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Consolidation solution for composite foundation considering a time- and depth-dependent stress increment along with three distribution patterns of soil permeability*

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Abstract: In actual engineering practice, the stress increment within a composite foundation caused by external loads may vary simultaneously with depth and time. In addition, column installation always leads to a decay of soil permeability towards the column. However, almost none of the consolidation theories for composite foundation comprehensively consider these factors until now. For this reason, a stress increment due to external loads changing simultaneously with time and depth was incorporated into the analysis, and three possible variation patterns of soil's horizontal permeability coefficient were considered to account for the detrimental influence of column installation. These three patterns included a constant distribution pattern (Pattern I), a linear distribution pattern (Pattern II), and a parabolic distribution pattern (Pattern III). Solutions were obtained for the average excess pore water pressures and the average degree of consolidation respectively. Then several special cases were discussed in detail based on the general solution obtained. Finally, comparisons were made, and the results show that the present solution is the most general rigorous solution in the literature, and it can be broken down into a number of previous solutions. The consolidation rate is accelerated with the increase in the value of the top to the bottom stress ratio. The consolidation rate calculated by the solution for Pattern I is less than that for Pattern II, which in turn is less than that for Pattern III.

Key words: Consolidation, Composite foundation, Stone column, Permeability, Ramp load, Disturbance effect
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1 Introduction

A system of stone columns combined with preloading is one of the most commonly used and effective techniques to achieve a rapid consolidation and enhance the strength and stiffness of soft clay. Usually, an external load equal to or greater than the expected foundation loading is applied over the soil surface to accelerate consolidation. The acceleration of consolidation was accredited to stone columns for shortening the drainage path and relieving the excess pore water pressures by rapid pore water dissipation through the column.

In a conventional approach, the external load is usually assumed to be applied instantly and the corresponding stress increment resulted within the foundation is always considered to be uniformly distributed along the column depth. In fact, a number of consolidation theories for composite foundation (Basu and Prezzi, 2007; Castro and Sagaseta, 2009; Xie *et al.*, 2009) still utilized such a stress increment independent of time and depth for its simplicity and effectiveness in solving engineering problems. However, in actual

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engineering projects, the external load is not applied instantly, but applied until the final load is reached. This is called ramp loading, and it is a commonly used loading method in practice (Wang *et al*., 2002; Leo, 2004; Zhu and Yin, 2004). On the other hand, except for the time-dependence, another significant characteristic of the external loads is that the stress increment caused is always depth-dependent. In fact, when the external load is applied on a limited area, it is only at the location close to the centerline that the stress increment is held constant in a vertical direction. Therefore, the stress increment due to the external loads should be a function dependent on time and depth. However, to our knowledge, almost none of the consolidation analyses of composite foundation reflected this characteristic of the stress increment due to external loads.

As is well known, column construction always results in the disturbance of soil zone adjacent to the column, which inevitably reduces the permeability of the disturbed soil zone. The conventional method to consider this disturbance effect is to assume a reduced permeability that is held constant throughout the entire disturbed soil zone. However, many researches (Indraratna and Redana, 1998; Chai and Miura, 1999; Sharma and Xiao, 2000) have indicated that the disturbance of soil in the adjacent soil zone increases towards the column. As a result of this, the permeability in the disturbed soil zone should be of continuous variation with a radial distance away from the column, and a linear (Zhang *et al*., 2006; Basu and Prezzi, 2007; Walker and Indraratna, 2007) or a parabolic distribution (Walker and Indraratna, 2006) of soil permeability within the disturbed soil zone was assumed. In the present paper, these three distribution patterns, including a constant distribution (Pattern I), a linear distribution (Pattern II), and a parabolic distribution (Pattern III) of the horizontal permeability coefficient in the disturbed soil zone, are incorporated to indicate the detrimental influence of column construction on the surrounding soil. Moreover, the influence of various distribution patterns on the consolidation behavior of composite foundation will be compared and discussed.

The excess pore water pressure within the soil could be larger than that within the column at the beginning of loading (Balaam and Booker, 1981; Han and Ye, 2002). Therefore, the conventional initial condition for the vertical drain, which assumed the excess pore water pressure within the soil is the same as that within the drain well, is no longer suitable for the consolidation of the stone column-reinforced composite foundation. For this reason, a new initial condition is proposed from the equal strain assumption and equilibrium condition, and is utilized in solving the governing equations of the consolidation problem in this paper.

The objective of this paper was to obtain a closed-form solution to the consolidation of composite foundation by comprehensively considering the above mentioned factors. These factors included a time- and depth-dependent stress increment caused by the external load together with three possible variation patterns of the soil permeability coefficient. Then, a new initial condition can be derived to solve the governing equations obtained for the consolidation problem. In particular, in order to reflect the edge effect in the actual engineering, four types of distribution patterns of the stress increment along the vertical direction were presented in detail after the general solution was obtained.

2 Derivation of the solutions

As shown in Fig. 1, a representative unit cell was selected to study the consolidation of the composite foundation, where a stone column is surrounded by the disturbed soil zone, which is in turn surrounded by the undisturbed soil zone. In order to obtain a simplified closed-form solution, the following basic assumptions were made during the derivation:

1. Each stone column has a circular influence zone, and no flow passes through the external radial surface and the bottom of the soil layer. In addition, the ground surface is always free-draining.

