification of failure modes in reliability, or the description of the effects of damaged components on the remnant part in robust estimation, concerns themselves with the load path in a structural system (Neves *et al.*, 2006; Qiu and Wang, 2010), and then the key point is the inner forces redistribution. The objectives of this paper are to present the relationship between the two matrixes and establish the quantitative relation of each matrix before and during a member failure. The transfer law shows how the deformation or the inner force of a removed component transfers to each remnant part.

2 Relationship between the internal force matrix and the deformation matrix

Consider a linear truss structure with n components. The deformation matrix mentioned above reveals the relationship between the compatible deformation and the free-form deformations resulting from indirect action (Kou *et al.*, 2008):

$$\boldsymbol{V} = \boldsymbol{R\varDelta}, \tag{1}$$

where $\boldsymbol{V}$, $\boldsymbol{\varDelta}$ and $\boldsymbol{R}$ represent the compatible deformation vector, the free-form deformation vector and the deformation matrix, respectively.

$$\boldsymbol{V} = [V_1, V_2, \cdots, V_n]^{\mathrm{T}}, \quad \boldsymbol{\varDelta} = [\varDelta_1, \varDelta_2, \cdots, \varDelta_n]^{\mathrm{T}},$$
$$\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix}.$$

The element R_{ij} in the matrix is defined as the compatible deformation of component i caused by unit free-form deformation of component j. $\boldsymbol{R}$ is calculated by

$$\boldsymbol{R} = \boldsymbol{A}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{WA})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W} - \boldsymbol{I}, \tag{2}$$

where $\boldsymbol{A}$, $\boldsymbol{W}$ and $\boldsymbol{I}$ denote the conversion matrix of displacements and deformations, the stiff matrix of components and the n order identity matrix, respectively.

On the other hand, the relationship between the internal forces of components and the axial balance force can be given by

$$\boldsymbol{N}=\boldsymbol{BP}, \tag{3}$$

where $\boldsymbol{N}$, $\boldsymbol{P}$ and $\boldsymbol{B}$ represent the inner force vector, the axial balance force vector and the internal force matrix, respectively.

$$\boldsymbol{N} = [N_1, N_2, \cdots, N_n]^{\mathrm{T}}, \quad \boldsymbol{P} = [P_1, P_2, \cdots, P_n]^{\mathrm{T}},$$
$$\boldsymbol{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ R_{21} & R_{22} & \cdots & B_{2n} \\ \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix},$$

where the element B_{ij} can be defined as the inner force of member i caused by the unit axial balance force at member j.

The two matrixes above are formed from

different aspects, but they resemble each other. It has been proven that the diagonal element B_{ii} or R_{ii} corresponds to the redundancy of member i (Liu and Liu 2005; Kou *et al*., 2008). However, all the elements in one matrix can be derived from the other one and the proving processes are as follows.

2.1 Relation of diagonal elements

From the deformation matrix, the element V_i in the vector $\boldsymbol{V}$ can be expressed as

$$V_i = \sum_{j=1}^{n} V_{ij} = \sum_{j=1}^{n} R_{ij} \Delta_j . \tag{4}$$

Let the free-form deformation just occur in member i and the extension of a member be positive. The compatible deformation of member i can be described as

$$V_{ii} = R_{ii} \Delta_i . \tag{5}$$

On the other hand, the structure bears axial balance force P_i at the member i and the tensile force is assumed to be positive. Let Δ_i be equal to the deformation of member i. With the displacement method, the equation can be obtained:

$$P_i = K_i \Delta_i , \tag{6}$$

where K_i is the stiffness of component i.

