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Velocity distribution and scaling properties of wall bounded flow*

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Abstract: The scaling and similarity of wall bounded turbulent flow were studied. The properties of such flows and the relationship between a power law and a logarithmic type of velocity distribution were investigated. Based on the physical mechanism involved, our results show that the power law and the logarithmic distribution are only different forms with the same hypothesis and hold only in the outer flow zone. Thus, a universal explanation for various empirical formulae of velocity distribution was obtained. Manning's formula was studied to explain theoretically the experiential result that the roughness coefficient is only a comprehensive parameter of the whole system without a corresponding physical factor. The physical mechanism of the velocity distribution of parallel to wall bounded flow was explored, the results show that the parameters in the formula of velocity distribution are indices of the system responding to flowing environmental factors to represent general case of boundary roughness and the flowing state, corresponding physical mechanism is vortex motion.

Key words: Velocity distribution, Scaling, Turbulence
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1 Introduction

The theoretical prediction of the velocity distribution of parallel to wall bounded flow, such as open-channel and pipe flows, remains an unsolved problem although experimental results and experiential or semi-experiential theories of velocity distribution have been accurate enough to satisfy engineering requirements. It is a key research issue both in the theory and in the engineering of fluid dynamics (Sreenivasan, 1999; Hunt, 2001). From the theoretical perspective, this problem should be approached by solving the hydrodynamic governing equation, i.e., the Navier-Stokes (N-S) equation. However, because of the un-closure problem of turbulence, the velocity distribution has been studied using other approaches, such as scaling and similarity (Barenblatt, 1996; Gioia and Bombardelli, 2001; Park and Chung, 2004; Brzek *et al*., 2007) or engineering analysis (Hogarth and Parlange, 2005). Velocity distribution is only a part of turbulence research. This problem should be further investigated using heuristic experiences of turbulence and related subjects, such as theoretical physics (She and Su, 1999; Goldenfeld, 2006; L'vov *et al*., 2008). Historically, the water surface equation of steady gradually-varied-flow, $\frac{\mathrm{d}H}{\mathrm{d}x}=(S-S_0)/(1-Fr^2)$, has been regarded as a simplified form of the Saint-Venant (S-V) equations (where H denotes the depth, x is the stream-wise distance, S is the bottom slope, S_0 is the hydraulic slope, and Fr is Froude's number). The formulas which are currently widely used, e.g., the Darcy-Weisbach formula and the Manning formula, came into being much earlier than the S-V equations, which were developed from experimental data analysis. There is still no theoreti-

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cal explanation of why the coefficients in these formulas are not universal.

In the early stages of turbulence research (from the 1910s to the 1930s), turbulence was characterized as a completely stochastic phenomenon in which a randomly fluctuating portion of the velocity field is superimposed. From the 1960s, it became apparent that large-scale motions govern the transport properties of a fluid flow while small-scale motions are responsible mainly for dissipative processes. Since then, one of the issues being discussed in turbulence research is how large-scale motions of the outer region are influenced by turbulent events (e.g., bursting) occurring in the inner region. The mechanism of the interaction between the inner and the outer layers remains controversial. But a sketch of the mean velocity profile can be drawn by distinguishing different layers: (1) a viscous sub-layer, $0<y^+<7$, where $u^+=y^+$; (2) a buffer layer, $7<y^+<50$, the region of maximum average production of turbulent kinetic energy; (3) an overlap layer, $y^+>50$, characterized by the approximate energy equilibrium; (4) a far outer layer, where the law-of-the-wake holds (we neglect this layer here), where $y^+ = u_* y / \upsilon$, $u^+ = u / u_*$, υ is the kinematic viscosity coefficient, and u_* is the shear velocity. The above sketch was validated from the flowing mechanism, for example, from the viewpoints of eddy and intermittency (Mehrafarin and Pourtolami, 2008), and from the phenomenon mechanism (Allen *et al*., 2007; Afzal, 2008). Some researchers believe that the velocity profile should have the form $u^+ = \frac{1}{\kappa}\ln(y^+) + B$ for the outer region and/or $u^+ = C(Re) y^{+\alpha(Re)} + D(Re)$ for the inner region. Thus, it is unclear whether a power law or a logarithmic type is the appropriate form to describe the velocity distribution.

In this paper, we examined these issues from the viewpoints of scaling transform and engineering applications. A zone extending some distance from the wall was considered. The scaling and similarity of wall bounded flow were developed, and the reasons why velocity distribution can be described with a power law and a logarithmic type were discussed. The results show that differences in velocity distribution between the hydraulic roughness boundary and the smoothness boundaries arise because the velocity distribution of the roughness is closely related to the local Reynolds number and is independent of the average Reynolds number. Similarities arise because the fluid flow maintains self-similarity in both cases.

