Journal of Zhejiang University-SCIENCE C (Computers & Electronics)
ISSN 1869-1951 (Print); ISSN 1869-196X (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Global stability analysis of computer networks with arbitrary topology and time-varying delays*

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Received Apr. 17, 2009; Revision accepted Sept. 5, 2009; Crosschecked Dec. 30, 2009

Abstract: In this paper, we determine the delay-dependent conditions of global asymptotic stability for a class of multi-dimensional nonlinear time-delay systems with application to computer communication networks. A nonlinear delayed model is considered for a rate-based congestion control system of a heterogeneous network with arbitrary topology and time-varying delays. We propose a Lyapunov-based method to obtain a sufficient condition under which global asymptotic stability of the equilibrium is guaranteed. The main contribution of the paper lies in considering time variations of delays in a heterogeneous network which may be applicable in actual networks. Moreover, we obtain conditions for Internet-style networks with multi-source multi-link topology. We first prove the stability for a class of nonlinear time-delay systems. Then, we apply the results to a Kelly's rate-based approximation of the congestion control system.

Key words: Internet congestion control, Global stability, Nonlinear time-delay system, Time-varying delay
doi:10.1631/jzus.C0910216 **Document code:** A **CLC number:** TP393

1 Introduction

A congestion control system in computer communication networks uses a mechanism at sources to dynamically adjust the sending rates of information to avoid congestion at links. Nonlinear delayed differential equations (NDDEs) have been widely used for modeling dynamic behavior in network congestion control systems (Low *et al*., 2002; Deb and Srikant, 2003; Shakkottai and Srikant, 2004; Srikant, 2004).

One of the most important challenges is to provide the conditions that will ensure the stability of a congestion control system to improve network performance. It is difficult to analyze global stability for the case of an arbitrary network with delay and nonlinearity. Several studies have dealt with the stability analysis of nonlinear undelayed or linearized delayed systems (Kelly *et al*., 1998; Low and Lapsley, 1999; Kelly, 2000; Vinnicombe, 2000; Johari and Tan, 2001; Massoulie, 2002; Wen and Arcak, 2004; Alpcan and Basar, 2005). However, the existence of time delays may cause instability or performance degradation (Ranjan *et al*., 2004). Moreover, the stability analyses based on linearized systems are usually misleading, because they do not take into account the nonlinearity of the systems.

Since the work of Kelly *et al*. (1998), there have been many studies on the stability of NDDE models of networks, in which the congestion control system has been viewed as a distributed control algorithm for achieving fair resource allocation. These methods formulate an optimization problem to maximize a utility function such that the aggregate rate of sources which share a link does not reach the capacity of the

* Project (No. T500-11453) supported in part by the Iran Telecommunication Research Center, Iran

link. The congestion control system includes a first-order primal algorithm for transmission control protocol (TCP) rate control and a first-order dual algorithm for the active queue management (AQM) scheme. It has been proven that in the absence of delays these algorithms are globally asymptotically stable (Kelly *et al*., 1998; Low and Lapsley, 1999; Kelly, 2000; Paganini, 2002). In subsequent studies the linear stability properties of the congestion control algorithms have been investigated in the presence of non-negligible delays (Vinnicombe, 2000; Johari and Tan, 2001; Massoulie, 2002). However, the stability conditions in these studies were based on a linearized system or local stability analysis in which the effect of nonlinearity is ignored.

Most earlier studies were concerned with the local or global stability of the congestion control algorithms with homogeneous delays (Low and Lapsley, 1999; Johari and Tan, 2001; Deb and Srikant, 2003; Wen and Arcak, 2004). However, in real networks, communication delays are rarely identical. Several recent studies have successfully dealt with heterogeneous delays (Liu *et al*., 2005; Paganini *et al*., 2005; Tian, 2005a; 2005b; Tan *et al*., 2006; Guo *et al*., 2007; Zhang *et al*., 2007; Zhang and Loguinov, 2008). Furthermore, Long *et al*. (2003) and Gao *et al*. (2005) investigated the local stability of a random exponential marking (REM) algorithm with time-varying delays.

