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Analytical layer-element solutions for a multi-layered transversely isotropic elastic medium subjected to axisymmetric loading*

Zhi-yong AI[†1], Nai-rui CANG[1], Jie HAN[2]

([1]*Key Laboratory of Geotechnical and Underground Engineering of MOE, Department of Geotechnical Engineering, Tongji University, Shanghai 200092, China*)

([2]*Department of Civil, Environmental, and Architectural Engineering, The University of Kansas, Lawrence, KS 66045, USA*)

[†]E-mail: zhiyongai@tongji.edu.cn

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Abstract: This paper presents an analytical layer-element method used to analyze the displacement of a multi-layered transversely isotropic elastic medium of arbitrary depth subjected to axisymmetric loading. Based on the basic constitutive equations and the HU Hai-chang's solutions for transversely isotropic elastic media, the state vectors of a multi-layered transversely isotropic medium were deduced. From the state vectors, an analytical layer element for a single layer (i.e., a symmetric and exact stiffness matrix) was acquired in the Hankel transformed domain, which not only simplified the calculation but also improved the numerical efficiency and stability due to the absence of positive exponential functions. The global stiffness matrix was obtained by assembling the interrelated layer elements based on the principle of the finite layer method. By solving the algebraic equations of the global stiffness matrix which satisfy the boundary conditions, the solutions for multi-layered transversely isotropic media in the Hankel transformed domain were obtained. The actual solutions of this problem in the physical domain were acquired by inverting the Hankel transform. This paper presents numerical examples to verify the proposed solutions and investigate the influence of the properties of the multi-layered medium on the load-displacement response.

Key words: Transverse isotropy, Analytical layer element, Multi-layered elastic medium, Axisymmetric loading, Hankel transform
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1 Introduction

Deformation of a material or medium subjected to an applied load is an important practical subject. A medium is often assumed to be a homogeneous, isotropic material when deformation is calculated in design. However, many materials in geotechnical engineering are multi-layered and transversely isotropic. For example, natural soil and rock are often formed through a sedimentation process and have different mechanical properties in different directions. Therefore, it is more realistic and important to consider these anisotropic properties when stress and displacement solutions for these materials are derived. Compared to a uniform elastic material model, the use of a multi-layered transversely isotropic model to describe the deformation of a layered medium is more reasonable. Many researchers have conducted numerous studies on this topic. For example, Pan and Chou (1976; 1979) derived static point-load Green's functions for a transversely isotropic full-space and a half-space media. Liao and Wang (1998) and Wang and Liao (1998; 1999) deduced closed-form solutions for the displacements and stresses in a transversely isotropic half-space medium subjected to various buried loadings. Pan (1989a; 1989b; 1997) derived Green's functions for a multi-layered transversely isotropic medium using vector functions and the

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propagator matrix method, and solved the deformation problem of a layered transversely isotropic elastic half-space medium under static surface loading. Yue (1995) and Yue *et al.* (2005) presented closed-form solutions for transversely isotropic media under concentrated forces and rectangular loadings. Chen *et al.* (1998) deduced a transfer matrix solution for a layered transversely isotropic medium subjected to non-axisymmetric loading. Rahman and Newaz (2000) studied the surface displacement of a transversely isotropic half-space medium with a thin film coating under static surface loading. Yu (2001) presented analytical solutions for problems associated with a transversely isotropic half-space medium under different static loadings. Utilizing the generalized Stroh formalism and 2D Fourier transforms, Pan *et al.* (2001) and Yuan *et al.* (2003) obtained 3D layered Green's functions for a medium with each layer being generally anisotropic. Wang *et al.* (2003) presented solutions for displacements and stresses in a vertically point-loaded transversely isotropic half-space medium with Young's and shear moduli varying exponentially with depth. Wang *et al.* (2006) investigated displacements and stresses in an inhomogeneous cross-anisotropic half-space medium under a uniform vertical circular load. Ding *et al.* (2006) conducted a comprehensive and systematic analysis of transversely isotropic materials.

Recently, a new analytical layer-element method, which is numerically stable, was proposed by Ai *et al.* (2010) and Ai and Zeng (2011) to solve axisymmetric and non-axisymmetric Biot's consolidation problems of layered soil. The analytical layer element does not have positive exponential functions and is also symmetric; therefore, it can be used to not only simplify the calculation but also improve the numerical efficiency and stability. One of the main objectives of this paper was to extend the analytical layer-element method to analyze a multi-layered transversely isotropic elastic medium subjected to axisymmetric loading. Another objective was to present an alternative formulation for the development of the analytical and numerical techniques for a multi-layered transversely isotropic elastic medium. Based on the HU Hai-chang's solutions (Hu, 1953), the state vectors of multi-layered transversely isotropic elastic media were deduced in this study. From the state vectors, the analytical layer element of a single layer (i.e., a symmetric and exact stiffness matrix) was first acquired in the Hankel transformed domain. The global stiffness matrix was then obtained by assembling the interrelated layer elements based on the principle of the finite layer method. Solving the algebraic equations of the global stiffness matrix by satisfying the boundary conditions resulted in the solutions of a multi-layered transversely isotropic elastic medium in the Hankel transformed domain. The actual solutions in the physical domain were acquired by taking the inversion of the Hankel transform. A numerical example available in the literature is presented to illustrate the correctness of the solutions obtained in this study. This paper also discusses the influence of the properties of the multi-layered medium on the load-displacement response.