2 Mean velocity distribution in overlap layer

Let $\phi = u / u_*$, $\eta = u_* y / \upsilon$. $\bar{\bar{Re}} = \bar{u} H / \upsilon$ is the Reynolds number formed by the mean velocity $\bar{u}$ and the depth of open channel flow or pipe diameter H. $Re = uy / \upsilon$ is the local Reynolds number formed by the local velocity u at y distance from the wall, $0 < y \le H$. $Re_* = u_* K_s / \upsilon$ is often called the inner Reynolds number formed by the thickness of roughness K_s. The velocity u is determined experientially by the factors of y, υ, K_s, u_*. It is known by dimension analysis that $\frac{\partial u}{\partial y} = \left(\frac{u_*}{y}\right)\phi(\eta, \bar{\bar{Re}}, Re_*)$ or $\frac{\partial \phi}{\partial \eta} = \left(\frac{1}{\eta}\right)\phi(\eta, \bar{\bar{Re}}, Re_*)$. The theories of complete similarity and non-complete similarity (Barenblatt, 1996) are applied, and then when the mean Reynolds number is very large the power type of velocity formula is obtained, and when $\bar{\bar{Re}} \to \infty$ asymptotically, the logarithmic formula is obtained. The power formula is

$$\phi = c\xi^{\beta} + D. \tag{1}$$

The logarithmic formula is

$$\phi = \frac{1}{\kappa}\ln\xi + c_1, \tag{2}$$

where

$$\xi = \frac{\eta}{Re_*} = \frac{y}{K_s}. \tag{3}$$

Let

$$\beta = \frac{c_2 \ln(Re / \bar{Re})}{\ln(Re / Re_*)} \times \frac{1}{\ln(\bar{Re} / Re_*)}, \tag{4}$$

where c_2 is a coefficient, $\bar{Re} = u_* H / \upsilon$. Clearly, the parameter β shows the extent to which the local flow is affected by the roughness and the flowing inertia at a certain distance from the wall. The parameter β can also be regarded as an index of the similarity of flow. The relationships between the local Reynolds number and the parameter β are complicated. However, near the wall flowing zone, the value of β will decrease when the value of the local Reynolds number increases with increasing distance from wall. The value of β will increase when the value of the inner Reynolds number increases. In terms of physical meaning, the Reynolds number is considered to be an index measuring the relative magnitude ratio of inertial force to viscous force. From the viewpoint of fluid dynamics, the physical meaning of parameter β is that it is an index for measuring the degree to which fluid flow has been excited, provided that the inner Reynolds number is taken as a reference value.

If the parameter β is substituted into the power type of velocity distribution, we obtain

$$\phi = c\left(\frac{y}{H}\right)^{\frac{c_2}{\ln(H/K_s)}} + D. \tag{5}$$

Let $\gamma = \dfrac{c_2}{\ln(H/K_s)}$, and use the condition y=0, u=0, D=0, we obtain:

$$\frac{u}{u_*} = c\left(\frac{y}{H}\right)^{\gamma}, \tag{6}$$

then, the power type of velocity distribution is obtained. The power law is obtained because of the intrinsic properties of self-similarity of flowing fluid. The value of parameter γ is determined by the physical condition of flowing fluid. From the viewpoint of hydraulic engineering, the parameter γ can be regarded as an index for measuring the degree of inhomogeneity of flowing fluid. It is known that for different flowing fluids, the value of parameter γ will be different. For similar flowing fluids, the value of parameter γ will also be different if the boundary condition of the flowing fluid is different. Usually, the value of parameter γ will be a statistical result.

If we repeat the exercise using a different method, using identical symbols for convenience, let

$$\xi = \frac{\eta}{u_* H/\upsilon} = \frac{y}{H},\ \beta = \frac{c_2 \ln(Re/Re_*)}{\ln(Re/\overline{Re})\ln(\overline{Re}/Re_*)}, \tag{7}$$

then, the following results can be obtained by procedures similar to those above:

$$\frac{u}{u_*} = c\left(\frac{y}{K_s}\right)^{\gamma}. \tag{8}$$

The meaning of parameter γ in the power type of velocity distribution discussed above corresponds well with the SL turbulent model (She and Su, 1999). Thus, one can understand the following formulas relating to the general basis of physical theory, e.g., Manning-Strickler's formula $\dfrac{\overline{u}}{u_*} = 7.68\left(\dfrac{H}{K_s}\right)^{1/6}$ and Englund's formula $\dfrac{\overline{u}}{u_*} = 9.45\left(\dfrac{H}{K_s}\right)^{1/8}$. Hence, the power type of velocity distribution and the logarithmic type in the roughness zone have the same physical basis. Both of them neglect viscosity while the smooth zone preserves viscosity.

