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Representing conics by low degree rational DP curves*

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Abstract: A DP curve is a new kind of parametric curve defined by Delgado and Peña (2003); it has very good properties when used in both geometry and algebra, i.e., it is shape preserving and has a linear time complexity for evaluation. It overcomes the disadvantage of some generalized Ball curves that are fast for evaluation but cannot preserve shape, and the disadvantage of the Bézier curve that is shape preserving but slow for evaluation. It also has potential applications in computer-aided design and manufacturing (CAD/CAM) systems. As conic section is often used in shape design, this paper deduces the necessary and sufficient conditions for rational cubic or quartic DP representation of conics to expand the application area of DP curves. The main idea is based on the transformation relationship between low degree DP basis and Bernstein basis, and the representation theory of conics in rational low degree Bézier form. The results can identify whether a rational low degree DP curve is a conic section and also express a given conic section in rational low degree DP form, i.e., give positions of the control points and values of the weights of rational cubic or quartic DP conics. Finally, several numerical examples are presented to validate the effectiveness of the method.

Key words: Conic sections, Bernstein basis, DP basis, Rational low degree Bézier curves, Rational low degree DP curves
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1 Introduction

Conic sections (or simply, conics) play a vital role in geometric shape design and machine manufacture. They are an essential part of the outline of most machine elements and an important design tool in the aircraft industry. They are also used in some artistic areas such as font design (Farin, 2001). In modern design systems, conics are expressed mostly in rational low degree polynomial form for the following reasons. First, rational parametric curves can unify the expressions of conic and polynomial parametric curves, and can also flexibly adjust the shape of the curve using weights. Second, their low degree is favorable for quick computing and storage compression. They have become an attractive and promising research direction, as shown by the amount of publications on rational low degree Bézier conics. Chou (1995) proved that the degree of the Bézier curve forming a full circle must be at least five for the curves to have all positive weights. This was generalized by Sánchez-Reyes (1997) to obtain rational Bézier circular arcs of arbitrary sweep angle and arbitrary even-degree. Wang and Wang (1992) investigated the necessary and sufficient conditions for representing conics in rational cubic Bézier form with proper parameterization. Fang (2002) presented a special representation for conics in rational quartic Bézier form such that the joined curves still have C^1 continuity in homogeneous space. Based on the fact that all rational Bézier conics except for degree two are degenerate (Sánchez-Reyes, 1997), Hu and Wang (2007) used a totally different method to present the necessary and sufficient conditions for the rational quartic Bézier representation of conics.

Following on from these in-depth studies, current investigations have tended to focus on the

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following two aspects of rational parametric curves: decreasing computational complexity and obtaining shape preservation. As for a parametric curve, evaluation is a fundamental operator in shape design, and low computational complexity for evaluation is of great benefit for improving the design efficiency. However, shape preservation is a basic requirement in shape design. Research shows that the parametric curve formed by normalized totally positive (NTP) basis imitates the shape of its control polygon because of the diminishing variation property of totally positive (TP) matrices (Delgado and Peña, 2003). Goodman and Said (1991) pointed out that "if the parametric curve is formed by NTP basis, then in many ways the shape of it mimics or 'preserves' the shape of the control polygon". But unfortunately, a parametric curve in NTP form may not have an evaluation algorithm with low computational complexity; whereas a parametric curve with low computational complexity may not be shape preserving. The Bézier and the Wang-Ball curves, respectively, are examples of the above two curves. Their time complexities for evaluation are quadratic and linear of the degree of the curve respectively (Phien and Dejdumrong, 2000), but only the Bézier curve is shape-preserving (Delgado and Peña, 2006). Therefore, designers have long been searching for a parametric curve which has both predominant properties, i.e., low computational cost for evaluation in algebra, as well as shape preservation in geometry. Confronting this challenge, Delgado and Peña (2003) constructed a new kind of parametric curve later named the DP curve by Jiang and Wang (2005). It is formed by NTP basis, and also has linear complexity for evaluation. Hence, it will suit a variety of applications in computer-aided design and manufacturing (CAD/CAM) systems. Jiang and Wang (2005) also studied the transformation relationship between DP basis and Bernstein basis, and presented the corresponding transformation matrices. With these matrices, they not only derived the subdivision formulae for DP surface, but also reduced the time complexity for evaluation from cubic (quadratic) to quadratic (linear), of the degree of the surface (curve) in Bernstein form.

Considering the valuable properties of DP basis, researchers have focused their research on the properties and applications of DP curves and surfaces (Aphirukmatakun and Dejdumrong, 2008; Delgado and Peña, 2008; Itsariyawanich and Dejdumrong, 2008). It is hoped that the application of DP curves may be extended to represent conics that are commonly used in shape design. Naturally, they should be generalized to rational form. Dejdumrong (2006) introduced the definition of a rational DP curve. This raised the question of how rational low degree DP conics could be constructed. That is, when a conic section is represented in rational low degree DP form, how can the control points and weights be determined? The significance of this problem lies in constructing a curve model to express conics with low computational cost and shape preservation.

To solve this problem, this paper concentrates on the necessary and sufficient conditions for conics represented by rational cubic or quartic DP curves. The idea is based on the necessary and sufficient conditions for rational low degree Bézier representation of conics (Wang and Wang, 1992; Hu and Wang, 2007) and the transformation matrices between DP basis and Bernstein basis (Jiang and Wang, 2005). We analyze, in great detail, the positions of the control points and values of the weights of conics in rational quartic DP form in different cases. Also, two algorithms are provided to judge whether a rational quartic DP curve is a conic section, and to express a given conic section in rational quartic DP form. Finally numerical examples are presented to confirm the effectiveness of the method.

2 Rational DP curves of low degree

Rational cubic Bézier and DP curves are defined by (Farin, 2001; Dejdumrong, 2006) respectively as

$$\boldsymbol{u}(t)=\sum_{i=0}^{3}B_i^3(t)\alpha_i\boldsymbol{U}_i\Big/\sum_{i=0}^{3}B_i^3(t)\alpha_i, \tag{1}$$

$$\boldsymbol{v}(t)=\sum_{i=0}^{3}C_i^3(t)\beta_i\boldsymbol{V}_i\Big/\sum_{i=0}^{3}C_i^3(t)\beta_i, \tag{2}$$

where α_i and β_i (i=0, 1, 2, 3) are the associated weights, and $\boldsymbol{U}_i$ and $\boldsymbol{V}_i$ (i=0, 1, 2, 3) are the associated control points. The cubic Bernstein basis functions $B_i^3(t)$ and DP basis functions $C_i^3(t)$ are $(1-t)^3$, $3(1-t)^2t$, $3(1-t)t^2$, t^3, and $(1-t)^3$, $t(1-t)(2-t)$, $t(1-t)(1+t)$, t^3, respectively. Rational quartic Bézier and DP curves are also defined by (Farin, 2001; Aphirukmatakun and Dejdumrong, 2008) respectively as

$$\boldsymbol{p}(t)=\sum_{i=0}^{4}B_i^4(t)\omega_i\boldsymbol{P}_i\Big/\sum_{i=0}^{4}B_i^4(t)\omega_i, \tag{3}$$

$$\boldsymbol{q}(t)=\sum_{i=0}^{4}C_i^4(t)\varphi_i\boldsymbol{Q}_i\Big/\sum_{i=0}^{4}C_i^4(t)\varphi_i, \tag{4}$$

where ω_i, φ_i (i=0, 1, …, 4) are the associated weights, and $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$ (i=0, 1, …, 4) are the associated control points. The quartic Bernstein basis functions $B_i^4(t)$ and DP basis functions $C_i^4(t)$ are $(1-t)^4$, $4(1-t)^3t$, $6(1-t)^2t^2$, $4(1-t)t^3$, t^4 and $(1-t)^4$, $(1-t)^3t$, $3(1-t)t$, $(1-t)t^3$, t^4, respectively. To preserve the convex property of the curves, herein we prescribe that all the weights are positive.