2. Stone columns and the surrounding soil only deform vertically, and have an equal strain at any depth.

3. Darcy's flow is obeyed.

4. The radial flow within the stone column is not considered. As revealed by Lu *et al*. (2010), this assumption implied that the column is assumed to be infinitely permeable in the radial direction; that is, the excess pore water pressure within the column is uniform at any depth.

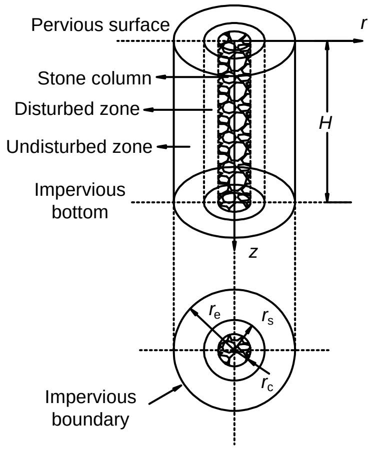

Fig. 1 Schematic diagram of a cylindrical unit representing a stone column and the surrounding soil
r_c, r_s, and r_e are the radii of the column, disturbed soil zone, and influence zone, respectively; H is the column length; and r and z denote the radial and vertical coordinates, respectively

As shown in Fig. 2, in actual engineering practice, the external load was usually applied gradually until the final value was obtained, and was then kept unchanged. Additionally, the stress increment due to the external load was also linearly distributed but not held constant in the vertical direction. In respect of the depth-dependent characteristic, four types of distribution patterns of stress increment are discussed, among which the most representative pattern is the trapezoidal one. Therefore, the average vertical stress increment in the composite foundation caused by the external load was assumed to be

$$\sigma(z,t)=\begin{cases}\left[P_T+(P_B-P_T)z/H\right]\dfrac{t}{t_c}, & t<t_c,\\ P_T+(P_B-P_T)z/H, & t\geq t_c,\end{cases} \quad (1)$$

where P_T and P_B are the final total average stresses at the top surface and bottom of a composite foundation, respectively, H is the column length, and t_c is the loading period.

In addition, three variation patterns of the horizontal permeability coefficient of the disturbed soil as shown in Fig. 3 are included to consider the detrimental influence of column construction on the surrounding soil. Similarly, a generalized expression of the horizontal permeability coefficient of soil for these three patterns is assumed to be

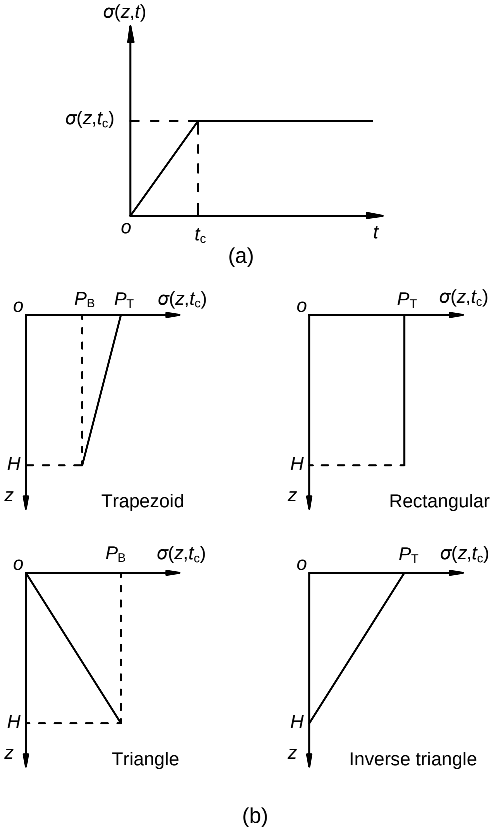

Fig. 2 Total stress due to external loads within the entire composite foundation
(a) Total stress varying with time; (b) Four types of distribution patterns of stress increment along the column depth

$$k_r(r)=k_h f(r), \quad (2)$$

where $k_r(r)$ is the horizontal permeability coefficient of the surrounding soil at any point, increasing from k_s to k_h; k_s is the horizontal permeability of soil at the column-soil interface; and k_h is the horizontal permeability of the undisturbed soil. $f(r)$ is a function depicting the variation pattern of the horizontal permeability throughout the surrounding soil zone, and different variation patterns of permeability coefficient have different $f(r)$ expressions.

The equilibrium condition is always valid for the soil-column system; that is, the total stress in a composite foundation is shared by both the column and the surrounding soil at any time:

$$\pi(r_e^2-r_c^2)\bar{\sigma}_s(z,t)+\pi r_c^2\bar{\sigma}_c(z,t)=\pi r_e^2\sigma(z,t), \quad (3)$$

where $\bar{\sigma}_c(z,t)$ and $\bar{\sigma}_s(z,t)$ are the average stress increments within the column and within the surrounding soil, respectively, and r_c and r_e are the radii of the column and the influence zone, respectively.