According to the deformation matrix, the compatible deformation of member i is V_{ii} after the member is constrained by other members. Then the total deformation of member i under axial balance force P_i is

$$\Delta_{Li} = V_{ii} + \Delta_i . \tag{7}$$

The internal force of member i can be described as

$$N_i = K_i \Delta_{Li} . \tag{8}$$

Substituting Eqs. (5)–(7) into Eq. (8), the following expression of the internal force of the member i can be written as

$$N_i = (1 + R_{ii}) P_i . \tag{9}$$

According to the definition of element B_{ii}, we obtain:

$$B_{ii} = 1 + R_{ii} . \tag{10}$$

2.2 Relation of non-diagonal elements

From the internal force matrix, the element N_i in the vector $\boldsymbol{N}$ can be expressed as

$$N_i = \sum_{j=1}^{n} N_{ij} = \sum_{j=1}^{n} B_{ij} P_j . \tag{11}$$

The compatible deformation of the member j aroused by free-form deformation in the member i can be written as

$$V_{ji} = R_{ji} \Delta_i . \tag{12}$$

Then the internal force of member j aroused by free-form deformation of the member i can be expressed as

$$N_{ji} = K_j V_{ji} . \tag{13}$$

In this study, the reciprocal theorem can be described as

$$P_j V_{ji} = P_i V_{ij} . \tag{14}$$

The axial balance force at the member j can be expressed as follows:

$$P_j = K_j \Delta_j . \tag{15}$$

Substituting Eq. (15) into Eq. (14), we obtain

$$K_j \Delta_j V_{ji} = P_i V_{ij} . \tag{16}$$

Substituting Eqs. (12) and (16) into Eq. (13), the inner force of component j can be obtained:

$$N_{ji} = R_{ij} P_i . \tag{17}$$

Comparing Eq. (17) with the definition of B_{ji}, the following expression can be written as

$$B_{ji} = R_{ij} . \tag{18}$$

This means that each non-diagonal element in matrix $\boldsymbol{B}$ is equal to the element which has the same position in the transposed matrix of $\boldsymbol{R}$.

2.3 Discussion

The two expressions Eqs. (10) and (18) can be abbreviated to one expression:

$$\boldsymbol{B} = [\boldsymbol{I} + \boldsymbol{R}]^{\mathrm{T}} . \tag{19}$$

When the system contains a large number of members, the computation of conventional methods to calculate matrix $\boldsymbol{B}$ is inefficient. The process includes the joint displacements and the inner forces calculation under the axial balance force at every member. Since the deformation matrix calculation is much more efficient, the relationship between the two matrixes can greatly improve the calculation rate of the internal force matrix.

3 Transfer law

When extraordinary loads occur, the inner forces in the structure will be redistributed because of the damaged member. The methods of matrix displacement and matrix force are adopted to solve the new inner forces as per usual. However, the two methods are not entirely efficient. Several simplified approaches have been put forward (Dong *et al*., 2000), and the methods depend on the diagonal elements of the internal force matrix. A new, more efficient method considering the load path is developed in this study to reveal the inner force redistribution when there are certain components damaged.

3.1 Transfer law of internal force matrix

Let the initial truss structure be subjected to an axial balance force P_j, and then the inner force of the member i is obtained

$$N_{ij} = B_{ij} P_j . \tag{20}$$

Similarly, the inner force of the member k can be obtained:

$$N_{kj} = B_{kj} P_j . \tag{21}$$

When damage occurs to the component k, the inner force is transferred from the damaged member to other components. When the inner force transfers from the component k to the component i the first occasion can be written as

$$N_{ik}^{(1)} = B_{ik} N_{kj} = B_{ik} B_{kj} P_j . \tag{22}$$

The inner force transferred to the damaged component k is

$$N_{kk}^{(1)} = B_{kk} N_{kj} = B_{kk} B_{kj} P_j . \tag{23}$$

However, the inner force cannot in reality be afforded by the damaged member. Again, it has to be transferred. At the second occasion, the component i obtains the inner force by

$$N_{ik}^{(2)} = B_{ik} N_{kk}^{(1)} = B_{ik} B_{kk} B_{kj} P_j , \tag{24}$$

$$N_{kk}^{(2)} = B_{kk} N_{kk}^{(1)} = B_{kk}^2 B_{kj} P_j . \tag{25}$$

The element B_{kk} is a number between 0 and 1. Thus, after repeating the steps above again and again, the inner force of the damaged member is equal to zero, which is the limit state of transfer. According to the sum formula of geometric progression, the result of the inner force transferred to the component i is

$$N_{ik} = \sum_{t=1}^{\infty} N_{ik}^{(t)} = B_{ik} B_{kj} P_j / (1 - B_{kk}) . \tag{26}$$