3 Transferring rational DP curves to standard form

A rational Bézier curve is shape-invariable under a special linear parameter transformation (Farin, 2001). That is, there exists a fractional linear parameter transformation such that an arbitrary rational Bézier curve can be represented in two forms, i.e., the same control points with different weights. Therefore, without loss of generality, we discuss only a standard rational DP curve, i.e., where the two end weights are both equal to 1. As for its nonstandard form, we can first convert it to rational Bézier form by the transformation relationship from DP basis to Bernstein basis. Then, applying a special fractional linear parameter transformation, the latter is expressed in standard form. Finally, by the transformation relationship from Bernstein basis to DP basis, the standard rational DP curve is obtained. This process is demonstrated in formulae as follows:

DP basis and Bernstein basis satisfy (Jiang and Wang, 2005)

$$\begin{aligned}&\left(C_0^n(t),\, C_1^n(t),\, ...,\, C_n^n(t)\right)^{\mathrm{T}}\\ &=\boldsymbol{M}_{(n+1)(n+1)}\left(B_0^n(t),\, B_1^n(t),\, ...,\, B_n^n(t)\right)^{\mathrm{T}}.\end{aligned}$$

When n=3 or 4, we have

$$\begin{cases}\boldsymbol{M}_{4\times4}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 2/3 & 1/3 & 0\\ 0 & 1/3 & 2/3 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\\ \boldsymbol{M}_{5\times5}=\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1/4 & 0 & 0 & 0\\ 0 & 3/4 & 1 & 3/4 & 0\\ 0 & 0 & 0 & 1/4 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.\end{cases} \tag{5}$$

Then the rational quartic DP curve is converted to the rational quartic Bézier curve written as Eq. (3), and the associated control points and weights satisfy

$$\begin{cases}\omega_0=\varphi_0,\quad 4\omega_1=\varphi_1+3\varphi_2,\quad \omega_2=\varphi_2,\\ 4\omega_3=3\varphi_2+\varphi_3,\quad \omega_4=\varphi_4;\\ \boldsymbol{P}_i=\boldsymbol{Q}_i,\ i=0, 2, 4,\\ \boldsymbol{P}_1=\dfrac{\varphi_1\boldsymbol{Q}_1+3\varphi_2\boldsymbol{Q}_2}{\varphi_1+3\varphi_2},\ \boldsymbol{P}_3=\dfrac{3\varphi_2\boldsymbol{Q}_2+\varphi_3\boldsymbol{Q}_3}{3\varphi_2+\varphi_3}.\end{cases} \tag{6}$$

According to the shape-invariability of the rational Bézier curve (Farin, 2001), making a fractional linear parameter transformation

$$t=\frac{\sqrt[4]{\omega_0}\,u}{\sqrt[4]{\omega_0}\,u+\sqrt[4]{\omega_4}\,(1-u)}$$

to the curve written as Eq. (3), we obtain the curve in standard form with associated weights

$$\omega_0^*=\omega_4^*=1,\quad \omega_1^*=\frac{\omega_1}{\sqrt[4]{\omega_0^3\omega_4}},\ \omega_2^*=\frac{\omega_2}{\sqrt{\omega_0\omega_4}},\quad \omega_3^*=\frac{\omega_3}{\sqrt[4]{\omega_0\omega_4^3}}.$$

Finally, according to Eq. (6), the rational quartic DP curve in Eq. (4) is converted to the standard form with associated control points and weights as follows:

$$\psi_0^*=\varphi_4^*=1,\qquad \varphi_1^*=\frac{\varphi_1+3\varphi_2}{\sqrt[4]{\varphi_0^3\varphi_4}}-\frac{3\varphi_2}{\sqrt{\varphi_0\varphi_4}},$$

$$\varphi_2^*=\frac{\varphi_2}{\sqrt{\varphi_0\varphi_4}},\quad \varphi_3^*=\frac{3\varphi_2+\varphi_3}{\sqrt[4]{\varphi_0\varphi_4^3}}-\frac{3\varphi_2}{\sqrt{\varphi_0\varphi_4}}.$$

$$\boldsymbol{Q}_i^*=\boldsymbol{Q}_i,\ \ i=0, 2, 4,$$

$$\boldsymbol{Q}_1^* = \frac{\varphi_1 \boldsymbol{Q}_1 + 3\varphi_2\left(1-\sqrt[4]{\varphi_0/\varphi_4}\right)\boldsymbol{Q}_2}{\varphi_1 + 3\varphi_2\left(1-\sqrt[4]{\varphi_0/\varphi_4}\right)},$$

$$\boldsymbol{Q}_3^* = \frac{3\varphi_2\left(1-\sqrt[4]{\varphi_4/\varphi_0}\right)\boldsymbol{Q}_2 + \varphi_3 \boldsymbol{Q}_3}{3\varphi_2\left(1-\sqrt[4]{\varphi_4/\varphi_0}\right) + \varphi_3}.$$

As for a rational cubic DP curve, its standard form is obtained in a similar fashion. Based on the above discussion, this paper studies only the curves in standard form, i.e., when their two end weights expressed as Eqs. (1)–(4) are all equal to 1.

4 Rational cubic DP conics

By the transformation Eq. (5) from cubic DP basis to Bernstein basis, the rational cubic DP curve in Eq. (2) is converted to the rational cubic Bézier curve in Eq. (1), in which the weights and control points meet the following expressions:

$$\begin{cases} \alpha_0=\beta_0,\ 3\alpha_1=2\beta_1+\beta_2,\ 3\alpha_2=\beta_1+2\beta_2,\ \alpha_3=\beta_3; \\ \boldsymbol{U}_i=\boldsymbol{V}_i,\ i=0, 3, \\ \boldsymbol{U}_1=\dfrac{2\beta_1\boldsymbol{V}_1+\beta_2\boldsymbol{V}_2}{2\beta_1+\beta_2}, \\ \boldsymbol{U}_2=\dfrac{\beta_1\boldsymbol{V}_1+2\beta_2\boldsymbol{V}_2}{\beta_1+2\beta_2}. \end{cases} \tag{7}$$

Lemma 1 For two triangles with an equal height, the ratio of their areas is equal to the ratio of their corresponding edge lengths. For two triangles with an equal edge length, the ratio of their areas is equal to the ratio of their corresponding heights.

Theorem 1 A rational cubic DP curve in Eq. (2) is a conic section if, and only if, the following three conditions hold simultaneously:

1. $\boldsymbol{V}_i$ (i=0, 1, 2, 3) compose a planar quadrilateral,
2. $\dfrac{\beta_0}{\beta_1\beta_2} = \dfrac{3T_0(\beta_2 T_1 + 2\beta_1 T_2)}{(2\beta_2 T_1 + \beta_1 T_2)^2}$,
3. $\dfrac{\beta_3}{\beta_1\beta_2} = \dfrac{3T_3(2\beta_2 T_1 + \beta_1 T_2)}{(\beta_2 T_1 + 2\beta_1 T_2)^2}$,

where T_i (i=0, 1, 2, 3) are the directed areas of $\Delta\boldsymbol{V}_1\boldsymbol{V}_2\boldsymbol{V}_3$, $\Delta\boldsymbol{V}_0\boldsymbol{V}_2\boldsymbol{V}_3$, $\Delta\boldsymbol{V}_0\boldsymbol{V}_1\boldsymbol{V}_3$, and $\Delta\boldsymbol{V}_0\boldsymbol{V}_1\boldsymbol{V}_2$, respectively (Fig. 1).

Proof According to Wang and Wang (1992), the necessary and sufficient conditions for rational cubic Bézier representation of conics are related only to the weights α_i and areas S_i (i=0, 1, 2, 3). Here S_i (i=0, 1, 2, 3) are the directed areas of $\Delta\boldsymbol{U}_1\boldsymbol{U}_2\boldsymbol{U}_3$, $\Delta\boldsymbol{U}_0\boldsymbol{U}_2\boldsymbol{U}_3$, $\Delta\boldsymbol{U}_0\boldsymbol{U}_1\boldsymbol{U}_3$, and $\Delta\boldsymbol{U}_0\boldsymbol{U}_1\boldsymbol{U}_2$, respectively (Fig. 1). Eq. (7) gives the relationship of the weights of the curves in Eqs. (1) and (2). Next, we define the relationship between S_i and T_i (i=0, 1, 2, 3).