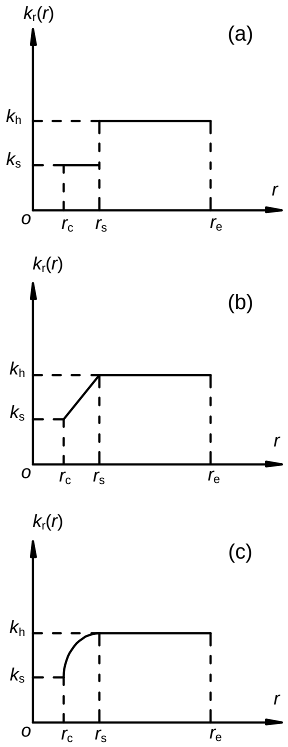

Fig. 3 Horizontal permeability coefficient of the surrounding soil. (a) Pattern I; (b) Pattern II; (c) Pattern III

With the equal strain assumption, the following relations can be obtained:

$$\frac{\bar{\sigma}_s(z,t)-\bar{u}_s(z,t)}{E_s}=\frac{\bar{\sigma}_c(z,t)-u_c(z,t)}{E_c}=\varepsilon_v, \quad (4)$$

where E_c and E_s are the compression moduli of the column and the surrounding soil, respectively. In actual engineering practice, E_c will vary with depth as the stone column is a granular material with high dependence on confining pressure. However, this characteristic is not considered in the analysis, and E_c is assumed to be independent of depth. ε_v is the vertical strain of the column and the surrounding soil, $u_c(z, t)$ is the excess pore water pressure within the column at any depth, and $\bar{u}_s(z,t)$ is the average excess pore water pressure within the surrounding soil, which is averaged in terms of radius as follows:

$$\bar{u}_s=\frac{1}{\pi(r_e^2-r_c^2)}\int_{r_c}^{r_e}2\pi r u_s(r,z,t)\mathrm{d}r, \quad (5)$$

where $u_s(r, z, t)$ is the excess pore water pressures within the surrounding soil at any point.

From Eqs. (3) and (4), ε_v can be derived as

$$\begin{aligned}\varepsilon_v&=\frac{n^2\sigma(z,t)-[(n^2-1)\bar{u}_s(z,t)+u_c(z,t)]}{E_s(n^2-1+Y)}\\&=\frac{n^2[\sigma(z,t)-\bar{u}(z,t)]}{E_s(n^2-1+Y)},\end{aligned} \quad (6)$$

where the radius ratio $n=r_e/r_c$; Y is the compression modulus ratio of the column to the surrounding soil, $Y=E_c/E_s$; and $\bar{u}(z,t)$ is the average excess pore water pressure within the entire foundation at any depth, which is given by

$$\begin{aligned}\bar{u}&=\frac{1}{\pi r_e^2}\left(\int_{r_c}^{r_e}2\pi r u_s\mathrm{d}r+\int_0^{r_c}2\pi r u_c\mathrm{d}r\right)\\&=\frac{(n^2-1)\bar{u}_s+\bar{u}_c}{n^2}=\frac{(n^2-1)\bar{u}_s+u_c}{n^2}.\end{aligned} \quad (7)$$

As stated by the 4th assumption mentioned previously, in Eq. (7), $\bar{u}_c$ is the average excess pore water pressure within the column at any depth, which is equal to u_c, because the column is assumed to be infinitely permeable in the radial direction, and hence the excess pore water pressure within the column is held constant along the radial direction.

The rate of vertical strain can be expressed from Eq. (6) as follows:

$$\frac{\partial\varepsilon_v}{\partial t}=-\frac{n^2}{E_s(n^2-1+Y)}\left(\frac{\partial\bar{u}}{\partial t}-\frac{\partial\sigma}{\partial t}\right). \quad (8)$$

Referring to the suggestion of Wang and Jiao (2004), the consolidation equation coupling the horizontal and vertical flows in the soil can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{k_r(r)}{\gamma_w}r\frac{\partial u_s}{\partial r}\right)+\frac{k_v}{\gamma_w}\frac{\partial^2\bar{u}_s}{\partial z^2}=-\frac{\partial\varepsilon_v}{\partial t}, \quad (9)$$

where k_v is the vertical permeability coefficient of soil, and γ_w is the unit weight of water.

The boundary conditions in the radial direction are as follows:

$$\frac{\partial u_s}{\partial r}=0,\quad r=r_e, \tag{10}$$

$$u_s=u_c,\ r=r_c. \tag{11}$$

Integrating Eq. (9) with respect to r and the boundary condition in Eq. (10) yields

$$\frac{\partial u_s}{\partial r}=\frac{\gamma_w}{2k_h}\left(\frac{\partial\varepsilon_v}{\partial t}+\frac{k_v}{\gamma_w}\frac{\partial^2\bar{u}_s}{\partial z^2}\right)\left[\frac{r_e^2}{rf(r)}-\frac{r}{f(r)}\right], \tag{12}$$

where $f(r)$ is a function with respect to the radial distance from the column, which can describe the variation of the horizontal permeability coefficient of soil.

Similarly, integrating Eq. (12) and the boundary condition in Eq. (11) yields

$$u_s=u_c+\frac{\gamma_w}{2k_h}\left(\frac{\partial\varepsilon_v}{\partial t}+\frac{k_v}{\gamma_w}\frac{\partial^2\bar{u}_s}{\partial z^2}\right)\left(r_e^2A_0(r)-B_0(r)\right), \tag{13}$$

where

$$A_0(r)=\int_{r_c}^{r}\frac{\mathrm{d}\xi}{\xi f(\xi)},\quad B_0(r)=\int_{r_c}^{r}\frac{\xi\mathrm{d}\xi}{f(\xi)}, \tag{14}$$

where ξ is the radial distance ranging from r_c to r.