Based upon the principle of linear superposition, the total inner force of component i can be given by

$$N'_{ij} = N_{ij} + N_{ik} = [B_{ij} + B_{ik} B_{kj} / (1 - B_{kk})] P_j . \tag{27}$$

Note that the definition of B'_{ij} and the transfer law of inner force matrix can be written as

$$B'_{ij}(k) = B_{ij} + B_{ik} B_{kj} / (1 - B_{kk}), \quad i \neq k, \ j \neq k. \tag{28}$$

Under the external load $\boldsymbol{F}$, the inner forces of the member i in the damaged structure without the member k can be expressed as

$$N(\boldsymbol{F})'_i = N(\boldsymbol{F})_i + N(\boldsymbol{F})_k B_{ik} / (1 - B_{kk}), \quad i \neq k, \quad (29)$$

where $N(\boldsymbol{F})_i$ and $N(\boldsymbol{F})_k$ are the inner forces of the members i and k, respectively, in the initial structure under the external load $\boldsymbol{F}$.

3.2 Transfer law of deformation matrix

Substituting the equations of $B'_{ij} = R'_{ji}$, $B_{ij}=R_{ji}$, $B_{ik}=R_{ki}$, $B_{kj}=R_{jk}$, and $B_{kk}=1+R_{kk}$ into Eq. (28), and changing the position of subscripts i and j, the transfer law of deformation matrix can be written as

$$R'_{ij}(k) = R_{ij} - R_{ik} R_{kj} / R_{kk}, \; i \neq k, j \neq k. \quad (30)$$

3.3 Discussion

If B_{kk} is equal to one or R_{kk} is equal to zero, the new inner forces or deformations are infinite. This means that when the component k is destroyed, the structure will change to a mechanism. The path-independence of the transfer law can be proven, as the different transfer order from one member to another retains the same result.

4 Examples and comparison

4.1 Relationship of the internal force matrix and the deformation matrix

Consider a plane truss with seven members (1)–(7) as shown in Fig. 1 for simplicity, and 1–5 are the node numbers. Assume the length L=4 m, the vertical force F_1=15 kN, the horizontal force F_2=2.5 kN and let all the members be the same in the elastic modulus and the section area.

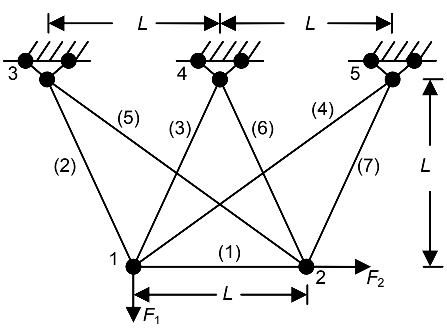

Fig. 1 Seven-bar truss structure

With the method provided by Kou *et al.* (2008), the solution of the deformation matrix is given by

$$\boldsymbol{R}=\begin{bmatrix} -0.259 & -0.195 & 0.101 & 0.153 & 0.153 & 0.101 & -0.195 \\ -0.219 & -0.273 & 0.302 & -0.047 & 0.129 & 0.085 & -0.165 \\ 0.113 & 0.302 & -0.479 & 0.284 & -0.066 & -0.044 & 0.085 \\ 0.275 & -0.075 & 0.459 & -0.618 & -0.162 & -0.107 & 0.207 \\ 0.275 & 0.207 & -0.107 & -0.162 & -0.618 & 0.459 & -0.075 \\ 0.113 & 0.085 & -0.044 & -0.066 & 0.284 & -0.479 & 0.302 \\ -0.219 & -0.165 & 0.085 & 0.129 & -0.047 & 0.302 & -0.273 \end{bmatrix}.$$

As a conventional method, the matrix displacement method is used to calculate the inner forces of all members when the member (1) subjected to the unit axial balance force. The result is given by

$$\boldsymbol{B}_1 = [0.741, -0.195, 0.101, 0.153, 0.153, 0.101, -0.195]^{\mathrm{T}}.$$

Columns (2)–(7) can be obtained when members (2)–(7) are subjected to unit axial balance forces.