There have been numerous studies reporting the global stability analysis of networks (Fan *et al*., 2004; Wen and Arcack, 2004; Ranjan *et al*., 2006; Peet and Lall, 2007). The methods used were usually based on Lyapunov-Razumhikin (L-R) functions (Deb and Srikant, 2003; Wang and Pagannini, 2006; Ying *et al*., 2006) or Lyapunov-Krasovskii (L-K) functions (Mazenc and Niculescu, 2003; Papachristodoulou *et al*., 2004; Alpcan and Basar, 2005). In these cases, sufficient condition for global stability was obtained with varying degrees of conservatism with respect to restrictions on system parameters or delays or both. The procedure typically followed in the most of the previous analyses was for smaller-scale cases involving simple network topologies such as a single bottleneck link with a fixed delay. Moreover, delays were considered as a fixed-value parameter. To the best of our knowledge, no analytical study has been published on the global stability of Kelly's model for heterogeneous networks with arbitrary topology and time-varying delays.

In this paper, we analyze an extension of Kelly's optimization framework for a rate allocation problem including time-varying delays, and thereby, we derive the sufficient condition for global asymptotic stability. Our analysis provides new results for global stability analysis in the case of a multi-source, multi-link and heterogeneous network.

Our approach was inspired by a novel technique for global stability analysis introduced by Fan and Zou (2004). The condition derived here depends on the system parameters and some bounds on the states and delays of the system. This condition can be used for designing a congestion control algorithm to achieve fair resource allocation. We first analyze the global stability of a general nonlinear delayed system as a model of the networks. Then our theoretical results are applied to a network congestion control system. Simulation results show the applicability and effectiveness of the results.

The paper is organized as follows: in the next section, as we introduce the rate control problem and the primal algorithms, we briefly review the dynamics of a system with time-varying propagation delays. Then, we present our main result—a global stability analysis for a class of nonlinear time-delay systems. After that, the stability criteria are given for the congestion control algorithm in general homogeneous and heterogeneous networks. Numerical simulations are given to verify the theoretical results. Finally, we conclude the paper in the last section.

2 Problem statement

2.1 Rate control problem

We consider a data network with general topology consisting of an interconnection of a set of sources (users) $S=\{1, 2, ..., n\}$, which use a set of links (resources) $L=\{1, 2, \dots, m\}$ (Fig. 1). Each source i has an associated non-negative transmission rate $x_i(t)\geq 0$ and each link l has a fixed capacity $c_l>0$. Associated with each source i is a route r_i, which is the collection of links through which information from that source i is flowing. The relation between sources and links can be defined as a full row rank routing matrix $\boldsymbol{R}$ of dimension $n\times m$ where

$$R_{li} = \begin{cases} 1, & l \in r_i, \\ 0, & l \notin r_i, \end{cases} \quad \text{for } i \in S,\ l \in L.$$

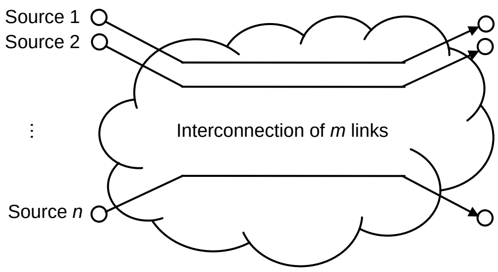

Fig. 1 Network with general topology

Each source i attains a utility of $U_i(x_i)$. The utility functions are used to select the desired rate allocation among the sources (i.e., the desired operating point of the system). The utility $U_i(x_i)$ must be an increasing, strictly concave, and continuously differentiable function over the range $x_i(t) \geq 0$ (Kelly *et al*., 1998). Each link l has a strictly increasing function of total rates going through it, namely the price function $p_l(y)$, which is the price of link l and may depend on link congestion, loss probability, etc. A congestion control scheme can be interpreted as a resource allocation problem formulated to maximize the aggregate utility over the source rates $\{x_i, i \in S\}$, subject to the constraints of total link capacity, i.e., for all links, $\sum_{l \in r_i} x_i \leq c_l$. Formally, the optimization problem is

$$\begin{aligned} &\max \left(\sum_{i \in S} U_i(x_i) - \int_0^{\sum_{i:l \in r_i} x_i} p_l(y) \mathrm{d}y \right) \\ &\text{s.t. } \sum_{l \in r_i} x_i(t) \leq c_l \quad \forall\, l \in L \qquad (1) \\ &x_i(t) \geq 0, \ i \in S, t \geq 0. \end{aligned}$$

It can be shown that all transmission rates of sources can be found such that the above optimization problem can be uniquely solved (Kelly *et al*., 1998). The unique solution of this optimization problem, called the primal solution, can be obtained using a gradient projection algorithm (Low and Lapsley, 1999).