2 Basic equations

The constitutive equations of a transversely isotropic medium can be expressed in terms of displacements in a cylindrical coordinate system as follows:

$$\sigma_r(r,z)=c_{11}\frac{\partial u_r}{\partial r}+c_{12}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+c_{13}\frac{\partial u_z}{\partial z},\quad (1a)$$

$$\sigma_z(r,z)=c_{13}\frac{\partial u_r}{\partial r}+c_{13}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\theta}{r}\right)+c_{33}\frac{\partial u_z}{\partial z},\quad (1b)$$

$$\tau_{z\theta}(r,z)=c_{44}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right),\quad (1c)$$

$$\tau_{zr}(r,z)=c_{44}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right),\quad (1d)$$

where u_r, u_θ, and u_z are the displacement components in r, θ, and z directions, respectively; σ_r and σ_z are the normal stress components in r and z directions, respectively; $\tau_{z\theta}$ and τ_{zr} are the shear stress components in the planes $z\theta$ and zr, respectively; c_{ij} are elastic coefficients, and $c_{11}=\lambda n(1-n\mu_{\rm vh}^2)$, $c_{12}=\lambda n(\mu_{\rm h}+n\mu_{\rm vh}^2)$, $c_{13}=\lambda n\mu_{\rm vh}(1+\mu_{\rm h})$, $c_{33}=\lambda(1-\mu_{\rm h}^2)$, $c_{44}=G_{\rm v}$, $n=E_{\rm h}/E_{\rm v}$, $\lambda=E_{\rm v}/[(1+\mu_{\rm h})(1-\mu_{\rm h}-2n\mu_{\rm vh}^2)]$, in which $E_{\rm v}$, $E_{\rm h}$, and $G_{\rm v}$ are Young's modulus in the vertical direction, Young's modulus in the horizontal direction, and shear modulus in planes normal to the plane of transverse isotropy, respectively, and μ_h and μ_{vh} are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel and normally to the plane, respectively.

According to Hu (1953), on a transversely isotropic medium the following expressions are available:

$$u_r(r,z) = -\frac{\partial^2 \psi}{\partial r \partial z}, \quad (2a)$$

$$u_z(r,z) = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) + \gamma\frac{\partial^2 \psi}{\partial z^2}, \quad (2b)$$

where $\alpha=c_{11}/(c_{13}+c_{44})$, $\gamma=c_{44}/(c_{13}+c_{44})$, and ψ is a stress function satisfying the following equation:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right)\psi = 0, \quad (3)$$

where s_1^2 and s_2^2 are two roots of the following equation:

$$c_{13}c_{44}s^4 - \left(c_{44}^2 + c_{11}c_{33} - (c_{13}+c_{44})^2\right)s^2 + c_{11}c_{44} = 0. \quad (4)$$

3 Analytical layer-element for a single material layer

Taking the Hankel transform (i.e., the first order) of Eqs. (1d) and (2a) with respect to τ_{zr} and u_r, respectively, and the Hankel transform (i.e., the zero order) of Eqs. (1b) and (2b) with respect to σ_z and u_z, respectively (Sneddon, 1972) yields:

$$\begin{cases} \bar{u}_r(\xi,z) = -\xi\dfrac{\partial \bar{\psi}}{\partial z}, \\ \bar{u}_z(\xi,z) = -\alpha\xi^2\bar{\psi} + \gamma\dfrac{\partial^2\bar{\psi}}{\partial z^2}, \\ \bar{\sigma}_z(\xi,z) = \xi c_{13}\bar{u}_r + c_{33}\dfrac{\partial \bar{u}_z}{\partial z}, \\ \bar{\tau}_{zr}(\xi,z) = c_{44}\left(\dfrac{\partial \bar{u}_r}{\partial z} - \xi\bar{u}_z\right), \end{cases} \quad (5)$$

where the superscript bar denotes a Hankel transform, and ξ is the parameter of the Hankel transform corresponding to r.

Similarly, taking the Hankel transform (i.e., the zero order) of Eq. (3) yields the following characteristic equation:

$$\left(-\xi^2 + \frac{1}{s_1^2}\lambda^2\right)\left(-\xi^2 + \frac{1}{s_2^2}\lambda^2\right) = 0, \quad (6)$$

where λ denotes the characteristic value.

According to Eq. (6), there are two situations with different s_1^2 and s_2^2 values:

1. When $s_1^2 \neq s_2^2$, $\lambda=\pm s_1\xi$, $\pm s_2\xi$, and the stress function ψ in the transformed domain can be expressed as

$$\bar{\psi}(\xi,z) = A\mathrm{e}^{-s_1\xi z} + B\mathrm{e}^{s_1\xi z} + C\mathrm{e}^{-s_2\xi z} + D\mathrm{e}^{s_2\xi z}, \quad (7)$$

where A, B, C, and D are four arbitrary constants.

Substituting Eq. (7) into Eq. (5) yields:

$$\bar{u}_r(\xi,z) = \xi^2(-s_1A\mathrm{e}^{-s_1\xi z} + s_1B\mathrm{e}^{s_1\xi z} - s_2C\mathrm{e}^{-s_2\xi z} + s_2D\mathrm{e}^{s_2\xi z}), \quad (8a)$$

$$\begin{aligned}\bar{u}_z(\xi,z) = \xi^2\Big((\gamma s_1^2-\alpha)A\mathrm{e}^{-s_1\xi z} + (\gamma s_1^2-\alpha)B\mathrm{e}^{s_1\xi z} \\ +(\gamma s_2^2-\alpha)C\mathrm{e}^{-s_2\xi z} + (\gamma s_2^2-\alpha)D\mathrm{e}^{s_2\xi z}\Big),\end{aligned} \quad (8b)$$

$$\begin{aligned}\bar{\sigma}_z(\xi,z) = \xi^3\Big\{ & s_1\left((\alpha-\gamma s_1^2)c_{33}-c_{13}\right)A\mathrm{e}^{-s_1\xi z} \\ & - s_1\left((\alpha-\gamma s_1^2)c_{33}-c_{13}\right)B\mathrm{e}^{s_1\xi z} \\ & + s_2\left((\alpha-\gamma s_2^2)c_{33}-c_{13}\right)C\mathrm{e}^{-s_2\xi z} \\ & - s_2\left((\alpha-\gamma s_2^2)c_{33}-c_{13}\right)D\mathrm{e}^{s_2\xi z}\Big\},\end{aligned} \quad (8c)$$

$$\begin{aligned}&\bar{\tau}_z(\xi,z) \\ &= \xi^3c_{44}\Big\{\left(s_1^2(1-\gamma)+\alpha\right)A\mathrm{e}^{-s_1\xi z} + \left(s_1^2(1-\gamma)+\alpha\right)B\mathrm{e}^{s_1\xi z} \\ &\quad + \left(s_2^2(1-\gamma)+\alpha\right)C\mathrm{e}^{-s_2\xi z} + \left(s_2^2(1-\gamma)+\alpha\right)D\mathrm{e}^{s_2\xi z}\Big\},\end{aligned} \quad (8d)$$