$$\boldsymbol{B}_2 = [-0.219, 0.727, 0.302, -0.047, 0.129, 0.085, -0.165]^{\mathrm{T}},$$
$$\boldsymbol{B}_3 = [0.113, 0.302, 0.521, 0.284, -0.066, -0.044, 0.085]^{\mathrm{T}},$$
$$\boldsymbol{B}_4 = [0.275, -0.075, 0.459, 0.382, -0.162, -0.107, 0.207]^{\mathrm{T}},$$
$$\boldsymbol{B}_5 = [0.275, 0.207, -0.107, -0.162, 0.382, 0.459, -0.075]^{\mathrm{T}},$$
$$\boldsymbol{B}_6 = [0.113, 0.085, -0.044, -0.066, 0.284, 0.521, 0.302]^{\mathrm{T}},$$
$$\boldsymbol{B}_7 = [-0.219, -0.165, 0.085, 0.129, -0.047, 0.302, 0.727]^{\mathrm{T}}.$$

Arraying these vectors according to the sequence of the unit axial balance forces, the internal force matrix can be written as

$$\boldsymbol{B}=[\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{B}_4, \boldsymbol{B}_5, \boldsymbol{B}_6, \boldsymbol{B}_7]$$
$$=\begin{bmatrix} 0.741 & -0.219 & 0.113 & 0.275 & 0.275 & 0.113 & -0.219 \\ -0.195 & 0.727 & 0.302 & -0.075 & 0.207 & 0.085 & -0.165 \\ 0.101 & 0.302 & 0.521 & 0.459 & -0.107 & -0.044 & 0.085 \\ 0.153 & -0.047 & 0.284 & 0.382 & -0.162 & -0.066 & 0.129 \\ 0.153 & 0.129 & -0.066 & -0.162 & 0.382 & 0.284 & -0.047 \\ 0.101 & 0.085 & -0.044 & -0.107 & 0.459 & 0.521 & 0.302 \\ -0.195 & -0.165 & 0.085 & 0.207 & -0.075 & 0.302 & 0.727 \end{bmatrix}.$$

Comparing the elements in the two matrixes $\boldsymbol{R}$ and $\boldsymbol{B}$, they are perfectly in accord with the expression: $\boldsymbol{B}=[\boldsymbol{I}+\boldsymbol{R})]^{\mathrm{T}}$. The indeterminacy degree of the whole structure is equal to $|\mathrm{tr}(\boldsymbol{R})|=\sum|R_{ii}|=3$, where $|R_{ii}|$ expresses the redundant degree of member i.

4.2 Transfer law and its application

The member (1) in subsection 4.1 is assumed to be damaged by an extraordinary load. The new inner force matrix calculated according to Eq. (28) is

$$\boldsymbol{B}'(1)=\begin{bmatrix} \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} \\ \mathrm{N} & 0.891 & 0.218 & 0.283 & 0 & 0 & 0 \\ \mathrm{N} & 0.218 & 0.565 & 0.566 & 0 & 0 & 0 \\ \mathrm{N} & -0.175 & 0.351 & 0.544 & 0 & 0 & 0 \\ \mathrm{N} & 0 & 0 & 0 & 0.544 & 0.351 & -0.175 \\ \mathrm{N} & 0 & 0 & 0 & 0.566 & 0.565 & 0.218 \\ \mathrm{N} & 0 & 0 & 0 & -0.283 & 0.218 & 0.891 \end{bmatrix}$$

The element "N" means that the member (1) does not exist and the element about it need not be computed. The element "0" in matrix $\boldsymbol{B}'(1)$ shows that after the break of member (1), the inner forces in members (2)–(4) cannot transmit to members (5)–(7), and vice versa. Since no diagonal element is equal to one, there is no bar required at this stage.