The average excess pore water pressure within the surrounding soil at any depth can be obtained by substituting Eq. (13) into Eq. (5):

$$\bar{u}_s=u_c+\frac{\gamma_w r_e^2F_c}{2k_h}\left(\frac{\partial\varepsilon_v}{\partial t}+\frac{k_v}{\gamma_w}\frac{\partial^2\bar{u}_s}{\partial z^2}\right), \tag{15}$$

$$F_c=\frac{2(A_1r_e^2-B_1)}{r_e^2r_c^2(n^2-1)}, \tag{16}$$

$$A_1=\int_{r_c}^{r_e}rA_0(r)\mathrm{d}r,\ B_1=\int_{r_c}^{r_e}rB_0(r)\mathrm{d}r, \tag{17}$$

where F_c is a dimensionless parameter to reflect the influence of the variation pattern of the horizontal permeability of soil along with the geometric feature of the unit cell. The detailed derivation of parameter F_c can be found in Xie *et al.* (2009).

Substituting Eqs. (7) and (8) into Eq. (15) yields

$$\bar{u}=u_c+\frac{\gamma_w r_e^2F_c}{2k_h}\left[-\frac{(n^2-1)}{E_s(n^2-1+Y)}\left(\frac{\partial\bar{u}}{\partial t}-\frac{\partial\sigma}{\partial t}\right)+\frac{k_v}{\gamma_w}\left(\frac{\partial^2\bar{u}}{\partial z^2}-\frac{1}{n^2}\frac{\partial^2u_c}{\partial z^2}\right)\right]. \tag{18}$$

To solve the above equations, the flow continuity assumption at the soil-column interface, which assumes that the water quantity flowing into the column is equal to that flowing out from the column, is needed. Considering the well resistance, the flow continuity assumption is given by

$$\left[2\pi r\frac{k_r(r)}{\gamma_w}\frac{\partial u_s}{\partial r}\right]\Bigg|_{r=r_c}=-\pi r_c^2\frac{k_c}{\gamma_w}\frac{\partial^2u_c}{\partial z^2}. \tag{19}$$

The following relation can be derived by substituting Eq. (12) into Eq. (19):

$$\frac{k_c}{\gamma_w}\frac{\partial^2u_c}{\partial z^2}=-(n^2-1)\left(\frac{\partial\varepsilon_v}{\partial t}+\frac{k_v}{\gamma_w}\frac{\partial^2\bar{u}_s}{\partial z^2}\right). \tag{20}$$

Substituting Eq. (15) into Eq. (20) and combining with Eq. (7) yield

$$\bar{u}=u_c-A\frac{\partial^2u_c}{\partial z^2}, \tag{21}$$

where A is a positive constant, $A=\dfrac{r_c^2F_ck_c}{2k_h}$.

Then, a nonhomogeneous partial differential equation with respect to the variable u_c can be obtained by substituting Eq. (21) into Eq. (18) as

$$B\frac{\partial^4u_c}{\partial z^4}+C\frac{\partial^2u_c}{\partial z^2}+A\frac{\partial^3u_c}{\partial t\partial z^2}-\frac{\partial u_c}{\partial t}=-\frac{\partial\sigma}{\partial t}, \tag{22}$$

where B and C are negative and positive constants, respectively, and they are given by

$$B=-2k_h\frac{E_sk_vr_c^2F_ck_c(n^2-1+Y)}{(n^2-1)\gamma_w},$$

$$C=\frac{E_s(n^2-1+Y)k_c}{n^2(n^2-1)\gamma_w}\left[1+(n^2-1)\frac{k_v}{k_c}\right].$$

Eqs. (21) and (22) are the governing equations for the consolidation problem in this study. To solve these, the corresponding boundary conditions in the vertical direction and the initial condition are needed. As shown in Fig. 1, the top surface of the foundation is pervious and the bottom is impervious. Therefore, the boundary conditions in the vertical directions are as follows:

$$\bar{u}(z,t)=0,\ u_{\mathrm{c}}(z,t)=0,\qquad z=0,\tag{23}$$

$$\frac{\partial\bar{u}(z,t)}{\partial z}=0,\ \frac{\partial u_{\mathrm{c}}(z,t)}{\partial z}=0,\qquad z=H.\tag{24}$$

At the beginning of loading, no deformations occur in the column and the surrounding soil. Therefore the vertical strains either in the column or in the surrounding soil are equal to zero. Then the following relations can be derived from Eq. (4):

$$\bar{\sigma}_{\mathrm{s}}=\bar{u}_{\mathrm{s}},\ \bar{\sigma}_{\mathrm{c}}=u_{\mathrm{c}},\qquad t=0.\tag{25}$$

Substituting Eq. (25) into Eq. (3) and combining with Eq. (7) give the new initial condition as follows:

$$\bar{u}=\sigma(z,0)=0,\qquad t=0.\tag{26}$$

Referring to the solution for the homogeneous partial differential equation corresponding to the nonhomogeneous one developed by Zhang *et al*. (2006), the solution to Eq. (22) can be assumed as follows by introducing a Fourier series:

$$u_{\mathrm{c}}=\sum_{m=1}^{\infty}T_m(t)\sin(Mz/H),\tag{27}$$

where $T_m(t)$ is a function dependent on time, H is the thickness of the composite foundation, and $M=(2m-1)\pi/2$ (m=1, 2, …). The average excess pore water pressure in a composite foundation at any depth can be obtained by substituting Eq. (27) into Eq. (21):

$$\bar{u}=\sum_{m=1}^{\infty}\left[1+A(M/H)^2\right]T_m(t)\sin(Mz/H).\tag{28}$$

Eqs. (27) and (28) satisfy the boundary conditions in Eqs. (23) and (24).