Analysis of the parameter κ in the logarithmic type of velocity distribution shows that κ has a similar meaning to parameter γ in the power distribution. So we demonstrate theoretically that κ is only one parameter instead of a constant, as it is usually considered. Thus, κ should be called the Karman parameter instead of the Karman constant. The parameter γ and the Karman constant may be the function of the parameters of an SL model in steady uniform open channel flow.

3 Manning's formula

The following Manning's formula is used widely in geophysics:

$$u = \frac{1}{n} R^{\frac{2}{3}} J^{\frac{1}{2}}, \tag{9}$$

where u is the mean velocity, R is the hydraulic radius. It is usually considered that R=H for wide shallow open channel flow (H is the depth of the channel), J is the hydraulic slope of the channel, n is the roughness coefficient, and g is gravitational acceleration. As $u_* = \sqrt{gRJ}$, it is easy to know that

$$\frac{u}{u_*} = \frac{H^{1/6}}{n\sqrt{g}}. \tag{10}$$

If Eq. (8) is used to calculate the mean velocity, then

$$\frac{u}{u_*} = \frac{c}{\gamma+1}\left(\frac{H}{K_s}\right)^{\gamma}. \tag{11}$$

Comparing Eq. (10) with Eq. (11) shows that the roughness coefficient n is a whole system parameter without a corresponding physical factor. So we have clearly demonstrated again that Manning's formula is valid only in the turbulent zone and the roughness coefficient is only a coefficient. Chezy's formula can also be explained in the same way.

4 Properties of scaling

If s denotes stream-wise direction, the S-V equations is given as

$$\begin{cases} \dfrac{\partial H}{\partial t} + \overline{u}\dfrac{\partial H}{\partial s} + \dfrac{A}{B}\dfrac{\partial \overline{u}}{\partial s} = 0, \\ \dfrac{\partial \overline{u}}{\partial t} + \overline{u}\dfrac{\partial \overline{u}}{\partial s} + g\dfrac{\partial H}{\partial s} = g(i-j), \end{cases} \tag{12}$$

where A is the area of cross-section, $B = \dfrac{\partial A}{\partial s}$, and i, j are the slops of bottom and energy, respectively.

Let x denote the variable of length. The meanings of other parameters are the same as those above. λ is a parameter, if Eq. (12) is transformed with the following formula:

$$\begin{cases} \widehat{x} = \lambda^{\beta_1} x, \\ \widehat{t} = \lambda^{\beta_2} t, \\ \widehat{u} = \lambda^{\beta_1-\beta_2} u, \\ \widehat{g} = \lambda^{\beta_1-2\beta_2} g, \end{cases} \tag{13}$$

then the form of Eq. (12) remains invariant, the dimensionless parameter of Froude's number, $Fr = u/\sqrt{gH}$, also remains invariant.

For a specific cross-section, the parameter values of c, D, κ and c_1 in a power law or a logarithmic type of velocity distribution cannot be determined theoretically because there are no general constant values which apply to all flowing states and zones. As shown above, the values of c, D, κ and c_1 are functions of both the flowing state and the flowing environment, such as boundary conditions and initial conditions. But from the viewpoint of the mechanism of phenomenology, the infinite hierarchy structures of fluid flow are only one deterministic process in which the energy of the average mean flow transfers to the energy of the fluctuation flow because of the available viscosity, and finally turns into molecular heat through both the vortex cascade down process and the function of molecular viscosity. Thus, there is a range of values which c, D, κ and c_1 take even in the case of curvature boundary because of the similarity of the phenomenology mechanism. The parameters of c and D are flowing responding values when the energy dissipation adjusts to a minimum, with flowing elements governed by the same physical mechanism and behaving with self-similarity under the inner and outer factors which originate from the different boundary and initial evolutional conditions. In summary, the parameters c and D are values of the system responding to flowing environmental factors. From the viewpoint of renormalization group transform, the parameters of c and D are comprehensive coefficients for measuring similarities of length scale and velocity scale to represent general cases of boundary roughness and flowing state. The corresponding physical mechanism is vortex motion.