Let

$$\begin{cases} \boldsymbol{U}_1 = [\bar{u}_r(\xi,0), \bar{u}_z(\xi,0), \bar{u}_r(\xi,z), \bar{u}_z(\xi,z)]^{\mathrm{T}}, \\ \boldsymbol{T}_1 = [-\bar{\tau}_{zr}(\xi,0), -\bar{\sigma}_z(\xi,0), \bar{\tau}_{zr}(\xi,z), \bar{\sigma}_z(\xi,z)]^{\mathrm{T}}, \end{cases} \quad (9)$$

where $\bar{u}_r(\xi,0), \bar{u}_z(\xi,0), \bar{\tau}_{zr}(\xi,0)$, and $\bar{\sigma}_z(\xi,0)$ are the values of $\bar{u}_r(\xi,z)$, $\bar{u}_z(\xi,z)$, $\bar{\tau}_{zr}(\xi,z)$, and $\bar{\sigma}_z(\xi,z)$ respectively, when z=0. Note that $\boldsymbol{U}_1$ and $\boldsymbol{T}_1$ are the state vectors composed of A, B, C, and D; therefore, they can be expressed as follows:

$$\boldsymbol{U}_1=\boldsymbol{M}[A, B, C, D]^{\mathrm{T}}, \boldsymbol{T}_1=\boldsymbol{N}[A, B, C, D]^{\mathrm{T}}, \tag{10}$$

$$\boldsymbol{M}=\begin{bmatrix} -s_1 & s_1 & -s_2 & s_2 \\ a_1 & a_1 & a_2 & a_2 \\ -s_1\mathrm{e}^{-s_1 z\xi} & s_1\mathrm{e}^{s_1 z\xi} & -s_2\mathrm{e}^{-s_2 z\xi} & s_2\mathrm{e}^{s_2 z\xi} \\ a_1\mathrm{e}^{-s_1 z\xi} & a_1\mathrm{e}^{s_1 z\xi} & a_2\mathrm{e}^{-s_2 z\xi} & a_2\mathrm{e}^{s_2 z\xi} \end{bmatrix}, \tag{11a}$$

$$\boldsymbol{N}=\begin{bmatrix} -a_3 & -a_3 & -a_4 & -a_4 \\ -a_5 & a_5 & -a_6 & a_6 \\ a_3\mathrm{e}^{-s_1 z\xi} & a_3\mathrm{e}^{s_1 z\xi} & a_4\mathrm{e}^{-s_2 z\xi} & a_4\mathrm{e}^{s_2 z\xi} \\ a_5\mathrm{e}^{-s_1 z\xi} & -a_5\mathrm{e}^{s_1 z\xi} & a_6\mathrm{e}^{-s_2 z\xi} & -a_6\mathrm{e}^{s_2 z\xi} \end{bmatrix}, \tag{11b}$$

where
$a_1=\gamma s_1^2-\alpha,\ a_2=\gamma s_2^2-\alpha,\ a_3=\left((1-\gamma)s_1^2+\alpha\right)\xi c_{44}$,
$a_4=\left((1-\gamma)s_2^2+\alpha\right)\xi c_{44}, a_5=\left((\alpha-\gamma s_1^2)c_{33}-c_{13}\right)\xi s_1$,
and $a_6=\left((\alpha-\gamma s_2^2)c_{33}-c_{13}\right)\xi s_2$.

From Eq. (10), the following expression can be obtained:

$$\boldsymbol{T}_1=\boldsymbol{N}\boldsymbol{M}^{-1}\boldsymbol{U}_1=\boldsymbol{K}\boldsymbol{U}_1, \tag{12}$$

where $\boldsymbol{K}$ is an exact and symmetric matrix of order 4×4, which establishes the relationship between the displacements and the stresses of a single medium layer in the transformed domain (Fig. 1).

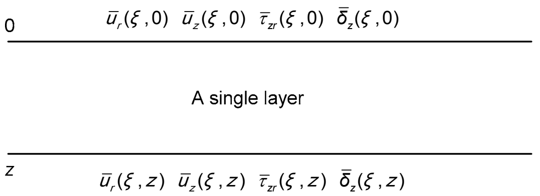

Fig. 1 Stresses and displacements in a single medium layer with depth *z*

The matrix $\boldsymbol{K}$ has the characteristics of the analytical solution, so it is called the analytical layer element. Appendix shows that all the elements of the matrix have only negative exponential functions, which improve the numerical efficiency and stability.