By substituting Eq. (27) into Eq. (22), the partial differential equation can be rewritten as

$$\begin{aligned}&B\sum_{m=1}^{\infty}T_m(t)(M/H)^4\sin(Mz/H)\\&-C\sum_{m=1}^{\infty}T_m(t)(M/H)^2\sin(Mz/H)\\&-A\sum_{m=1}^{\infty}T_m'(t)(M/H)^2\sin(Mz/H)\\&-\sum_{m=1}^{\infty}T_m'(t)\sin(Mz/H)=-\frac{\partial\sigma(z,t)}{\partial t}.\end{aligned}\tag{29}$$

For the orthogonality of the Fourier series, the following equation can be obtained by multiplying $\sin(Mz/H)$ on both sides of Eq. (29):

$$T_m'(t)+\beta_mT_m(t)=Q_m(t),\tag{30}$$

$$\beta_m=-\frac{B(M/H)^4-C(M/H)^2}{1+A(M/H)^2},\tag{31}$$

$$Q_m(t)=\frac{2\int_0^H\frac{\partial\sigma(z,t)}{\partial t}\sin(Mz/H)\mathrm{d}z}{H\left[1+A(M/H)^2\right]}.\tag{32}$$

Combining with Eq. (1), Eq. (32) can be simplified as

$$Q_m(t)=\begin{cases}\dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\left[1+A(M/H)^2\right]}, & t<t_{\mathrm{c}},\\ 0, & t\ge t_{\mathrm{c}}.\end{cases}\tag{33}$$

Substituting Eq. (28) into the initial condition in Eq. (26) yields

$$T_m(0)=0.\tag{34}$$

Eqs. (30) and (34) are the ordinary differential equation and its initial condition, respectively. By using Eq. (34), the solution for Eq. (30) can be readily obtained as

$$T_m(t)=\mathrm{e}^{-\beta_m t}\int_0^t Q_m(\tau)\mathrm{e}^{-\beta_m\tau}\mathrm{d}\tau.\tag{35}$$

where τ denotes the time, and it ranges from zero to a certain moment t.

The solution of $T_m(t)$ can be further obtained in detail by substituting Eq. (33) into Eq. (35) as

$$T_m=\begin{cases}\dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\beta_m\left[1+A\left(M/H\right)^2\right]}\\ \times(1-\mathrm{e}^{-\beta_m t}), & t<t_{\mathrm{c}},\\ \dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\beta_m\left[1+A\left(M/H\right)^2\right]}\\ \times(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t}), & t\geq t_{\mathrm{c}}.\end{cases}\tag{36}$$

By substituting Eq. (36) into Eqs. (27) and (28), the average excess pore water pressures within the column and within the entire foundation at any depth can be then obtained as follows:

$$u_{\mathrm{c}}=\begin{cases}\sum_{m=1}^{\infty}\dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\beta_m\left[1+A(M/H)^2\right]}\\ \times(1-\mathrm{e}^{-\beta_m t})\sin(Mz/H), & t<t_{\mathrm{c}},\\ \sum_{m=1}^{\infty}\dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\beta_m\left[1+A(M/H)^2\right]}\\ \times(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t})\sin(Mz/H), & t\geq t_{\mathrm{c}},\end{cases}\tag{37}$$

$$\bar{u}=\begin{cases}\sum_{m=1}^{\infty}\dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\beta_m}\\ \times(1-\mathrm{e}^{-\beta_m t})\sin(Mz/H), & t<t_{\mathrm{c}},\\ \sum_{m=1}^{\infty}\dfrac{2\left[P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M\right]}{Mt_{\mathrm{c}}\beta_m}\\ \times(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t})\sin(Mz/H), & t\geq t_{\mathrm{c}}.\end{cases}\tag{38}$$

After obtaining the solution for excess pore water pressure, the solution for the average degree of consolidation can be derived. Usually, the average degree of consolidation is defined as the average effective stress to the final average total stress in a composite foundation:

$$U(t)=\frac{\int_0^H\left[\sigma(z,t)-\bar{u}(z,t)\right]\mathrm{d}z}{\int_0^H\sigma(z,\infty)\mathrm{d}z}.\tag{39}$$

Substituting Eq. (38) into Eq. (39) and combining with Eq. (1), the average degree of consolidation can be determined as follows:

$$U(t)=\begin{cases}\dfrac{t}{t_{\mathrm{c}}}-\dfrac{4}{P_{\mathrm{B}}+P_{\mathrm{T}}}\sum_{m=1}^{\infty}\dfrac{P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M}{M^2t_{\mathrm{c}}\beta_m}\\ \times(1-\mathrm{e}^{-\beta_m t}), & t<t_{\mathrm{c}},\\ 1-\dfrac{4}{P_{\mathrm{B}}+P_{\mathrm{T}}}\sum_{m=1}^{\infty}\dfrac{P_{\mathrm{T}}-(-1)^m(P_{\mathrm{B}}-P_{\mathrm{T}})/M}{M^2t_{\mathrm{c}}\beta_m}\\ \times(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t}), & t\geq t_{\mathrm{c}}.\end{cases}\tag{40}$$

Until now, all the solutions for the consolidation problem under consideration are obtained. In the following section, the solutions for several special cases are presented in detail.