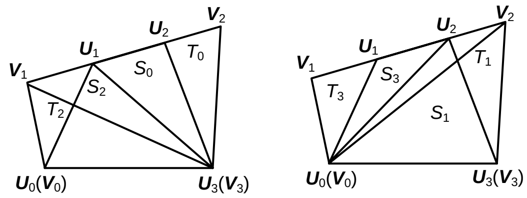

Fig. 1 Directed areas of S_i and T_i (i=0, 1, 2, 3)

By Eq. (7), it follows that

$$\frac{\|\boldsymbol{V}_1\boldsymbol{U}_1\|}{\|\boldsymbol{U}_1\boldsymbol{V}_2\|} = \frac{\beta_2}{2\beta_1},\ \frac{\|\boldsymbol{V}_1\boldsymbol{U}_2\|}{\|\boldsymbol{U}_2\boldsymbol{V}_2\|} = \frac{2\beta_2}{\beta_1}.$$

Then we have

$$\|\boldsymbol{V}_1\boldsymbol{U}_1\|:\|\boldsymbol{U}_1\boldsymbol{U}_2\|:\|\boldsymbol{U}_2\boldsymbol{V}_2\| = \beta_2(\beta_1+2\beta_2)/\beta_1:(3\beta_2):(2\beta_1+\beta_2). \tag{8}$$

Observing Fig. 1, according to Lemma 1 and by Eq. (8), it gives

$$\frac{S_0}{T_0} = \frac{S_3}{T_3} = \frac{3\beta_1\beta_2}{(\beta_1+2\beta_2)(2\beta_1+\beta_2)}.$$

Following a similar reasoning, the two equations

$$\frac{T_1 - S_2}{T_1 - T_2} = \frac{2\beta_1}{2\beta_1+\beta_2} \text{ and } \frac{T_1 - S_1}{T_1 - T_2} = \frac{\beta_1}{\beta_1+2\beta_2}$$

hold. Therefore, S_1 and S_2 can be expressed by T_1 and T_2 as

$$S_1 = \frac{2\beta_2 T_1 + \beta_1 T_2}{2\beta_2+\beta_1} \text{ and } S_2 = \frac{\beta_2 T_1 + 2\beta_1 T_2}{\beta_2 + 2\beta_1},$$

respectively. Substituting the above formulae, Eq. (7) and Eq. (8) into Theorem 2 in Wang and Wang (1992), and eliminating α_i, S_i (i=0, 1, 2, 3), Theorem 1 is proven.

5 Rational quartic DP conics

5.1 Necessary and sufficient conditions for rational quartic Bézier representation of conics

To deduce the necessary and sufficient conditions for rational quartic DP representation of conics, we first introduce the conditions for rational quartic Bézier representation of conics. Hu and Wang (2007) conducted a thorough study of degree-reducible and improperly parameterized cases. They distinguished five cases according to the control polygon and the weights, as shown in Lemma 2. This is the fundamental theoretical basis for deriving the necessary and sufficient conditions for the rational quartic DP case.

Lemma 2 Suppose a rational quartic Bézier curve is expressed as Eq. (3), and $\boldsymbol{Q}$ is the intersection point of the lines $\boldsymbol{P}_0\boldsymbol{P}_1$ and $\boldsymbol{P}_3\boldsymbol{P}_4$. Then, if one of the following five conditions holds, the rational quartic Bézier curve represents a conic (Hu and Wang, 2007).

1. (1a) The five points $\boldsymbol{P}_i$ (i=0, 1, …, 4) are coplanar.

(1b) $\boldsymbol{P}_i$ (i=0, 1, …, 4) are different to each other and the positions of $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$ are determined by one of the following three conditions:

($1b_1$) $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ are internal points of division of the line segments $\boldsymbol{P}_0\boldsymbol{Q}$ and $\boldsymbol{P}_4\boldsymbol{Q}$, respectively, and $\boldsymbol{P}_2$ is inside the triangle $\Delta\boldsymbol{P}_0\boldsymbol{Q}\boldsymbol{P}_4$ (Fig. 2a);

($1b_2$) $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ are on the extension lines of the oriented line segments $\boldsymbol{Q}\boldsymbol{P}_0$ and $\boldsymbol{Q}\boldsymbol{P}_4$, respectively, and $\boldsymbol{P}_2$ is in the domain determined by the extension lines of the oriented line segments $\boldsymbol{Q}\boldsymbol{P}_0$, $\boldsymbol{Q}\boldsymbol{P}_4$, and the line segment $\boldsymbol{P}_0\boldsymbol{P}_4$ (Fig. 2b);

($1b_3$) $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ are on the extension lines of the oriented line segments $\boldsymbol{P}_0\boldsymbol{Q}$ and $\boldsymbol{P}_4\boldsymbol{Q}$, respectively, and $\boldsymbol{P}_2$ is in the domain determined by the extension lines of the oriented line segments $\boldsymbol{Q}\boldsymbol{P}_0$, $\boldsymbol{Q}\boldsymbol{P}_4$, and the line segment $\boldsymbol{P}_0\boldsymbol{P}_4$ (Fig. 2c).

(1c) $\dfrac{\omega_1^2}{\omega_0\omega_2}=\dfrac{3}{8}\dfrac{B_2B_3}{B_1D_0}.$

(1d) $\dfrac{\omega_2^2}{\omega_1\omega_3}=\dfrac{4}{9}\dfrac{B_3D_0}{B_2D_1}=\dfrac{4}{9}\dfrac{B_1D_2}{B_2D_3}.$

(1e) $\dfrac{\omega_3^2}{\omega_2\omega_4}=\dfrac{3}{8}\dfrac{B_1B_2}{B_3D_2}.$

2. (2a), (2b) are the same as (1a) and (1b), respectively;

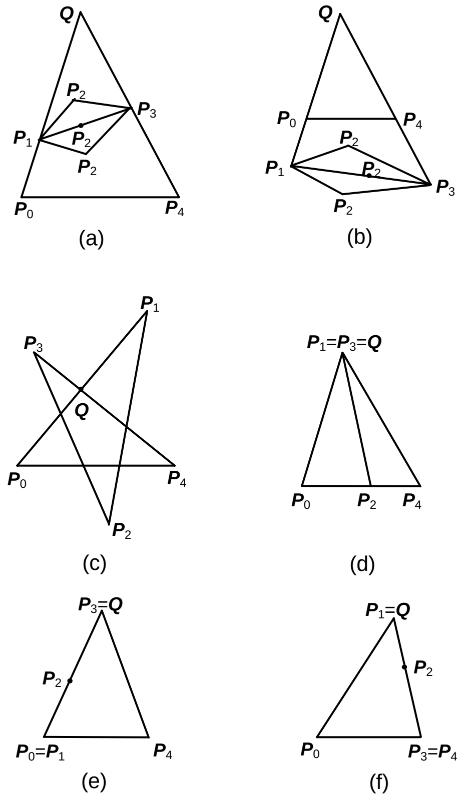

Fig. 2 The control points and control polygons of a rational quartic Bézier conic section in six different forms

(2c) $\dfrac{\omega_1^2}{\omega_0\omega_2}=\dfrac{3}{2}\dfrac{B_3D_1}{D_0^2}=\dfrac{3}{2}\dfrac{B_3^2D_3}{B_1D_0D_2};$

(2d) $\dfrac{\omega_3^2}{\omega_2\omega_4}=\dfrac{3}{2}\dfrac{B_1D_3}{D_2^2}=\dfrac{3}{2}\dfrac{B_1^2D_1}{B_3D_0D_2};$

(2e) $\dfrac{\omega_2^2}{\omega_1\omega_3}=\dfrac{2}{9\left(\sqrt{\dfrac{B_2^2D_1D_3}{B_1B_3D_0D_2}}-\dfrac{2D_1D_3}{D_0D_2}\right)}.$

3. (3a) is the same as (1a);

(3b) $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ both coincide with $\boldsymbol{Q}$, and $\boldsymbol{P}_2$ is an internal point of division of the line segment $\boldsymbol{P}_0\boldsymbol{P}_4$ (Fig. 2d);

(3c) $\omega_2=\dfrac{1}{6}\left(\sqrt{D_3/D_1}+\sqrt{D_1/D_3}\right);$

(3d) $\omega_1/\omega_3=\sqrt{D_3/D_1}.$

4. (4a) is the same as (1a);

(4b) $\boldsymbol{P}_1$, $\boldsymbol{P}_3$ coincide with $\boldsymbol{P}_0$, $\boldsymbol{Q}$, respectively, and $\boldsymbol{P}_2$ is an internal point of division of the line segment $\boldsymbol{P}_1\boldsymbol{P}_3$ (Fig. 2e);

(4c) $\dfrac{3\omega_2}{2\omega_1^2}-1=\dfrac{\omega_3}{2\omega_1^3}=\dfrac{B_2}{D_1}$.