2. When $s_1^2=s_2^2$, $s_1=s_2=s$, $\lambda=\pm s\xi$ (i.e., double root), and the stress function ψ in the transformed domain has the following expression:

$$\bar{\psi}(\xi,z)=(A+Bz)\mathrm{e}^{-s\xi z}+(C+Dz)\mathrm{e}^{s\xi z}. \tag{13}$$

Substituting Eq. (13) into Eq. (5) yield:

$$\begin{aligned}\bar{u}_r(\xi,z)=&\ \xi\left(-s\xi A+(1-s\xi z)B\right)\mathrm{e}^{-s\xi z}\\&+\xi\left(s\xi C+(1+s\xi z)D\right)\mathrm{e}^{s\xi z},\end{aligned} \tag{14a}$$

$$\begin{aligned}\bar{u}_z(\xi,z)=&\left((\gamma s^2-\alpha)\xi^2 A+\left(-2\gamma s\xi+(\gamma s^2-\alpha)\xi^2 z\right)B\right)\mathrm{e}^{-s\xi z}\\&+\left((\gamma s^2-\alpha)\xi^2 C+\left(2\gamma s\xi+(\gamma s^2-\alpha)\xi^2 z\right)D\right)\mathrm{e}^{s\xi z},\end{aligned} \tag{14b}$$

$$\begin{aligned}&\bar{\sigma}_z(\xi,z)\\&=\xi^2\left\{\left((\alpha-\gamma s^2)c_{33}-c_{13}\right)s\xi A\right.\\&\left.+\left(\left((3\gamma s^2-\alpha)c_{33}+c_{13}\right)+\left((\alpha-\gamma s^2)c_{33}-c_{13}\right)s\xi z\right)B\right\}\mathrm{e}^{-s\xi z}\\&+\xi^2\left\{\left((\alpha-\gamma s^2)c_{33}-c_{13}\right)s\xi C\right.\\&\left.+\left(\left((3\gamma s^2-\alpha)c_{33}+c_{13}\right)+\left((\alpha-\gamma s^2)c_{33}-c_{13}\right)s\xi z\right)D\right\}\mathrm{e}^{s\xi z},\end{aligned} \tag{14c}$$

$$\begin{aligned}&\bar{\tau}_{zr}(\xi,z)\\&=\xi^2 c_{44}\left\{\left[\left(s^2(1-\gamma)+\alpha\right)A\xi\right.\right.\\&\qquad\left.+\left(\left(s^2(1-\gamma)+\alpha\right)\xi z-2s(1-\gamma)\right)B\right]\mathrm{e}^{-s\xi z}\\&\qquad+\left[\left(s^2(1-\gamma)+\alpha\right)C\xi\right.\\&\qquad\left.\left.+\left(\left(s^2(1-\gamma)+\alpha\right)\xi z+2s(1-\gamma)\right)D\right]\mathrm{e}^{s\xi z}\right\},\end{aligned} \tag{14d}$$

Similarly, let

$$\begin{cases}\boldsymbol{U}_2=\left[\bar{u}_r(\xi,0),\bar{u}_z(\xi,0),\bar{u}_r(\xi,z),\bar{u}_z(\xi,z)\right]^{\mathrm{T}},\\ \boldsymbol{T}_2=\left[-\bar{\tau}_{zr}(\xi,0),-\bar{\sigma}_z(\xi,0),\bar{\tau}_{zr}(\xi,z),\bar{\sigma}_z(\xi,z)\right]^{\mathrm{T}},\end{cases} \tag{15}$$

or

$$\begin{cases}\boldsymbol{U}_2=\boldsymbol{M}\left[A,B,C,D\right]^{\mathrm{T}},\\ \boldsymbol{T}_2=\boldsymbol{N}\left[A,B,C,D\right]^{\mathrm{T}},\end{cases} \tag{16}$$

$$\boldsymbol{M}=\begin{bmatrix} -s\xi & 1 & s\xi & 1 \\ a_1 & -2\gamma s & a_1 & 2\gamma s \\ -s\xi\mathrm{e}^{-sz\xi} & a_2\mathrm{e}^{-sz\xi} & s\xi\mathrm{e}^{sz\xi} & a_3\mathrm{e}^{sz\xi} \\ a_1\mathrm{e}^{-sz\xi} & a_4\mathrm{e}^{-sz\xi} & a_1\mathrm{e}^{sz\xi} & a_5\mathrm{e}^{sz\xi} \end{bmatrix}, \tag{17a}$$

$$\boldsymbol{N}=\begin{bmatrix} -a_6 & a_7 & -a_6 & -a_7 \\ a_8 & -a_9 & -a_8 & -a_9 \\ a_6\mathrm{e}^{-sz\xi} & a_{10}\mathrm{e}^{-sz\xi} & a_6\mathrm{e}^{sz\xi} & a_{11}\mathrm{e}^{sz\xi} \\ -a_8\mathrm{e}^{-sz\xi} & a_{12}\mathrm{e}^{-sz\xi} & a_8\mathrm{e}^{sz\xi} & a_{13}\mathrm{e}^{sz\xi} \end{bmatrix}, \tag{17b}$$

where $a_1=\gamma s^2-\alpha,\ a_2=1-sz\xi,\ a_3=1+s\xi z,$
$a_4=(\gamma s^2-\alpha)\xi z-2\gamma s,\ a_5=(\gamma s^2-\alpha)\xi z+2\gamma s,$
$a_6=\xi^2 c_{44}\left(s^2(1-\gamma)+\alpha\right),\ a_7=2s(1-\gamma)\xi c_{44},$
$a_8=\left((\gamma s^2-\alpha)c_{33}+c_{13}\right)\xi^2 s,$
$a_9=\left((3\gamma s^2-\alpha)c_{33}+c_{13}\right)\xi,$
$a_{10}=\left(\left((1-\gamma)s^2+\alpha\right)\xi z-2s(1-\gamma)\right)\xi c_{44},$
$a_{11}=\left(\left((1-\gamma)s^2+\alpha\right)\xi z+2s(1-\gamma)\right)\xi c_{44},$
$a_{12}=\left(\left((2\gamma s^2+a_1)c_{33}+c_{13}\right)-(a_1c_{33}+c_{13})s\xi z\right)\xi,$ and
$a_{13}=\left(\left((2\gamma s^2+a_1)c_{33}+c_{13}\right)+(a_1c_{33}+c_{13})s\xi z\right)\xi.$

The following equation can be obtained:

$$\boldsymbol{T}_2=\boldsymbol{N}\boldsymbol{M}^{-1}\boldsymbol{U}_2=\boldsymbol{K}\boldsymbol{U}_2. \tag{18}$$

Similarly, the elements of the matrix $\boldsymbol{K}$ in Eq. (18) are provided in the Appendix.