3 Solutions for several special cases

3.1 Solution for the case of rectangular distribution of stress increment

When $P_{\mathrm{T}}=P_{\mathrm{B}}$, the trapezoidal distribution pattern can be reduced to the rectangular pattern. Therefore, the solutions for the case of rectangular pattern can be obtained by letting $P_{\mathrm{T}}=P_{\mathrm{B}}$ in Eqs. (37), (38), and (40):

$$u_{\mathrm{c}}=\begin{cases}\dfrac{2P_{\mathrm{T}}}{t_{\mathrm{c}}}\sum_{m=1}^{\infty}\dfrac{(1-\mathrm{e}^{-\beta_m t})\sin(Mz/H)}{M\beta_m\left[1+A(M/H)^2\right]}, & t<t_{\mathrm{c}},\\ \dfrac{2P_{\mathrm{T}}}{t_{\mathrm{c}}}\sum_{m=1}^{\infty}\dfrac{(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t})\sin(Mz/H)}{M\beta_m\left[1+A(M/H)^2\right]}, & t\geq t_{\mathrm{c}},\end{cases}\tag{41}$$

$$\bar{u}=\begin{cases}\dfrac{2P_{\mathrm{T}}}{t_{\mathrm{c}}}\sum_{m=1}^{\infty}(1-\mathrm{e}^{-\beta_m t})\sin(Mz/H), & t<t_{\mathrm{c}},\\ \dfrac{2P_{\mathrm{T}}}{t_{\mathrm{c}}}\sum_{m=1}^{\infty}\dfrac{(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t})\sin(Mz/H)}{M\beta_m}, & t\geq t_{\mathrm{c}},\end{cases}\tag{42}$$

$$U(t)=\begin{cases}\dfrac{t}{t_{\mathrm{c}}}-\dfrac{2}{t_{\mathrm{c}}}\sum_{m=1}^{\infty}\dfrac{(1-\mathrm{e}^{-\beta_m t})}{M^2\beta_m}, & t<t_{\mathrm{c}},\\ 1-\dfrac{2}{t_{\mathrm{c}}}\sum_{m=1}^{\infty}\dfrac{(\mathrm{e}^{-\beta_m(t-t_{\mathrm{c}})}-\mathrm{e}^{-\beta_m t})}{M^2\beta_m}, & t\geq t_{\mathrm{c}}.\end{cases}\tag{43}$$

The above solution for the case of Pattern I, the same as that developed by Wang *et al*. (2002), only considered the first pattern (Pattern I) of the horizontal permeability coefficient of soil within the disturbed soil zone, without taking into account the gradual variation of the horizontal permeability coefficient of soil in the disturbed soil zone.

Furthermore, if letting $t_c \to 0$, Eq. (43) can be reduced to

$$U(t) = 1 - \sum_{m=1}^{\infty} \frac{2}{M^2} \mathrm{e}^{-\beta_m t}. \tag{44}$$

This is the solution provided by Zhang *et al*. (2006), assuming that the external load was applied instantly and the stress increment in the composite foundation was kept constant along the column depth. However, Zhang *et al*. (2006) only considered the linear variation pattern of the horizontal permeability coefficient of soil in the disturbed soil zone (Pattern II). From the above analysis, it can be seen that the present solution can be reduced to the solution provided by Zhang *et al*. (2006) after two forms of degeneration by letting $P_T=P_B$ and $t_c \to 0$.

3.2 Solution for the case of triangular distribution of stress increment

Similarly, if letting $P_T=0$, the solution for the case of the triangular distribution pattern can be derived from Eqs. (37), (38), and (40) as follows:

$$u_c = \begin{cases} \dfrac{2P_B}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{-(-1)^m \sin(Mz/H)}{M^2 \beta_m \left[1 + A(M/H)^2\right]} \\ \quad \times (1 - \mathrm{e}^{-\beta_m t}), & t < t_c, \\ \dfrac{2P_B}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{-(-1)^m \sin(Mz/H)}{M^2 \beta_m \left[1 + A(M/H)^2\right]} \\ \quad \times (\mathrm{e}^{-\beta_m (t-t_c)} - \mathrm{e}^{-\beta_m t}), & t \ge t_c, \end{cases} \tag{45}$$

$$\bar{u} = \begin{cases} \dfrac{2P_B}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{-(-1)^m \sin(Mz/H)}{M^2 \beta_m}, \\ \quad \times (1 - \mathrm{e}^{-\beta_m t}), & t < t_c, \\ \dfrac{2P_B}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{-(-1)^m \sin(Mz/H)}{M^2 \beta_m}, \\ \quad \times (\mathrm{e}^{-\beta_m (t-t_c)} - \mathrm{e}^{-\beta_m t}), & t \ge t_c, \end{cases} \tag{46}$$

$$U(t) = \begin{cases} \dfrac{t}{t_c} + \dfrac{4}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{(-1)^m}{M^3 \beta_m} (1 - \mathrm{e}^{-\beta_m t}), & t < t_c, \\ 1 + \dfrac{4}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{(-1)^m}{M^3 \beta_m} (\mathrm{e}^{-\beta_m (t-t_c)} - \mathrm{e}^{-\beta_m t}), & t \ge t_c. \end{cases} \tag{47}$$