5. (5a) is the same as (1a);

(5b) $\boldsymbol{P}_1$, $\boldsymbol{P}_3$ coincide with $\boldsymbol{Q}$, $\boldsymbol{P}_4$, respectively, and $\boldsymbol{P}_2$ is an internal point of division of the line segment $\boldsymbol{P}_1\boldsymbol{P}_3$ (Fig. 2f);

(5c) $\dfrac{3\omega_2}{2\omega_3^2}-1=\dfrac{\omega_1}{2\omega_3^3}=\dfrac{B_2}{D_3}$. Herein, B_i is the directed area of $\Delta\boldsymbol{P}_0\boldsymbol{P}_i\boldsymbol{P}_4$ (i=1, 2, 3), and D_i (i=0, 1, 2, 3) are directed areas of $\Delta\boldsymbol{P}_1\boldsymbol{P}_3\boldsymbol{P}_4$, $\Delta\boldsymbol{P}_2\boldsymbol{P}_3\boldsymbol{P}_4$, $\Delta\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_3$, and $\Delta\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$, respectively (Fig. 3).

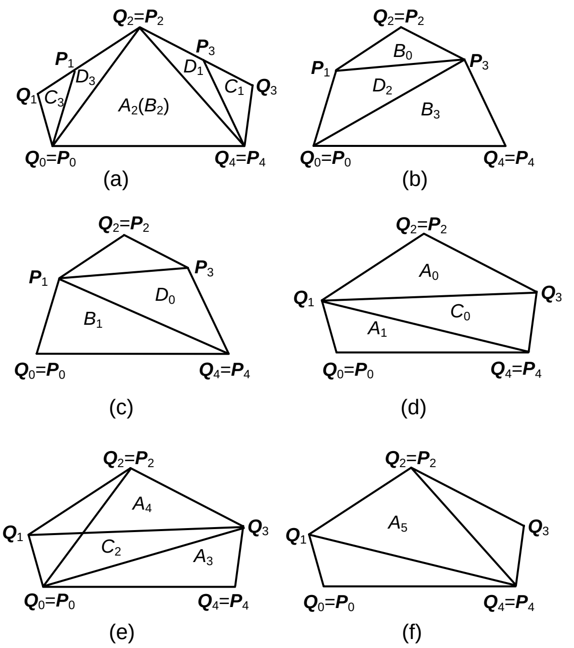

Fig. 3 The directed areas A_i (i=0, 1, …, 5), B_i, C_i and D_i (i=0, 1, 2, 3)

Remark 1 For conditions 1 and 3, the rational quartic Bézier curve is degree-reducible; for conditions 2, 4 and 5, the curve is improperly reparameterized.

5.2 Rational quartic DP representation for conics by basis transformation

According to the relationship (Eq. (6)) between the rational quartic DP curve and the Bézier curve, it is easy to see that $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ are internal points of division of the line segments $\boldsymbol{Q}_1\boldsymbol{Q}_2$ and $\boldsymbol{Q}_2\boldsymbol{Q}_3$ respectively, and $\boldsymbol{P}_i$=$\boldsymbol{Q}_i$ (i=0, 2, 4). Therefore, for the triangles which are composed of the control points of the rational quartic Bézier curve (Eq. (3)) and those of the rational quartic DP curve (Eq. (4)), we first deduce the relationship of their magnitude. We then convert various formulae relating to the rational quartic Bézier curve into those relating to the rational quartic DP curve. Hence, the necessary and sufficient conditions for representing conics by the curve in Eq. (4) are obtained.

In Fig. 3, A_i is the directed area of $\Delta\boldsymbol{Q}_0\boldsymbol{Q}_i\boldsymbol{Q}_4$ (i=1, 2, 3) respectively, C_i (i=0, 1, 2, 3) are directed areas of $\Delta\boldsymbol{Q}_1\boldsymbol{Q}_3\boldsymbol{Q}_4$, $\Delta\boldsymbol{Q}_2\boldsymbol{Q}_3\boldsymbol{Q}_4$, $\Delta\boldsymbol{Q}_0\boldsymbol{Q}_1\boldsymbol{Q}_3$, and $\Delta\boldsymbol{Q}_0\boldsymbol{Q}_1\boldsymbol{Q}_2$ respectively, A_0 and B_0 are directed areas of $\Delta\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3$ and $\Delta\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3$ respectively, and B_i (i=1, 2, 3) and D_i (i=0, 1, 2, 3) are defined in Lemma 2.

Theorem 2 Suppose a rational quartic DP curve is expressed as Eq. (4). Then, if one of the following five conditions holds, the rational quartic DP curve in Eq. (4) represents a conic.

1. (a_1) The five points $\boldsymbol{Q}_i$ (i=0, 1, …, 4) are coplanar.

(b_1) $\boldsymbol{Q}_i$ (i=0, 1, …, 4) are different to each other, and the position of $\boldsymbol{Q}_2$ is determined by one of the following two conditions:

(b_*) $\boldsymbol{Q}_2$ is on the same side of the line $\boldsymbol{Q}_0\boldsymbol{Q}_4$ as the curve in Eq. (4);

(b_{**}) $\boldsymbol{Q}_2$ is on the other side of the line $\boldsymbol{Q}_0\boldsymbol{Q}_4$ from the curve in Eq. (4).

(c_1) $$\frac{6\varphi_0\varphi_2}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)}=\frac{3\varphi_2A_2+\varphi_1A_1}{3\varphi_2A_2+\varphi_3A_3}\frac{D_0}{A_2}.$$

(d_1) $$\frac{6\varphi_2\varphi_4}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)}=\frac{3\varphi_2A_2+\varphi_3A_3}{3\varphi_2A_2+\varphi_1A_1}\frac{D_2}{A_2}.$$

(e_1) $$\frac{36\varphi_2^2}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)}=\frac{3\varphi_2A_2+\varphi_3A_3}{\varphi_3C_1}\frac{D_0}{A_2}=\frac{3\varphi_2A_2+\varphi_1A_1}{\varphi_1C_3}\frac{D_2}{A_2}.$$

In (c_1)–(e_1),

$$D_0=\frac{\varphi_3C_1}{3\varphi_2+\varphi_3}-\frac{\varphi_1\varphi_3A_0}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)}+\frac{\varphi_1(C_3+A_2-A_1)}{3\varphi_2+\varphi_1},\tag{9}$$

$$D_2=\frac{\varphi_1C_3}{3\varphi_2+\varphi_1}-\frac{\varphi_1\varphi_3A_0}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)}+\frac{\varphi_3(C_1+A_2-A_3)}{3\varphi_2+\varphi_3}.\tag{10}$$

2. (a_2)–(b_2) are the same as (a_1)–(b_1), respectively;

(c_2)

$$\frac{24\varphi_0\varphi_2}{(\varphi_1+3\varphi_2)^2(3\varphi_2+\varphi_3)^2}=\frac{D_0^2}{(3\varphi_2A_2+\varphi_3A_3)\varphi_3C_1}=\frac{3\varphi_2A_2+\varphi_1A_1}{(3\varphi_2A_2+\varphi_3A_3)^2}\frac{D_0D_2}{\varphi_1C_3}.$$

(d_2)

$$\frac{(\varphi_1+3\varphi_2)(3\varphi_2+\varphi_3)}{72\varphi_2^2}=\sqrt{\frac{\varphi_1\varphi_3A_2^2}{(3\varphi_2A_2+\varphi_3A_3)(3\varphi_2A_2+\varphi_1A_1)}\frac{C_1C_3}{D_0D_2}-\frac{2\varphi_1\varphi_3}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)}\frac{C_1C_3}{D_0D_2}}.$$

(e_2)

$$\frac{24\varphi_2\varphi_4}{(3\varphi_2+\varphi_3)^2(3\varphi_2+\varphi_1)^2}=\frac{D_2^2}{(3\varphi_2A_2+\varphi_1A_1)\varphi_1C_3}=\frac{3\varphi_2A_2+\varphi_3A_3}{(3\varphi_2A_2+\varphi_1A_1)^2}\frac{D_0D_2}{\varphi_3C_1}.$$

In (c_2)–(e_2), D_0 and D_2 are shown as in Eqs. (9) and (10) respectively.