The inner forces of the initial structure are

$$\boldsymbol{N}=[0.837, 1.560, 0.331, -0.345, 0.979, 0.646, -1.253]^{\mathrm{T}}.$$

By adding the redistribution axial force vector which is derived from the internal force matrix and the original inner force of the damaged component to the original axial force vector, the internal force vector of the new structure can be obtained immediately. The inner forces of the damaged structure without component (1) are

$$N_i'=N_i+B_{i1}N_1/(1-B_{11}),\ i\neq 1,$$
$$\boldsymbol{N}'=[\mathrm{N},\ 0.930,\ 0.656,\ 0.147,\ 1.471,\ 0.971,\ -1.883]^{\mathrm{T}}.$$

Again, if the second member (4) is removed, the internal force matrix without members (1) or (4) can be obtained by

$$\boldsymbol{B}''(4)=\begin{bmatrix} \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} \\ \mathrm{N} & 1 & 0 & \mathrm{N} & 0 & 0 & 0 \\ \mathrm{N} & 0 & 1 & \mathrm{N} & 0 & 0 & 0 \\ \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} \\ \mathrm{N} & 0 & 0 & \mathrm{N} & 0.544 & 0.351 & -0.175 \\ \mathrm{N} & 0 & 0 & \mathrm{N} & 0.566 & 0.565 & 0.218 \\ \mathrm{N} & 0 & 0 & \mathrm{N} & -0.283 & 0.218 & 0.891 \end{bmatrix}.$$

From the new matrix $\boldsymbol{B}''$, the diagonal elements $B''_{2,2}=B''_{3,3}=1$ denote that the components (2) and (3) are necessary bars after components (1) and (4) are taken away. The removal of member (2) or (3) will cause the whole structure to change to a mechanism. Of course, the result is the same when it is recalculated using the methods of matrix displacement. Actually, members (2) and (3) are statically determinate in the disturbed system; therefore, no compatible deformation originates from construction errors, settlement of supports, and so on.

The inner forces of the damaged structure without components (1) and (4) are

$$N_i''=N_i'+B_{i4}'N_4'/(1-B_{44}'),\ i\neq 1 \text{ and } i\neq 4,$$
$$\boldsymbol{N}''=[\mathrm{N},\ 0.839,\ 0.839,\ \mathrm{N},\ 1.471,\ 0.971,\ -1.883]^{\mathrm{T}}.$$

Generally speaking, based on Eqs. (19), (28) and (29), the inner force increments of the remnant components are calculated after a certain member is removed. By adding the inner force increments to the former inner forces, the internal forces of the new structure are obtained.

4.3 Comparison with the finite element method

With the finite element method (FEM), the results of the first stage and the second stage are also shown in Table 1.

Table 1 shows that the results obtained by the transfer law method are the same as that calculated with the FEM. As the simplified method avoids rebuilding and decomposing the global stiffness matrix and computing the joints displacements, it is more efficient in internal force redistribution. Furthermore, according to the diagonal element of the inner force matrix, we can predict whether the structure will turn into a mechanism due to the removal of that member.

However, if we use the FEM, this judgment depends on the determinant value of the global stiffness matrix, and this is not very efficient. Therefore, the simplified method has advantages over the FEM in the identification of the failure modes in structure reliability. The structural system behavior under the loss of an arbitrary member or several members can be measured by the inner force matrix, and the key member identification is easier. Designers take measures to strengthen the key member or provide backup load paths, and so on. For this reason, the method can be used in robustness analysis to guarantee the structure against the accidental load.

Table 1 Results of the transfer law method and FEM

Member	Inner forces (kN)	
Stage 1: without member (1)		
	Transfer law method	FEM
2	0.930	0.930
3	0.656	0.656
4	0.147	0.147
5	1.471	1.471
6	0.971	0.971
7	−1.883	−1.883
Stage 2: without members (1) and (4)		
Member	Inner forces (kN)	
	Transfer law method	FEM
2	0.839	0.839
3	0.839	0.839
5	1.471	1.471
6	0.971	0.971
7	−1.883	−1.883

5 Conclusions

This paper deals with the internal force matrix and the deformation matrix, both of which can be used to estimate the redundancy of a component and analyze the topological relationship of a structure. Based on the reciprocal theorem, the relationship between the two matrixes is obtained. The expression greatly reduces the calculation of the internal force matrix whose diagonal elements are used in the redistribution of inner forces and as the important indices for assessment of structural components. The transfer law establishes the quantitative relation from the former matrix to the latter one due to the removal of components, and explains how the inner force transfers from the broken member to the remnant structure. The principle of linear superposition and the reciprocal theorem are universal in linear elastic structure; hence, the expressions above are applicable to all linear truss structures, no matter how complicated. Finally, the methods can be applied to rigid frame structures.

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