3.3 Solution for the case of inverse triangular distribution of stress increment

Similarly, the solutions for the case of the inverse triangular distribution of stress increment can be obtained by letting $P_B=0$ from Eqs. (37), (38), and (40) as follows:

$$u_c = \begin{cases} \dfrac{2P_T}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{\left[1 + (-1)^m / M\right] \sin(Mz/H)}{M \beta_m \left[1 + A(M/H)^2\right]} \\ \quad \times (1 - \mathrm{e}^{-\beta_m t}), & t < t_c, \\ \dfrac{2P_T}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{\left[1 + (-1)^m / M\right] \sin(Mz/H)}{M \beta_m \left[1 + A(M/H)^2\right]} \\ \quad \times (\mathrm{e}^{-\beta_m (t-t_c)} - \mathrm{e}^{-\beta_m t}), & t \ge t_c, \end{cases} \tag{48}$$

$$\bar{u} = \begin{cases} \dfrac{2P_T}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{\left[1 + (-1)^m / M\right] \sin(Mz/H)}{M \beta_m} \\ \quad \times (1 - \mathrm{e}^{-\beta_m t}), & t < t_c, \\ \dfrac{2P_T}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{\left[1 + (-1)^m / M\right] \sin(Mz/H)}{M \beta_m} \\ \quad \times (\mathrm{e}^{-\beta_m (t-t_c)} - \mathrm{e}^{-\beta_m t}), & t \ge t_c, \end{cases} \tag{49}$$

$$U(t) = \begin{cases} \dfrac{t}{t_c} - \dfrac{4}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{M + (-1)^m}{M^3 \beta_m} \times (1 - \mathrm{e}^{-\beta_m t}), & t < t_c, \\ 1 - \dfrac{4}{t_c} \displaystyle\sum_{m=1}^{\infty} \dfrac{M + (-1)^m}{M^3 \beta_m} \times (\mathrm{e}^{-\beta_m (t-t_c)} - \mathrm{e}^{-\beta_m t}), & t \ge t_c. \end{cases} \tag{50}$$

4 Comparisons

In this section, the influence of various distribution patterns of stress increment in the vertical direction is firstly compared in Fig. 4. T_h and T_{hc} denote the horizontal time factors corresponding to the time moments t and t_c, $T_h = c_h t/(4r_e^2)$, and $T_{hc} = c_h t_c/(4r_e^2)$, where c_h is the horizontal consolidation coefficient of soil. It can be seen that the average degree of consolidation is at its maximum when the stress increment takes the inverse triangular distribution pattern. On the contrary, the average degree of consolidation is at its minimum for the case of triangular distribution. In addition, the trapezoidal distribution ($P_T>P_B$) always gives a larger average degree of consolidation than the rectangular distribution. In general, the average degree of consolidation increases with the increase in the value of P_T/P_B.

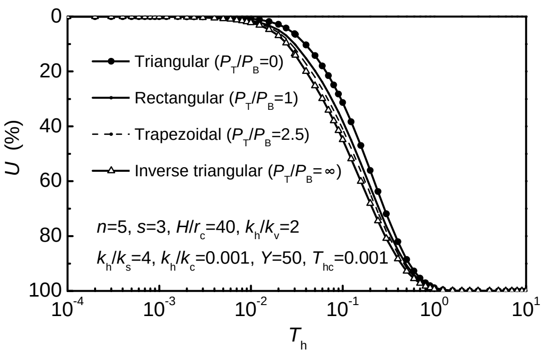

Fig. 4 Average degree of consolidation with different distribution patterns of the total stress along the column depth (Pattern I)

As shown in Fig. 5, a comparison is made to show the difference between the three variation patterns of horizontal permeability coefficient of soil in the disturbed soil zone. It can be seen that under the same calculation conditions, the average degree of consolidation for Pattern I is less than that for Pattern II, which is in turn less than that for Pattern III.

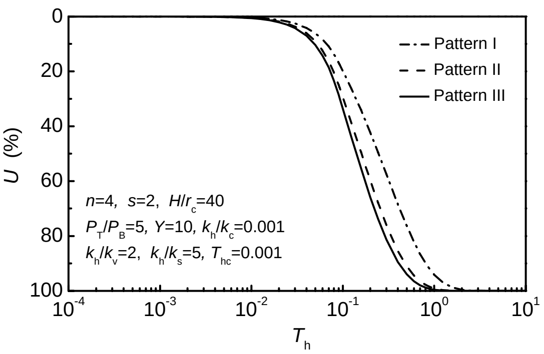

Fig. 5 Average degree of consolidation calculated by the solutions for three variation patterns of the horizontal permeability coefficient of soil in the disturbed soil zone

Finally, a comparison is made between the present solution for Pattern III and some available solutions (Han and Ye, 2002; Wang *et al*., 2002; Zhang *et al*., 2006) (Fig. 6). The distributions of horizontal permeability of soil in solutions of Wang *et al*. (2002) and Zhang *et al*. (2006) are the same as Patterns I and II in this paper, respectively. For this reason, the consolidation rate by the present solution for Pattern III is greater than that by the solution of Zhang *et al*. (2006), which in turn is greater than that by the solution of Wang *et al*. (2002). As the solution of Han and Ye (2002) ignored the vertical flow in the surrounding soil, the consolidation rate is less than that for the other solutions.