5 Velocity profile near the wall

It is known that the exact equation of velocity distribution for a turbulent channel flow is

$$\frac{\partial u^+}{\partial y^+} = 1 - \frac{y^+}{H^+} + \overline{u'_+ v'_+}, \tag{14}$$

where u'_+ and v'_+ are dimensionless fluctuation velocities scaled by u_*. Neglecting viscous diffusion (i.e., at a high enough Reynolds number), the budget of turbulent kinetic energy for the boundary layer approximation gives (Hinze, 1975)

$$u^+ \frac{\partial(0.5q^2)}{\partial x^+} + v^+ \frac{\partial(0.5q^2)}{\partial y^+} + \overline{u'_+v'_+}\frac{\partial u^+}{\partial y^+} + \frac{\partial}{\partial y^+}\left(0.5\overline{q^2v'_+} + \overline{pv'_+}\right) + \varepsilon = 0, \tag{15}$$

where $q^2 = (u'_+)^2_{ii}$, and the dummy repeated subscript means the Einstein summation. p represents the dimensionless temporal fluctuation from the mean pressure, and ε the turbulent dissipation rate (similar to the local rate of conversion of turbulent energy into heat (Townsend, 1976)). In a uniform 2D open channel, the assumptions of $v^+ = 0$ and negligible convection yield:

$$\overline{u'_+v'_+}\frac{\partial u^+}{\partial y^+} + \frac{\partial}{\partial y^+}\left(0.5\overline{q^2v'_+} + \overline{pv'_+}\right) + \varepsilon = 0. \tag{16}$$

According to the turbulent modeling theory (Wilcox, 1994), the first term on the left-hand side of Eq. (16) is identified by the production or generation of turbulent kinetic energy, i.e., with energy transferred from the mean motion towards the turbulent fluctuations.

Zones of the flow near the boundary exhibit such large local rates of energy production and dissipation that both terms are in approximate local equilibrium. Thus, these zones are called equilibrium layers. Hence,

$$\overline{u'_+v'_+}\frac{\partial u^+}{\partial y^+} + \varepsilon = 0, \tag{17}$$

$$\frac{\partial}{\partial y^+}\left(0.5\overline{q^2v'_+} + \overline{pv'_+}\right) \approx 0. \tag{18}$$

Townsend (1976) established three conditions for the existence of an equilibrium layer: (1) the existence of a local equilibrium between production and turbulent dissipation; (2) that this layer has to be thin enough for production and dissipation rates to be independent of the largest scales of the flow; and (3) that the shear stress variation across this layer has to be small to ensure that length scales associated with the vertical (wall normal) distribution of shear stresses are not important. Moreover, he obtained the logarithmic law by proper scaling of Eq. (17). The energy balance in the wall region indicates that an excess of turbulent kinetic energy exists, so that part of the energy transferred towards the fluctuations is converted locally into heat, although this time by turbulence dissipation, whereas the rest is transported away from the boundary by turbulent diffusion due to pressure and vertical fluctuations feeding further regions of the flow. Note that in the intermediate region, each term in Eq. (18) is negligible by itself when compared with that in Eq. (17).

If we assume that $u^+=y^+$, then $y^+ / H^+ = -\overline{u'_+v'_+}$, and $\varepsilon = y^+ / H^+$. Compared with the form $u^+= C(Re)y^{+\alpha(Re)}+D(Re)$, one can obtain $C(Re)^{+\alpha(Re)} = \alpha(Re) = 1$, and $D(Re)=0$. We know that the laminar flow hold $u^+=y^+$, and others (Park and Chung, 2004) consider that the turbulent flow hold $u^+ = y^+ + 0.25\sigma(y^+)^4$ (here σ is a constant) or other forms. In fact, bursting plays an important role in forming a velocity profile, but its effect cannot be seen. We believe that the velocity field has different scaling in different regions and cannot be in the quasi-equilibrium state described above. Existing formulae cannot be considered as a general form, as the effects of Reynolds stress on flowing field are partially determined by boundary conditions and coming flows. The flowing mechanism near the wall is different from that in other regions. Symmetry-breaking plays an important role and flowing is far from in equilibrium. There are many problems which need further study (Lee and Wu, 2008).

6 Conclusions

The mathematical properties and physical basis of a power law and a logarithmic type of velocity distribution of wall bounded flow were studied and showed that: (1) the physical basis of both is the same; (2) each adopts a different mathematical descriptive method; (3) both neglect viscosity and hold only in the outer flow zone. Thus, there is a general framework to explain many empirical velocity distribution formulas. Clearly, Manning's formula is only a specific form of them. The fact that roughness is only a coefficient of the whole system without corresponding physical factors, formerly known only from practice, was theoretically demonstrated.

The self-similarity and scaling of open channel flow were discussed to show that the values of parameters in a power law and a logarithmic type of velocity distribution cannot be determined theoretically. Thus, the Karman constant should be correctly re-named the Karman parameter.