4 Analytical layer-element solution for multi-layered materials

Fig. 2 shows an elastic medium divided into n layers according to the number of the natural layers and the calculation points. Let the thickness of the ith layer be $h_i=H_i-H_{i-1}$, in which H_i and H_{i-1} are the depths from the surface to the bottom and top of the ith layer, respectively. An axisymmetric loading $p(r, H_j)$ is applied at the depth H_j. Obviously, j=0 means that the loading is applied on the surface of the elastic medium.

It is assumed that the surface of the elastic medium is free and the bottom of the elastic medium is fixed, i.e.,

$$\sigma_z(r,0)=\tau_{zr}(r,0)=0, \tag{19a}$$

$$u_r(r,H_n)=u_z(r,H_n)=0. \tag{19b}$$

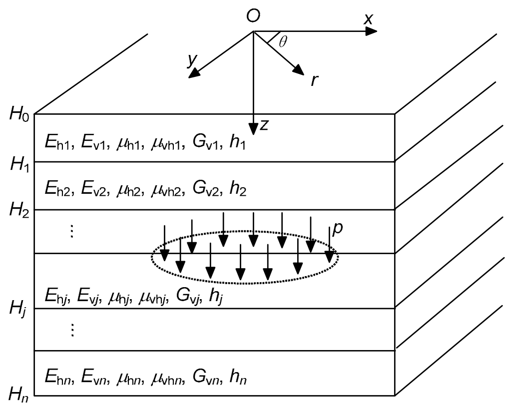

Fig. 2 A multi-layered medium under a uniform vertical circular loading

Applying Eq. (12) or (18) to each layer yields:

$$\boldsymbol{T}=\boldsymbol{K}^{(i)}\boldsymbol{U}, \tag{20}$$

where $\boldsymbol{K}^{(i)}=\boldsymbol{K}(\xi,\ h_i)$ denotes the analytical layer element of the ith layer.

The global stiffness matrix of the multi-layered medium is assembled after considering the continuity conditions of the interfaces between the adjacent layers:

$$\begin{pmatrix} -\bar{\boldsymbol{T}}(\xi,H_0) \\ 0 \\ \vdots \\ \bar{\boldsymbol{T}}(\xi,H_j) \\ 0 \\ \vdots \\ \bar{\boldsymbol{T}}(\xi,H_n) \end{pmatrix}=\begin{pmatrix} \boldsymbol{K}^{(1)} & & & & 0 \\ & \boldsymbol{K}^{(2)} & & & \\ & & \ddots & & \\ & & & \boldsymbol{K}^{(n-1)} & \\ 0 & & & & \boldsymbol{K}^{(n)} \end{pmatrix}_{(2n+2)\times(2n+2)}\begin{pmatrix} \bar{\boldsymbol{U}}(\xi,H_0) \\ \bar{\boldsymbol{U}}(\xi,H_1) \\ \vdots \\ \bar{\boldsymbol{U}}(\xi,H_j) \\ \vdots \\ \bar{\boldsymbol{U}}(\xi,H_{n-1}) \\ \bar{\boldsymbol{U}}(\xi,H_n) \end{pmatrix}, \tag{21}$$

where $[-\bar{\boldsymbol{T}}(\xi,H_0),0,...0,\bar{\boldsymbol{T}}(\xi,H_j),0,...,0,\bar{\boldsymbol{T}}(\xi,H_n)]^{\mathrm{T}}$ is the external force vector in the transformed domain, and $[\bar{\boldsymbol{U}}(\xi,H_0),\bar{\boldsymbol{U}}(\xi,H_1),...,\bar{\boldsymbol{U}}(\xi,H_n)]^{\mathrm{T}}$ is the displacement vector at the interfaces in the transformed domain.

Combined with the boundary conditions in Eq. (19), the unknown variables $[\bar{\boldsymbol{U}}(\xi,H_0),\bar{\boldsymbol{U}}(\xi,H_1),$ $...,\bar{\boldsymbol{U}}(\xi,H_n)]^{\mathrm{T}}$ in the transformed domain can be acquired by solving Eq. (21). The real solutions can be obtained by taking the inversion of the Hankel transform of these variables.

5 Numerical results and discussion

To obtain the solutions in the actual domain, it is necessary to take the inversion of the Hankel transform of solutions derived in the preceding sections. In this study, the technique suggested by Ai *et al.* (2002) was adopted to achieve the inversion of the Hankel transform, and then a computer program based on the presented theory was developed to realize this process.

5.1 Verification

The accuracy of the derived solutions from this study was verified using data from a simple published problem. This problem involves a transversely isotropic elastic half-space medium subjected to a uniform vertical circular loading on the surface. The calculated results based on the derived solutions in this study are compared with those obtained by Wang *et al.* (2006). The results from these two studies are in good agreement (Fig. 3).