3. (a_3) is the same as (a_1);

(b_3) $\boldsymbol{Q}_2$ is an internal point of division of the line segment $\boldsymbol{Q}_0\boldsymbol{Q}_4$, and is located on the line segment $\boldsymbol{Q}_1\boldsymbol{Q}_3$ or the extension line $\boldsymbol{Q}_3\boldsymbol{Q}_1$;

(c_3)

$$6\varphi_2=\sqrt{\frac{\varphi_1(3\varphi_2+\varphi_3)}{\varphi_3(3\varphi_2+\varphi_1)}\frac{C_3}{C_1}}+\sqrt{\frac{\varphi_3(3\varphi_2+\varphi_1)}{\varphi_1(3\varphi_2+\varphi_3)}\frac{C_1}{C_3}};$$

(d_3) $\dfrac{\varphi_1(3\varphi_2+\varphi_3)^3}{\varphi_3(3\varphi_2+\varphi_1)^3}=\dfrac{C_1}{C_3}$;

(e_3) $\dfrac{\varphi_3(3\varphi_2+\varphi_1)}{\varphi_1(3\varphi_2+\varphi_3)}=\dfrac{A_1}{A_3}$.

4. (a_4) is the same as (a_1);

(b_4) $\boldsymbol{Q}_i$ (i=0, 1, 2, 3) are collinear, and the sequence is $\boldsymbol{Q}_1$, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_3$;

(c_4)

$$\frac{\varphi_3}{3\varphi_2+\varphi_3}\left(\frac{24\varphi_2}{(3\varphi_2+\varphi_1)^2}-1\right)=\frac{8\varphi_3}{(3\varphi_2+\varphi_1)^3}=\frac{A_2}{C_1};$$

(d_4) $3\varphi_2/\varphi_1=-A_1/A_2$.

5. (a_5) is the same as (a_1);

(b_5) $\boldsymbol{Q}_i$ (i=1, 2, 3, 4) are collinear, and the sequence is $\boldsymbol{Q}_3$, $\boldsymbol{Q}_4$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_1$;

(c_5)

$$\frac{\varphi_1}{3\varphi_2+\varphi_1}\left(\frac{24\varphi_2}{(3\varphi_2+\varphi_3)^2}-1\right)=\frac{8\varphi_1}{(3\varphi_2+\varphi_3)^3}=\frac{A_2}{C_3};$$

(d_5) $3\varphi_2/\varphi_3=-A_3/A_2$.

Proof At first, when the rational quartic Bézier curve in Eq. (3) is converted to the rational quartic DP curve, condition (1a) in Lemma 2 is equivalent in that the five control points $\boldsymbol{Q}_i$ (i=0, 1, …, 4) of the curve in Eq. (4) are coplanar; i.e., condition (a_1) holds. Also, by Eq. (6), $\boldsymbol{Q}_i$ coincides with $\boldsymbol{P}_i$ (i=0, 2, 4) respectively, and then

$$B_2=A_2. \tag{11}$$

As for the non-degenerate triangles $\Delta\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$, $\Delta\boldsymbol{P}_0\boldsymbol{Q}_1\boldsymbol{P}_2$, $\Delta\boldsymbol{P}_2\boldsymbol{P}_3\boldsymbol{P}_4$, and $\Delta\boldsymbol{P}_2\boldsymbol{Q}_3\boldsymbol{P}_4$ (Fig. 3a), according to Lemma 1 and Eq. (6), we have

$$\begin{cases}\dfrac{B_0}{A_0}=\dfrac{\varphi_1\varphi_3}{(3\varphi_2+\varphi_1)(3\varphi_2+\varphi_3)},\\ \dfrac{D_3}{C_3}=\dfrac{\varphi_1}{3\varphi_2+\varphi_1},\ \dfrac{D_1}{C_1}=\dfrac{\varphi_3}{3\varphi_2+\varphi_3}.\end{cases} \tag{12}$$

Next, we discuss the representation of condition 3 in Lemma 2 when the curve in Eq. (3) is transformed to the curve in Eq. (4). As shown in Fig. 2d, by Eqs. (6) and (12), conditions (3c) and (3d) in Lemma 2 are equivalent to conditions (c_3) and (d_3), respectively. To meet (3b) in Lemma 2, $\boldsymbol{Q}_2$ should be an internal point of division of the line segment $\boldsymbol{Q}_0\boldsymbol{Q}_4$. As $\boldsymbol{P}_1$ coincides with $\boldsymbol{P}_3$ and by Eq. (6), the condition that $\boldsymbol{Q}_2$ is on the line segment $\boldsymbol{Q}_1\boldsymbol{Q}_3$ or the extension of the line segment $\boldsymbol{Q}_3\boldsymbol{Q}_1$ is equivalent to condition (b_3). As shown in Fig. 4b, by Eq. (6) and Lemma 1, we have

$$\frac{\|\boldsymbol{P}_2\boldsymbol{P}_1\|}{\|\boldsymbol{P}_2\boldsymbol{Q}_1\|}=\frac{\varphi_1}{3\varphi_2+\varphi_1},\ \frac{\|\boldsymbol{P}_2\boldsymbol{P}_1\|}{\|\boldsymbol{P}_2\boldsymbol{Q}_3\|}=\frac{\varphi_3}{3\varphi_2+\varphi_3},\ \frac{\|\boldsymbol{P}_2\boldsymbol{Q}_1\|}{\|\boldsymbol{P}_2\boldsymbol{Q}_3\|}=\frac{A_1}{A_3}.$$

Inserting the first two terms into the third term of the above formulae yields condition (e_3) in Theorem 2. Therefore condition 3 in Lemma 2 is equivalent to condition 3 in Theorem 2.

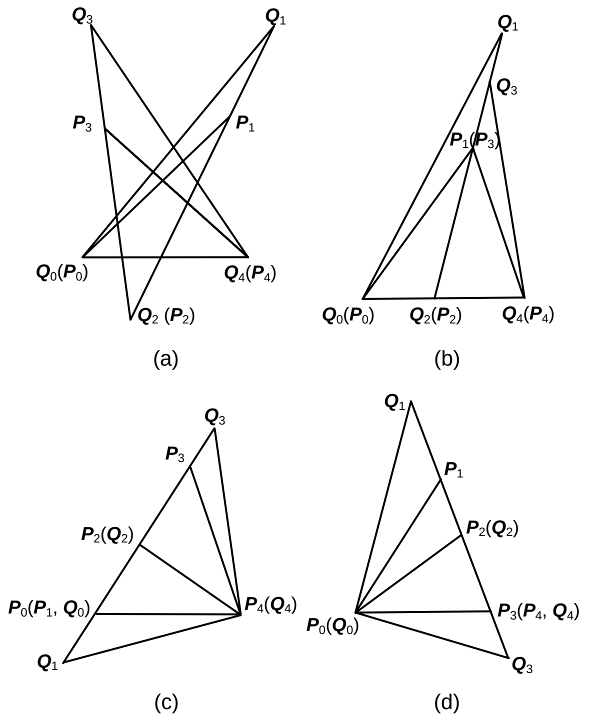

Fig. 4 Relationship of the control points of curves in Eqs. (3) and (4) in the case of Figs. 1c–1f

Thirdly, we discuss the representation of condition 4 in Lemma 2 when the curve in Eq. (3) is transformed to the curve in Eq. (4). As shown in Fig. 4c, as $\Delta \boldsymbol{P}_2\boldsymbol{P}_3\boldsymbol{P}_4$ and $\Delta \boldsymbol{P}_2\boldsymbol{Q}_3\boldsymbol{P}_4$ are non-degenerate, we substitute Eqs. (6), (11) and the second term of Eq. (12) into (4c) in Lemma 2, and replace ω_i (i=1, 2, 3), B_2 and D_1 by φ_i (i=1, 2, 3), A_2 and C_1 respectively, then obtain (c_4) in Theorem 2. To meet (4b) in Lemma 2, Eq. (6) implies $\boldsymbol{Q}_i$ (i=0, 1, 2, 3) are collinear, and the sequence is $\boldsymbol{Q}_1$, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_3$; i.e., condition (b_4) holds. Also, as $\boldsymbol{P}_0$ coincides with $\boldsymbol{P}_1$, (d_4) in Theorem 2 holds by Eq. (6). Then condition 4 in Lemma 2 is equivalent to condition 4 in Theorem 2.