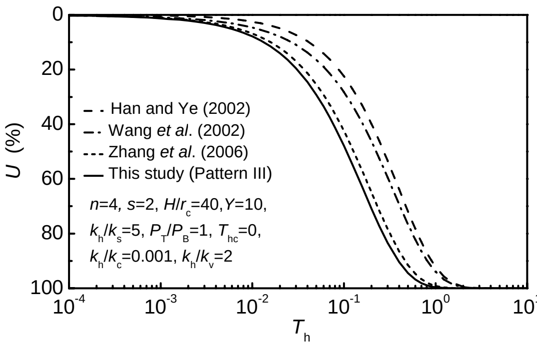

Fig. 6 Comparison of the present solution with some available solutions

5 Conclusions

This paper presented a general closed-form solution to calculate the average degree of consolidation of a composite foundation by considering several important factors. These factors included a time- and depth-dependent stress increment and three variation patterns of the horizontal permeability coefficient of the soil in the disturbed soil zone. Moreover, four types of distribution patterns of stress increment with depth are discussed, and the solutions for them are presented in detail. Several comparisons were made, and the results show that the present solution is the most general closed-form solution, which can be reduced to many solutions in the literature. The average degree of consolidation increases with the increase in the value of the ratio of the top to the bottom total stress. The solution for Pattern III always gives a larger consolidation rate than those of the other two patterns, whereas Pattern I produces the minimum consolidation rate amongst the three patterns.

References

Balaam, N.P., Booker, J.R., 1981. Analysis of rigid rafts supported by granular piles. *International Journal for Numerical and Analytical Method in Geomechanics*, **5**(4): 379-403. [doi:10.1002/nag.1610050405]

Basu, D., Prezzi, M., 2007. Effect of the smear and transition zones around PVD installed in a triangular pattern on the rate of soil consolidation. *International Journal of Geomechanics*, **7**(1):34-43. [doi:10.1061/(ASCE)1532-3641(2007)7:1(34)]

Castro, J., Sagaseta, C., 2009. Consolidation around stone columns: influence of column deformation. *International Journal for Numerical and Analytical Methods in Geomechanics*, **33**(7):851-877. [doi:10.1002/nag.745]

Chai, J.C., Miura, N., 1999. Investigation of factors affecting vertical drain behavior. *Journal of Geotechnical and Geoenvironmental Engineering*, **125**(3):216-226. [doi:10.1061/(ASCE)1090-0241(1999)125:3(216)]

Han, J., Ye, S.L., 2002. A theoretical solution for consolidation rates of stone column-reinforced foundations accounting for smear and well resistance effects. *International Journal of Geomechanics*, **2**(2):135-151. [doi:10.1061/(ASCE)1532-3641(2002)2:2(135)]

Indraratna, B., Redana, I.W., 1998. Laboratory determination of smear zone due to vertical drain installation. *Journal of Geotechnical and Geoenvironmental Engineering*, **124**(2):180-184. [doi:10.1061/(ASCE)1090-0241(1998)124:2(180)]

Leo, C.J., 2004. Equal strain consolidation by vertical drains. *Journal of Geotechnical and Geoenvironmental Engineering*, **130**(3):316-327. [doi:10.1061/(ASCE)1090-0241(2004)130:3(316)]

Lu, M.M., Xie, K.H., Guo, B., 2010. Consolidation theory for a composite foundation considering radial and vertical flows within the column and the variation of soil permeability within the disturbed soil zone. *Canadian Geotechnical Journal*, **47**(2):207-217. [doi:10.1139/T09-086]

Sharma, J.S., Xiao, D., 2000. Characterization of a smear zone around vertical drains by large-scale laboratory tests. *Canadian Geotechnical Journal*, **37**(6):1265-1271. [doi:10.1139/cgj-37-6-1265]

Walker, R., Indraratna, B., 2006. Vertical drain consolidation with parabolic distribution of permeability in smear zone. *Journal of Geotechnical and Geoenvironmental Engineering*, **132**(7):937-941. [doi:10.1061/(ASCE)1090-0241(2006)132:7(937)]

Walker, R., Indraratna, B., 2007. Vertical drain consolidation with overlapping smear zones. *Geotechnique*, **57**(5): 463-467. [doi:10.1680/geot.2007.57.5.463]

Wang, R.C., Xie, K.H., Guan, S.H., 2002. Analytical solutions for consolidation of composite ground with granular columns under time-dependent loading. *Journal of Zhejiang University (Engineering Science)*, **36**(1):12-16 (in Chinese).

Wang, X.S., Jiao, J.J., 2004. Analysis of soil consolidation by vertical drains with double porosity model. *International Journal for Numerical and Analytical Method in Geomechanics*, **28**(14):1385-1400. [doi:10.1002/nag.391]

Xie, K.H., Lu, M.M., Liu, G.B., 2009. Equal strain consolidation for stone-column reinforced foundation. *International Journal for Numerical and Analytical Methods in Geomechanics*, **33**(15):1721-1735. [doi:10.1002/nag.790]

Zhang, Y.G., Xie, K.H., Wang, Z., 2006. Consolidation Analysis of Composite Ground Improved by Granular Columns Considering Variation of Permeability Coefficient of Soil. Proceedings of Sessions of Geo Shanghai, Ground Modification and Seismic Mitigation, ASCE, Shanghai, China, **152**:135-142. [doi:10.1061/40864(196)19]

Zhu, G.F., Yin, J.H., 2004. Consolidation analysis of soil with vertical and horizontal drainage under ramp loading considering smear effects. *Geotextiles and Geomembranes*, **22**(1-2):63-74. [doi:10.1016/S0266-1144(03)00052-9]