References

Afzal, N., 2008. Turbulent boundary layer with negligible wall stress. *Journal of Fluids Engineering*, **130**(5):051205. [doi:10.1115/1.2903754]

Allen, J.J., Shockling, M.A., Kunkel, G.J., Smits, A.J., 2007. Turbulent flow in smooth and rough pipes. *Philosophical Transactions of The Royal Society A: Mathematical Physical and Engineering Sciences*, **365**(1852):699-714. [doi:10.1098/rata.2006.1939]

Barenblatt, G.I., 1996. Similarity, Self-similarity, and Intermediate Asymptotics. Cambridge University Press, Cambridge, p.64-94.

Brzek, B., Cal, R.B., Johansson, G., Castillo, L., 2007. Inner and outer scalings in rough surface zero pressure gradient turbulent boundary layers. *Physics of Fluids*, **19**(6): 065101. [doi:10.1063/1.2732439]

Gioia, G., Bombardelli, F.A., 2001. Scaling and similarity in rough channel flows. *Physical Review Letters*, **88**(1): 014501. [doi:10.1103/PhysRevLett.88.014501]

Goldenfeld, 2006. Roughness-induced critical phenomena in a turbulent flow. *Physical Review Letters*, **96**(4):044503. [doi:10.1103/PhysRevLett.96.044503]

Hinze, J.O., 1975. Turbulence. McGraw-Hill, New York, p.386-388.

Hogarth, W.L., Parlange, J.Y., 2005. Interpolation between Darcy-Weisbach and Darcy for laminar and turbulent flows. *Advances in Water Resources*, **28**(10):1028-1031.

Hunt, J.C.R., 2001. Development in turbulence research: a review based on the 1999 Programme of the Isaac Newton Institute, Cambridge. *Journal of Fluid Mechanics*, **436**:353-391. [doi:10.1017/S002211200100430X]

Lee, C.B., Wu, J.Z., 2008. Transition in wall bounded flows. *Applied Mechanics Reviews*, **61**(3):030802-1. [doi:10.1115/1.2909605]

L'vov, V.S., Procaccia, I., Rudenko, O., 2008. Universal model of finite Reynolds number turbulent flow in channels and pipes. *Physical Review Letters*, **100**(5):054504. [doi:10.1103/PhysRevLett.100.054504]

Mehrafarin, M., Pourtolami, N., 2008. Intermittency and rough-pipe turbulence. *Physical Review E*, **77**(5): 055304(R). [doi:10.1103/PhysRevE.77.055304]

Park, J.Y., Chung, M.K., 2004. Revisit of viscous sublayer scaling law. *Physics of Fluids*, **16**(2):478-481. [doi:10.1063/1.1637600]

She, Z.S., Su, W.D., 1999. Hierarchical structure and scaling in turbulence. *Advance in Mechanics*, **29**(3):289-303 (in Chinese).

Sreenivasan, K.R., 1999. Fluid turbulence. *Reviews of Modern Physics*, **71**(2):S383-S395. [doi:10.1103/RevModPhys.71.S383]

Townsend, A.A., 1976. The Structure of Turbulent Shear Flow. Cambridge University Press, Cambridge, p.262-275.

Wilcox, D.C., 1994. Turbulence Modeling for CFD. DCW Industries, Inc, California, p.243-272.

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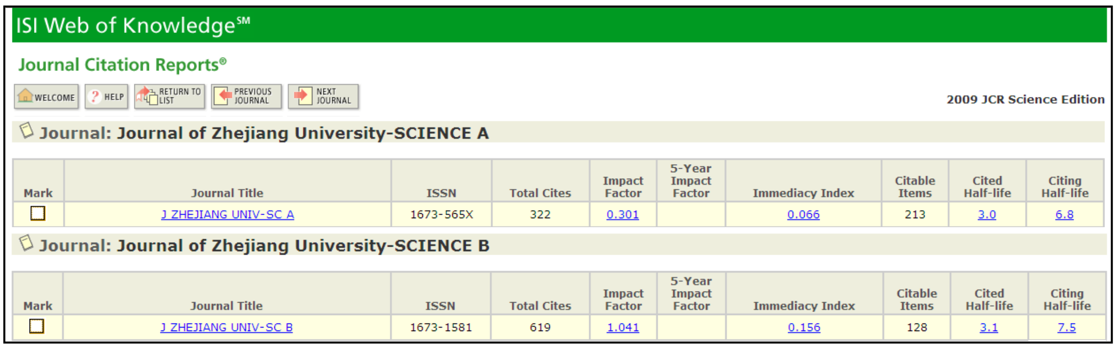

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