To discuss the representation of condition 5 in Lemma 2 when the curve in Eq. (3) is converted to the curve in Eq. (4), we first observe Figs. 4c and 4d. It is easy to detect that the sequence of the control points in condition 4 in Lemma 2 is symmetric to the control points in condition 5 about the point $\boldsymbol{P}_2$. Therefore, by the equivalent condition 4 in Lemma 2, we deduce the corresponding equivalent condition 5 in Lemma 2. Specifically, the equivalent condition, condition 5 in Theorem 2, is obtained by replacing the subscripts 0 and 1 by 4 and 3 respectively in condition 4 in Theorem 2.

As for conditions 1 and 2 in Lemma 2, whatever is the distribution of the control points of the rational quartic Bézier curve in Eq. (3) (Figs. 2a, 2b or 2c), according to Fig. 3a and Fig.4a, they all satisfy

$$D_0 + B_0 = D_1 + S_{\Delta \boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_4}. \tag{13}$$

Also, by Eq. (6) and Lemma 1, we have

$$S_{\Delta \boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_4} = \frac{\varphi_1}{3\varphi_2+\varphi_1} S_{\Delta \boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_4} = \frac{\varphi_1(C_3+A_2-A_1)}{3\varphi_2+\varphi_1}.$$

Substituting the above formula, the first, and third terms of Eq. (12) into Eq. (13), and eliminating B_0, D_1, and $S_{\Delta \boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_4}$, we have the representation of D_0 in Eq. (9) by C_1, C_2, and A_i (i=0, 1, 2). Then, D_2 is handled in a similar fashion to obtain Eq. (10).

As for the condition (1b) about control points, we analyze the relationship between $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ and $\boldsymbol{Q}_1$, $\boldsymbol{Q}_3$ in Eq. (6). It is easy to convert ($1b_1$) and ($1b_2$) to the condition that $\boldsymbol{Q}_2$ is on the same side of the line $\boldsymbol{Q}_0\boldsymbol{Q}_4$ as the curve in Eq. (4), i.e., (b_*); and convert ($1b_3$) to the condition that $\boldsymbol{Q}_2$ is on the other side of the line $\boldsymbol{Q}_0\boldsymbol{Q}_4$ from the curve in Eq. (4), i.e., (b_{**}).

Finally, substituting Eqs. (9)–(13) to (1c)–(1e) or (2c)–(2e) in Lemma 2, we can obtain the conditions (c_1)–(e_1) and (c_2)–(e_2). Then conditions 1 and 2 hold.

To sum up, the five cases above show that the theorem is proven.

6 Algorithms for modeling conics in DP form

For conciseness, we discuss the algorithms in the quartic case only.

Algorithm 1 (Judge whether a rational quartic DP curve is a conic section) A rational quartic DP curve with control points $\boldsymbol{Q}_i$ and weights φ_i (i=0, 1, …, 4).

Step 1: If $\boldsymbol{Q}_i$ (i=0, 1, …, 4) satisfy (a_1), then go to Step 2; else return 'No'.

Step 2: If $\boldsymbol{Q}_2$ is located on the line segment $\boldsymbol{Q}_0\boldsymbol{Q}_4$, then judge whether the control points and the weights satisfy (c_3)–(e_3). If so, return 'Yes'; else go to Step 3.

Step 3: If $\boldsymbol{Q}_i$ (i=0, 1, 2, 3) are collinear and the sequence is $\boldsymbol{Q}_1$, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_3$, judge whether the control points and the weights satisfy (c_4)–(d_4). If so, return

'Yes'; else go to Step 4.

Step 4: If $\boldsymbol{Q}_i$ (i=1, 2, 3, 4) are collinear and the sequence is $\boldsymbol{Q}_3$, $\boldsymbol{Q}_4$, $\boldsymbol{Q}_2$, $\boldsymbol{Q}_1$, judge whether the control points and the weights satisfy (c_5)–(d_5). If so, return 'Yes'; else go to Step 5.

Step 5: If $\boldsymbol{Q}_2$ is on the same side of the line $\boldsymbol{Q}_0\boldsymbol{Q}_4$ as the curve in Eq. (4), judge whether the control points and the weights satisfy (c_1)–(e_1) or (c_2)–(e_2). If so, return 'Yes'; else go to Step 6.

Step 6: If $\boldsymbol{Q}_2$ is on the other side of the line $\boldsymbol{Q}_0\boldsymbol{Q}_4$ from the curve in Eq. (4), judge whether the control points, the weights satisfy (c_1)–(e_1) or (c_2)–(e_2). If so, return 'Yes'; else return 'No'.

By Theorem 2, Algorithm 2 provides the rational quartic DP form of a given conic section.

Algorithm 2 (Designing a given conic section in rational quartic DP form)

Step 1: If the two end tangent lines are not parallel, calculate the corresponding control points $\boldsymbol{R}_i$ (i=0, 1, 2) and weights 1, u_1, and u_2 in rational quadratic DP form. Considering the curve in Eq. (4) being in standard form, there are $\boldsymbol{Q}_0=\boldsymbol{R}_0$, $\boldsymbol{Q}_4=\boldsymbol{R}_2$, $\varphi_0=\varphi_4=1$, and go to Step 2; else go to Step 5.

Step 2: Input the kind of the control points we need: conditions 1–5. If condition 1 or 3, go to Step 3; else go to Step 4.

Step 3: Set $\alpha=\frac{1+u_2^2-4u_1u_2}{4u_2(1-u_1)}$, $\beta=\frac{1+u_2^2-4u_1}{4u_2(u_2-u_1)}$,

$$\xi=\sqrt{u_2^{-1}+u_2}\Big/2,\ \eta=\sqrt{(1+u_2^2)u_2}\Big/2,\ \psi=-\frac{1+u_2^2}{4u_1u_2}.$$

1. Condition 1, when $u_1>0$, we choose $a_1>0$ so that the positions of the control points are as (b_*), if a_1 satisfies one of the following three situations:

(1) If $u_1<\min\{1, u_2\}$, then $a_1>\max\{\alpha, \beta\}$;

(2) If $u_1>\max\{1, u_2\}$, then $a_1<\min\{\alpha, \beta\}$;

(3) If $\min\{1, u_2\}<u_1<\max\{1, u_2\}$, then $\min\{\alpha, \beta\}<a_1<\max\{\alpha, \beta\}$.

We choose $a_1<0$ so that the positions of the control points are as (b_{**}), if a_1 satisfies one of the following four situations:

(1) If $\max\{\xi^2, u_2^{-1}\eta^2\}<u_1<\min\{1, u_2\}$, then $\max\{\alpha, \beta, \psi\}<a_1<0$;

(2) If $u_1>\max\{1, u_2, \xi, \eta\}$, then $\psi<a_1<\min\{\alpha, \beta, 0\}$;

(3) If $\max\left\{1, u_2^{-1}\eta^2, \xi, \left(1+u_2-\sqrt{2u_2}\right)\Big/2\right\}<u_1<\min\left\{u_2, \left(1+u_2+\sqrt{2u_2}\right)\Big/2\right\}$, then $\max\{\beta, \psi\}<a_1<\alpha$;

(4) If $\max\left\{u_2, \eta, \xi^2, \left(1+u_2-\sqrt{2u_2}\right)\Big/2\right\}<u_1<\min\left\{1, \left(1+u_2+\sqrt{2u_2}\right)\Big/2\right\}$, then $\max\{\alpha, \psi\}<a_1<\beta$.

When $-\min\{\xi, \eta\}<u_1<0$, we choose a_1 satisfying $\max\{\alpha, \beta\}<a_1<\psi$, so that the positions of the control points are as (b_*).