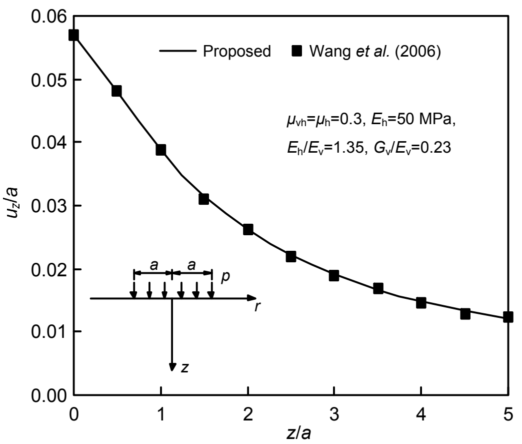

Fig. 3 An elastic half-space medium under a uniform vertical circular loading

5.2 Examples of multi-layered media

An example was considered in this study to evaluate the displacement response of a multi-layered medium under static loading. Fig. 4 shows the calculated vertical displacements in a multi-layered medium under uniform vertical circular loading based on the exact solution from this study and the approximate solution. The approximate solution used the average moduli of all the three layers including Young's moduli in vertical and horizontal directions and the shear modulus. The ratio of the thickness to the radius of the loading area was H_1:H_2:H_3:a=3:4:3:1. Young's moduli of all three layers in vertical and horizontal directions were: E_{v1}=10 MPa, E_{h1}=40 MPa, E_{v2}=5 MPa, E_{h2}=15 MPa, E_{v3}=30 MPa, E_{h3}=25 MPa, and the average moduli in vertical direction and horizontal direction were $E_v^*=(E_{v1}+E_{v2}+E_{v3})/3$ and $E_h^*=(E_{h1}+E_{h2}+E_{h3})/3$, respectively. The shear modulus of each layer was $G_{v1}=G_{v2}=G_{v3}$=20 MPa and the average shear modulus was $G_v^*=(G_{v1}+G_{v2}+G_{v3})/3$. Poisson's ratios of all layers in horizontal and vertical directions were $\mu_{hi}=\mu_{vhi}$=0.3 (i=1, 2, 3), and their average Poisson's ratios in horizontal direction and vertical direction were $\mu_h^*=(\mu_{h1}+\mu_{h2}+\mu_{h3})/3$ and $\mu_{vh}^*=(\mu_{vh1}+\mu_{vh2}+\mu_{vh3})/3$, respectively. The total thickness of the three-layered medium was $H=H_1+H_2+H_3$. The surface of the medium was assumed to be free while the bottom was fixed.

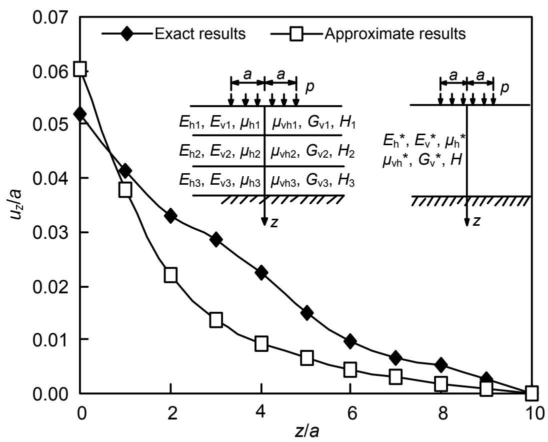

Fig. 4 Vertical displacements in a three-layered medium under a uniform vertical circular loading

Fig. 4 shows that the average approximate method was not very accurate as an exact solution to calculate the displacement of the multi-layered medium under a uniform vertical circular load.

To investigate the effect of the modulus ratio ($n=E_h/E_v$) in the horizontal to the vertical direction, four conditions listed in Table 1 were investigated and other parameters were kept the same as those used for the results in Fig. 4. Fig. 5 shows that the calculated vertical displacement on the surface varied with the condition, and that the properties of the surface layer had more effect on the vertical displacement than those of the other layers.

Table 1 Modulus ratios of the three-layered medium

Condition	Layer 1, n=1.5		Layer 2, n=2.0		Layer 3, n=1.0	
	E_v (MPa)	E_h (MPa)	E_v (MPa)	E_h (MPa)	E_v (MPa)	E_h (MPa)
1	20	30	20	40	10	10
2	20	30	30	60	10	10
3	30	45	20	40	20	20
4	30	45	20	40	10	10

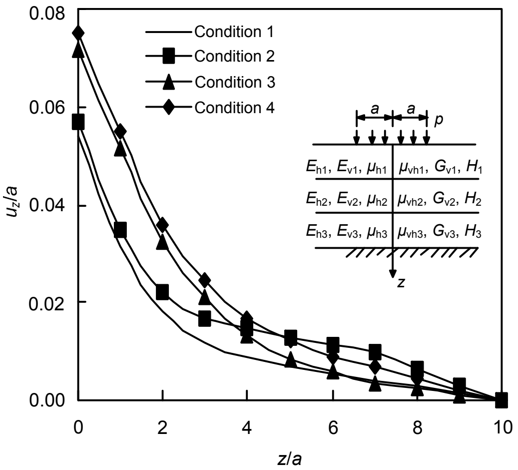

Fig. 5 Vertical surface displacement at different modulus ratios

To investigate the shear modulus effect, its value was varied as listed in Table 2 for three conditions. The elastic moduli of the three layers were selected as follows: $E_{v1}=E_{v2}=E_{v3}=10$ MPa, $E_{h1}=E_{h2}=E_{h3}=30$ MPa, and the other parameters were kept the same as those used for Fig. 4. Fig. 6 shows that the shear modulus effect happened within the depth of twice the radius of the loading area.

Table 2 Shear moduli of the three-layered medium

Condition	Layer 1	Layer 2	Layer 3
	G_{v1} (MPa)	G_{v2} (MPa)	G_{v3} (MPa)
1	10	10	10
2	20	10	10
3	10	20	30