The control points and weights are obtained using the following formulae:

$$\begin{cases}\boldsymbol{Q}_1=\dfrac{(4a_1-u_2^{-1})\boldsymbol{R}_0+4u_1(1-a_1)\boldsymbol{R}_1-u_2\boldsymbol{R}_2}{4a_1-u_2^{-1}+4u_1(1-a_1)-u_2},\\ \boldsymbol{Q}_2=\dfrac{\boldsymbol{R}_0+4a_1u_1u_2\boldsymbol{R}_1+u_2^2\boldsymbol{R}_2}{1+4a_1u_1u_2+u_2^2},\\ \boldsymbol{Q}_3=\dfrac{\boldsymbol{R}_0+4u_1(a_1u_2-1)\boldsymbol{R}_1-u_2^2(4a_1-1)\boldsymbol{R}_2}{1+4u_1(a_1u_2-1)-u_2^2(4a_1-1)},\\ \varphi_1=\left(4(a_1+u_1-a_1u_1)-(u_2^{-1}+u_2)\right)\Big/2,\\ \varphi_2=(1+4a_1u_1u_2+u_2^2)/(6u_2),\\ \varphi_3=(4u_1+4a_1u_2^2-4a_1u_1u_2-u_2^2-1)/2u_2.\end{cases}\quad(14)$$

2. Condition (3), if $u_1>\max\{\xi^2, u_2^{-1}\eta^2\}$, then the positions of the control points are as (b_3), and hence the weights and control points are obtained as follows:

$$\begin{cases}\varphi_1=(4u_1u_2-1-u_2^2)/(2u_2),\\ \varphi_2=(1+u_2^2)/(6u_2),\\ \varphi_3=(4u_1-1-u_2^2)/(2u_2),\\ \boldsymbol{Q}_1=(u_2^{-1}\boldsymbol{R}_0-4u_1\boldsymbol{R}_1+u_2\boldsymbol{R}_2)/(u_2^{-1}-4u_1+u_2),\\ \boldsymbol{Q}_2=(\boldsymbol{R}_0+u_2^2\boldsymbol{R}_2)/(1+u_2^2),\\ \boldsymbol{Q}_3=(\boldsymbol{R}_0-4u_1\boldsymbol{R}_1+u_2^2\boldsymbol{R}_2)/(1-4u_1+u_2^2).\end{cases}$$

Step 4: Convert the weights $\{1, u_1, u_2\}$ to $\{1, u_1\big/\sqrt{u_2}, 1\}$ by a linear parameter transformation $t=s\Big/\left(s+\sqrt{u_2}(1-s)\right)$. Denote the new middle weight as u_1.

1. Condition 2, when $u_1>0$, we choose $a_1>0$ and $b_1\neq0$ satisfying

$$0<2b_1^2+2a_1^2+(4a_1b_1+1)u_1<4\min\{(b_1+a_1u_1),(a_1+b_1u_1)\}.$$

Then the positions of the control points are as (b_i), and the weights and control points are

$$\begin{cases} \varphi_1 = 4b_0(b_1 + a_1u_1) - \left(2b_1^2 + 2a_1^2 + (4a_1b_1 + b_0)u_1\right), \\ \varphi_2 = \left(2b_1^2 + 2a_1^2 + (4a_1b_1 + b_0)u_1\right)/3, \\ \varphi_3 = 4(a_1 + b_1u_1) - \left(2b_1^2 + 2a_1^2 + (4a_1b_1 + b_0)u_1\right), \\ \boldsymbol{Q}_1 = \dfrac{2b_1(2b_0 - b_1)\boldsymbol{R}_0 + (4a_1b_0 - 4a_1b_1 - b_0)u_1\boldsymbol{R}_1 - 2a_1^2\boldsymbol{R}_2}{2b_1(2b_0 - b_1) + (4a_1b_0 - 4a_1b_1 - b_0)u_1 - 2a_1^2}, \\ \boldsymbol{Q}_2 = \dfrac{2b_1^2\boldsymbol{R}_0 + (4a_1b_1 + b_0)u_1\boldsymbol{R}_1 + 2a_1^2\boldsymbol{R}_2}{2b_1^2 + (4a_1b_1 + b_0)u_1 + 2a_1^2}, \\ \boldsymbol{Q}_3 = \dfrac{2b_1^2\boldsymbol{R}_0 + (4a_1b_1 + b_0 - 4b_1)u_1\boldsymbol{R}_1 - 2a_1(2 - a_1)\boldsymbol{R}_2}{2b_1^2 + (4a_1b_1 + b_0 - 4b_1)u_1 - 2a_1(2 - a_1)}, \end{cases} \tag{15}$$

where b_0=1.

If we choose a_1<0 and b_1 satisfying $0<2b_1^2+2a_1^2+(4a_1b_1-1)u_1<4\min\{-(b_1+a_1u_1), (a_1+b_1u_1)\}$, then the positions of the control points are as (b_{**}), and the weights and control points are as Eq. (15), where $b_0=-1$.

When $-1<u_1<0$, we choose a_1>0 and b_1 satisfying $0<2b_1^2+2a_1^2+(4a_1b_1+1)u_1<4\min\{(b_1+a_1u_1), (a_1+b_1u_1)\}$. Then the positions of the control points are as (b_*), and the weights and control points are as Eq. (15), where b_0=1.

2. Condition 4, choose a_1>0 satisfying $2a_1^2+u_1<4a_1\min\{1, u_1\}$. Then the positions of the control points are as (b_4), and the weights and control points are

$$\begin{cases} \varphi_1 = 4a_1 - 2a_1^2 - u_1, \quad \varphi_2 = (2a_1^2 + u_1)/3, \\ \varphi_3 = 4a_1u_1 - 2a_1^2 - u_1; \\ \boldsymbol{Q}_1 = \left(2a_1(a_1 - 2)\boldsymbol{R}_0 + u_1\boldsymbol{R}_1\right)/(2a_1^2 - 4a_1 + u_1), \\ \boldsymbol{Q}_2 = (2a_1^2\boldsymbol{R}_0 + u_1\boldsymbol{R}_1)/(2a_1^2 + u_1), \\ \boldsymbol{Q}_3 = \left(2a_1^2\boldsymbol{R}_0 - u_1(4a_1 - 1)\boldsymbol{R}_1\right)/(2a_1^2 - 4a_1u_1 + u_1). \end{cases}$$

3. Condition 5, choose a_1>0 satisfying $2a_1^2+u_1<4a_1\min\{1, u_1\}$. Then the positions of the control points are as (b_5), and the weights and control points are

$$\begin{cases} \varphi_1 = 4a_1u_1 - 2a_1^2 - u_1, \varphi_2 = \dfrac{2a_1^2 + u_1}{3}, \varphi_3 = 4a_1 - 2a_1^2 - u_1; \\ \boldsymbol{Q}_1 = \left(-u_1(4a_1 - 1)\boldsymbol{R}_1 + 2a_1^2\boldsymbol{R}_2\right)/(-4a_1u_1 + u_1 + 2a_1^2), \\ \boldsymbol{Q}_2 = (u_1\boldsymbol{R}_1 + 2a_1^2\boldsymbol{R}_2)/(u_1 + 2a_1^2), \\ \boldsymbol{Q}_3 = \left(u_1\boldsymbol{R}_1 + 2a_1(a_1 - 2)\boldsymbol{R}_2\right)/(u_1 + 2a_1^2 - 4a_1). \end{cases}$$

Step 5: Under these conditions, the curve is a semi-circle or semi-ellipse. Suppose the curve is expressed as $x^2/a^2+y^2/b^2=1$, with end parameter angles of θ, $\pi+\theta$ respectively. Input the kind of the curve we need: case 1 or case 2. Obviously the two control endpoints are $\begin{bmatrix} a\cos\theta \\ b\sin\theta \end{bmatrix}$, $\begin{bmatrix} -a\cos\theta \\ -b\sin\theta \end{bmatrix}$.