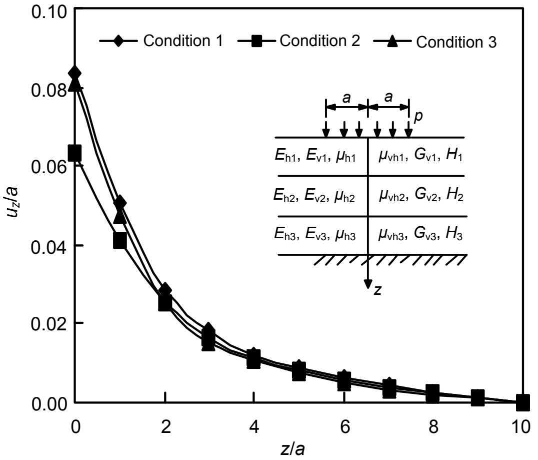

Fig. 6 Vertical surface displacement at different shear moduli

6 Conclusions

In this study, the stress functions were substituted into the basic equilibrium equations for a transversely isotropic elastic medium. The analytical layer element for a single layer was obtained by a Hankel transform. After considering the continuity conditions between adjacent layers and the boundary conditions of the layered medium, a global stiffness matrix was acquired by assembling the interrelated layer elements. The unknown displacement variables in the transformed domain were determined by solving the global stiffness matrix equation. The real solutions were obtained by taking the inversion of the Hankel transform of these variables. The analytical layer element presented in this paper had no positive exponential function and was symmetric; therefore, it simplified the calculations and improved the numerical efficiency and stability. A numerical example available in the literature was used to verify the accuracy of the derived solutions. The numerical analysis of the multi-layered transversely isotropic elastic medium subjected to uniform vertical circular loading showed that the elastic modulus and shear modulus of the layers had an obvious effect on the vertical surface displacement of the multi-layered medium.

References

Ai, Z.Y., Zeng, W.Z., 2011. Analytical layer-element method for non-axisymmetric consolidation of multilayered soils. *International Journal for Numerical and Analytical Methods in Geomechanics*, in press. [doi:10.1002/nag.1000]

Ai, Z.Y., Yue, Z.Q., Tham, L.G., Yang, M., 2002. Extended Sneddon and Muki solutions for multilayered elastic materials. *International Journal of Engineering Science*, **40**(13):1453-1483. [doi:10.1016/S0020-7225(02)00022-8]

Ai, Z.Y., Cheng, Y.C., Zeng, W.Z., 2010. Analytical layer-element solution to axisymmetric consolidation of multilayered soils. *Computers and Geotechnics*, **38**(2):227-232. [doi:10.1016/j.compgeo.2010.11.011]

Chen, G.J., Zhao, X.H., Yu, L., 1998. Transferring matrix method for solutions of non-axisymmetric load applied to layered cross-anisotropic elastic body. *Chinese Journal of Geotechnical Engineering*, **26**(5):522-527 (in Chinese).

Ding, H., Chen, W., Zhang, L., 2006. Elasticity of Transversely Isotropic Materials. Springer, the Netherlands.

Hu, H.C., 1953. On the three-dimensional problems of the theory of elasticity of a transversely isotropic body. *Acta Physica Sinica*, **9**(2):130-147 (in Chinese).

Liao, J.J., Wang, C.D., 1998. Elastic solutions for a transversely isotropic half-space subjected a point load. *International Journal for Numerical and Analytical Methods in Geomechanics*, **22**(6):425-447. [doi:10.1002/(SICI)1096-9853(199806)22:6<425::AID-NAG925>3.0.CO:2-H]

Pan, E., 1989a. Static response of a transversely isotropic and layered half-space to general surface loads. *Physics of the Earth and Planetary Interiors*, **54**(3-4):353-363. [doi:10.1016/0031-9201(89)90252-5]

Pan, E., 1989b. Static response of a transversely isotropic and layered half-space to general dislocation sources. *Physics of the Earth and Planetary Interiors*, **58**(2-3):103-117. [doi:10.1016/0031-9201(89)90046-0]

Pan, E., 1997. Static Green's functions in multilayered half-spaces. *Applied Mathematical Modelling*, **21**(8):509-521. [doi:10.1016/S0307-904X(97)00053-X]

Pan, E., Yang, B., Cai, X., Yuan, F.G., 2001. Stress analyses around holes in composite laminates using boundary element method. *Engineering Analysis with Boundary Elements*, **25**(1):31-40. [doi:10.1016/S0955-7997(00)00066-7]

Pan, Y.C., Chou, T.W., 1976. Point force solution for an infinite transversely isotropic solid. *Journal of Applied Mechanics*, **43**(4):608-612. [doi:10.1115/1.3423941]

Pan, Y.C., Chou, T.W., 1979. Green's function solutions for semi-infinite transversely isotropic materials. *International Journal of Engineering Science*, **17**(5):545-551. [doi:10.1016/0020-7225(79)90124-1]

Rahman, M., Newaz, G., 2000. Boussinesq type solution for a transversely isotropic half-space coated with a thin film. *International Journal of Engineering Science*, **38**(7):807-822. [doi:10.1016/S0020-7225(99)00052-X]

Sneddon, I.N., 1972. The Use of Integral Transform. McGraw-Hill, New York.

Wang, C.D., Liao, J.J., 1998. Stress influence chart for transversely isotropic rocks. *International Journal of Rock Mechanics and Mining Sciences*, **35**(6):771-785. [doi:10.1016/S0148-9062(98)00015-1]

Wang, C.D., Liao, J.J., 1999. Elastic solutions for a transversely isotropic half-space subjected buried axisymmetric loads. *International Journal for Numerical and Analytical Methods in Geomechanics*, **23**(2):115-139. [doi:10.1002/(SICI)1096-9853(199902)23:2<115::AID-NAG961>3.0.CO;2-A]

Wang, C.D., Tzeng, C.S., Pan, E., Liao, J.J., 2003. Displacements and stresses due to a vertical point load in an inhomogeneous transversely isotropic half-space. *International Journal of Rock Mechanics and Mining Sciences*, **40**(5):667-685. [doi:10.1016/S1365-1609(03)00058-3]

Wang, C.D., Pan, E., Tzeng, C.S., Han, F., Liao, J.J., 2006. Displacements and stresses due to a uniform vertical circular load in an inhomogeneous cross-anisotropic half-space. *International Journal of Geomechanics*, **6**(1):1-10. [doi:10.1061/(ASCE)1532-3641(2006)6:1(1)]

Yu, H.Y., 2001. A concise treatment of indentation problems in transversely isotropic half-space. *International Journal of Solids and Structures*, **38**(10-13):2213-2232. [doi:10.1016/S0020-7683(00)00163-3]

Yuan, F.G., Yang, S., Yang, B., 2003. Three-dimensional Green's functions for composite laminates. *International Journal of Solids and Structures*, **40**(2):331-342. [doi:10.1016/S0020-7683(02)00545-0]

Yue, Z.Q., 1995. Elastic fields in two joined transversely isotropic solids due to concentrated forces. *International Journal of Engineering Science*, **33**(3):351-369. [doi:10.1016/0020-7225(94)00063-P]

Yue, Z.Q., Xiao, H.T., Tham, L.G., Lee, C.F., Yin, J.H., 2005. Stresses and displacements of a transversely istropic elastic half-space due to rectangular loadings. *Engineering Analysis with Boundary Elements*, **29**(6):647-671. [doi:10.1016/j.enganabound.2005.01.015]

Appendix: Elements of *K* (a symmetric matrix)

1. When $s_1^2 \neq s_2^2$:

$$k_{11} = \alpha\xi c_{44}(s_1^2 - s_2^2)(A_2A_3s_2A_5 - A_1A_4s_1A_6)/B,$$
$$k_{12} = \left(\left((A_7 - A_8)^2 + (A_7A_8 - 1)^2\right)A_{10} + A_3A_4A_9\right)\xi c_{44}/B,$$
$$k_{13} = -2\alpha\xi c_{44}(s_1^2 - s_2^2)(A_8A_3s_2A_5 - A_7A_4s_1A_6)/B,$$
$$k_{14} = 2\alpha\xi c_{44}s_1s_2(s_1^2 - s_2^2)(A_7A_8 - 1)(A_7 - A_8)/B,$$
$$k_{21} = k_{12},$$
$$k_{22} = \xi\gamma c_{33}s_1s_2(s_1^2 - s_2^2)(A_1A_4s_2A_5 - A_3A_2s_1A_6)/B,$$
$$k_{23} = -k_{14},$$
$$k_{24} = 2\gamma\xi c_{44}s_1s_2(s_1^2 - s_2^2)(A_7A_4s_2A_5 - A_8A_3s_1A_6)/B,$$
$$k_{31} = k_{13},\ k_{32} = k_{23},\ k_{33} = k_{11},\ k_{34} = -k_{12},$$
$$k_{41} = k_{14},\ k_{42} = k_{24},\ k_{43} = k_{34},\ k_{44} = k_{22},$$

where

$A_1=e^{-2z\xi s_1}+1,\ \ A_2=e^{-2z\xi s_2}+1,\ \ A_3=1-e^{-2z\xi s_1},$

$A_4=1-e^{-2z\xi s_2},\ \ A_5=\gamma s_1^2-\alpha,\ \ A_6=\gamma s_2^2-\alpha,$

$A_7=e^{-z\xi s_1},\ \ A_8=e^{-z\xi s_2},$

$A_9=(\gamma-1)\gamma s_1^4 s_2^2+s_1^2\left(\alpha^2+2\alpha(1-2\gamma)s_2^2\right)+A_{11},$

$A_{10}=-s_1s_2\left(\alpha A_{12}+s_1^2\left(\alpha-2\alpha\gamma+2(\gamma-1)\gamma s_2^2\right)\right),$

$A_{11}=(\gamma-1)\gamma s_2^4+\alpha^2 s_2^2,\ \ A_{12}=2\alpha+(1-2\gamma)s_2^2,$

$A_{13}=(A_7-A_8)^2+(A_7A_8-1)^2$, and

$B=-2s_1s_2A_5A_6A_{13}+A_3A_4(A_5^2s_2^2+A_6^2s_1^2).$

2. When $s_1^2=s_2^2$:

$k_{11}=2s\alpha\xi c_{44}\left(\alpha A_{10}+s^2\gamma(1-A_1^4+4A_1^2sz\xi)\right)/B,$

$k_{12}=\xi c_{44}\left(A_1^4A_4+A_4-(A_1^2-1)^2A_5+A_{11}\right)/B,$

$k_{13}=4A_1s\alpha\xi\left(s^2\gamma(A_9A_1^2+A_8)-\alpha(A_8A_1^2+A_9)\right)/B,$

$k_{14}=4A_1(A_1^2-1)s^3z\alpha\xi^2c_{44}/B,$

$k_{21}=k_{12},$

$k_{22}=-2s^3\gamma\xi c_{33}\left(s^2\gamma A_{10}+\alpha(1-A_1^4+4A_1^2sz\xi)\right)/B,$

$k_{23}=-k_{14},$

$k_{24}=4A_1s^3\gamma\xi\left(s^2\gamma(A_8A_1^2+A_9)-\alpha(A_9A_1^2+A_8)\right)/B,$

$k_{31}=k_{13},\ \ k_{32}=k_{23},\ \ k_{33}=k_{11},\ \ k_{34}=-k_{12},$

$k_{41}=k_{14}, k_{42}=k_{24}, k_{43}=k_{34}, k_{44}=k_{22},$

where

$A_1=e^{-z\xi s},\ \ A_2=1+2s^2z^2\xi^2,\ \ A_3=2s^2z^2\xi^2-1,$

$A_4=s^2\alpha+s^4\gamma-2s^2\alpha\gamma,\ \ A_5=\alpha^2+s^4\gamma^2,$

$A_6=2z^2\alpha^2\xi^2+s^2\gamma\left(2s^2z^2\xi^2(\gamma-1)-1\right),$

$A_7=\alpha(A_2-2\gamma A_3),\ \ A_8=1+sz\xi,\ \ A_9=sz\xi-1,$

$A_{10}=1-A_1^4-4A_1^2sz\xi,\ \ A_{11}=4s^2A_1^2(A_6+A_7)$, and

$B=-(A_1^4+1-2A_1^2A_2)A_5-2s^2\alpha\gamma(A_1^4+1+2A_1^2A_3).$