Case 1: Choose $a_1>\xi^2\min\{1, u_2^{-1}\}$, and then the weights are 1, $2a_1-2\xi^2$, $2\xi^2/3$, $2a_1u_2-2\xi^2$, 1; and the other three middle control points are

$$\frac{1}{4a_1 - u_2^{-1} - u_2}\begin{bmatrix} a((4a_1 + u_2 - u_2^{-1})\cos\theta - 4\sqrt{u_2}(1 - a_1)\sin\theta) \\ b((4a_1 + u_2 - u_2^{-1})\sin\theta + 4\sqrt{u_2}(1 - a_1)\cos\theta) \end{bmatrix},$$

$$\frac{1}{1 + u_2^2}\begin{bmatrix} a((1 - u_2^2)\cos\theta - 4a_1\sqrt{u_2^3}\sin\theta) \\ b((1 - u_2^2)\sin\theta + 4a_1\sqrt{u_2^3}\cos\theta) \end{bmatrix},$$

$$\frac{-1}{4a_1u_2^2 - u_2^2 - 1}\begin{bmatrix} a((4a_1u_2^2 + 1 - u_2^2)\cos\theta + 4\sqrt{u_2}(1 - a_1u_2)\sin\theta) \\ b((4a_1u_2^2 + 1 - u_2^2)\sin\theta - 4\sqrt{u_2}(1 - a_1u_2)\cos\theta) \end{bmatrix}.$$

Case 2: Choose a_1, and b_1>0 satisfying $\max\{(a_1-1)^2+b_1^2, (b_1-1)^2+a_1^2\}<1$, and then the weights are 1, $2(2b_1-a_1^2-b_1^2)$, $2(a_1^2+b_1^2)/3$, $2(2a_1-a_1^2-b_1^2)$, 1; and the other three middle control points are

$$\frac{1}{2b_1 - a_1^2 - b_1^2}\begin{bmatrix} a((a_1^2 - b_1^2 + 2b_1)\cos\theta - (2a_1b_1 - 2a_1 - 1/2)\sin\theta) \\ b((a_1^2 - b_1^2 + 2b_1)\sin\theta + (2a_1b_1 - 2a_1 - 1/2)\cos\theta) \end{bmatrix},$$

$$\frac{1}{2a_1^2 + 2b_1^2}\begin{bmatrix} a(2(b_1^2 - a_1^2)\cos\theta - (4a_1b_1 + 1)\sin\theta) \\ b(2(b_1^2 - a_1^2)\sin\theta + (4a_1b_1 + 1)\cos\theta) \end{bmatrix},$$

$$\frac{-1}{2a_1 - a_1^2 - b_1^2}\begin{bmatrix} a((b_1^2 - a_1^2 + 2a_1)\cos\theta + (2b_1 - 2a_1b_1 - 1/2)\sin\theta) \\ b((b_1^2 - a_1^2 + 2a_1)\sin\theta - (2b_1 - 2a_1b_1 - 1/2)\cos\theta) \end{bmatrix}.$$

7 Numerical experiments

Example 1 Given a rational cubic/quartic DP curve, judge whether it is a conic section.

1. The control points: (−1, 0), (−1.8759, 0.2190), (0.3861, 0.8911), (1.6752, −0.1911), and (1, 0); the

weights: 1, 0.9133, 0.6733, 1.0467, and 1.

2. The control points: (−1, 0), (−3.1667, −0.8333), (−0.6176, 0.1471), (2.5286, 1.3571), and (1, 0); the weights: 1, 0.72, 1.36, 1.68, and 1.

3. The control points: (−1, 0), (−0.4222, 1.0667), (0.9333, 0.8), and (1, 0); the weights: 1, 1.5, 2, and 1.5.

For case 1, according to Algorithm 1, as $\boldsymbol{Q}_2$ is on the same side of the line segment $\boldsymbol{Q}_0\boldsymbol{Q}_4$ as the curve, we check whether the control points and weights satisfy conditions (c_1)–(e_1) or (c_2)–(e_2). The directed areas needing to be calculated are

$$A_0=1.6572,\ A_1=0.2190,\ A_2=0.8911,\ A_3=-0.1911,\ C_1=0.2422,\ C_3=0.5420.$$

Obviously, conditions (c_2)–(e_2) hold. Thus, this curve is a conic section (Fig. 5a).

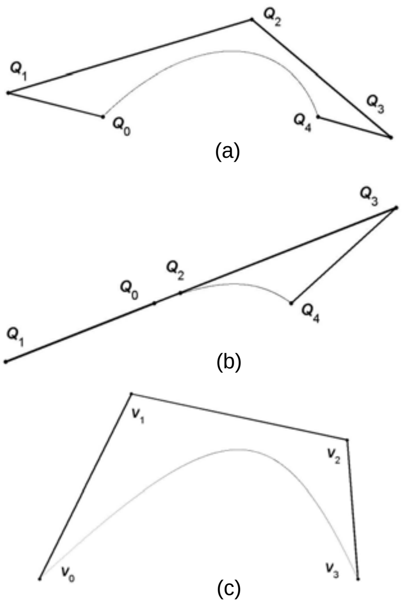

Fig. 5 A rational quartic DP representation for a conic section

For case 2, according to Algorithm 1, as $\boldsymbol{Q}_i$ (i=0, 1, 2, 3) are collinear, and the sequence is $\boldsymbol{Q}_1$, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_2$, and $\boldsymbol{Q}_3$, we check whether the control points and weights satisfy (c_4)–(d_4). The directed areas needing to be calculated are

$$A_1=-0.8333,\ A_2=0.1471,\ C_1=1.2101.$$

Obviously, conditions (c_4)–(d_4) hold. Thus, this curve is a conic section (Fig. 5b).

For case 3, according to Theorem 1, this curve satisfies case 1. To check whether it satisfies cases 2 and 3, we calculate the following directed areas:

$$T_0=0.5333,\ T_1=0.8,\ T_2=1.0667,\ T_3=0.8.$$

Obviously, for this curve, cases 2 and 3 hold. Thus, it is a conic section (Fig. 5c).

Example 2 A semi-ellipse satisfies $x^2/25+y^2/9=1$ and its two end parameter angles are π/4, 5π/4 respectively. We will represent its rational quartic DP form.

By Algorithm 2, we go directly to Step 5. Choose the kind of the curve as condition 2, and a_1=1/3, b_1=1/2, and then the control points are (3.5355, 2.1213), (5.6876, 2.3058), (−6.7991, 5.7112), (−11.2494, −4.4355), and (−3.5355, −2.1213), and the weights are 1, 1.2778, 0.2407, 0.6111, and 1 (Fig. 6).

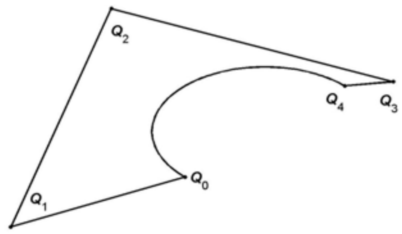

Fig. 6 A rational quartic DP representation for a semi-ellipse

Example 3 Given a rational quadratic DP curve whose control points are (−1, 0), (0.2, 0.8), and (1, 0) and weights are 1, 1.5, and 0.5, we will represent it in rational quartic DP form.

By Algorithm 2, if the curve belongs to condition 1, set a_1=0.5, and then its weights are 1, 1.25, 0.9167, 3.75, and 1, and control points are (−1, 0), (0.04, 0.96), (−0.1636, 0.4364), (0.5733, 0.96), and (1, 0) (Fig. 7a). If the curve belongs to condition 2, set a_1=−0.1, and then its weights are 1, 1.85, 0.3167, 4.95, and 1, and control points are (−1, 0), (0.8703, 1.4270), (−0.8526, −0.2526), (0.3859, 1.0182), and (1, 0) (Fig. 7c). If the curve belongs to condition 5, then its weights are 1, 1.75, 0.4167, 4.75, 1, and the control points are (−1, 0), (0.7714, 1.3714), (−0.6, 0), (0.4105, 1.0105), and (1, 0) (Fig. 7b).

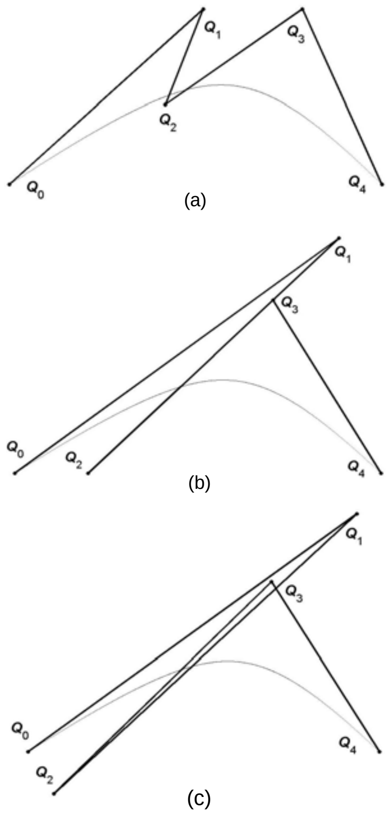

Fig. 7 A rational quartic DP representation for a conic section: (a) Condition 1; (b) Condition 5; (c) Condition 2

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