2.2 The primal algorithm

The primal and dual algorithms are represented by dynamics at source and link ends respectively, while they have static adaptation at the other end. The primal-dual algorithm has dynamics at sources as well as links and generally takes the following form:

$$\dot{x}_i(t) = f\left(x_i(t), \sum_{l \in r_i} p_l(t)\right),$$
$$\dot{p}_l(t) = g\left(p_l(t), \sum_{j:l \in r_j} x_j(t)\right),$$

where $f(\cdot)$ and $g(\cdot)$ are the control functions for updating the transmission rate and price at the source and the link, respectively. In this paper, we consider the primal algorithm which allows individual sources to dynamically adjust their transmitting rates while the price functions adapt according to a static law (Ranjan *et al*., 2004).

Let $w_i(t)$ denote a user's willingness to pay (per unit time) at time t. Suppose that each link l charges a price per unit flow of $\mu_l(t) = p_l\left(\sum_{i:l \in r_i} x_i(t)\right)$ at time t, where $p_l(\cdot)$ is a strictly increasing function of the total rate going through link l. Consider the system of differential equations

$$\dot{x}_i(t) = \kappa_i \left(w_i(t) - x_i(t_i) \sum_{l \in r_i} \mu_l(t) \right), \qquad (2)$$

where κ_i is a positive gain parameter. Kelly *et al*. (1998) showed that a unique equilibrium point of the primal algorithm Eq. (2) is the solution for the utility maximization problem in Eq. (1). This equation can be motivated as follows: each source first computes a price per unit time which it is willing to pay, namely $w_i(t)$. Then, it adjusts its rate based on the feedback provided by the links in the network to equalize its willingness to pay and the total price. For example, in Kelly *et al*. (1998), $w_i(t)$ is set to $x_i(t) \cdot U_i'\left(x_i(t)\right)$. The feedback from a link can be interpreted as a congestion indicator, requiring a reduction in the flow rates going through the link in the case of an overload. From Eq. (2) one can see that the operating point of the system depends on both the sources' utility and their link price functions. Therefore, the design of the rate control algorithm is equivalent to selecting the sources' utility function and the price function of the links in the network.

The feedback information from the links to the sources, which is typically carried by acknowledgements, is delayed owing to link propagation. For all i and $l\in r_i$, let $\tau_{i,l}^{\mathrm{f}}(t)$ denote the forward time-varying delay that source i packets endure before reaching link l and $\tau_{i,l}^{\mathrm{r}}(t)$ denote the reverse delay of the feedback signal from link l to the source i. If $l\notin r_i$, we assume that $\tau_{i,l}^{\mathrm{f}}(t)=\tau_{i,l}^{\mathrm{r}}(t)=0.$ Suppose that the links in $r_i=\{l_{i,1}, l_{i,2}, \ldots, l_{i,R_i}\}$ are arranged in the same order as source i's packets visit, where R_i denotes the cardinality of r_i. Let us define $\tau_i(t)=\tau_{i,l_{i,k}}^{\mathrm{f}}(t)+\tau_{i,l_{i,k}}^{\mathrm{r}}(t),$ k=1, 2, …, R_i as the total round-trip delay before the receipt of the acknowledgement of a packet. Under this general model, the source dynamics are given by

$$\dot{x}_i(t)=\kappa_i\left(x_i(t)U_i'\left(x_i(t)\right)-x_i\left(t-\tau_i(t)\right)\sum_{l\in r_i}\mu_l\left(t-\tau_{i,l}^{\mathrm{r}}(t)\right)\right). \quad (3)$$

The feedback signal generated by the link price functions is delayed by $\tau_{i,l}^{\mathrm{r}}(t),$ before sender i receives it. The price of link l at time t depends on the rates of the sources at time $t-\tau_{j,l}^{\mathrm{f}}(t)$ owing to the delay from the senders to the links. Therefore, we arrive at the following general model of the price functions:

$$\mu_l\left(t-\tau_{i,l}^{\mathrm{r}}(t)\right)=p_l\left(\sum_{j:l\in r_j}x_j\left(t-\tau_{i,l}^{\mathrm{r}}(t)-\tau_{j,l}^{\mathrm{f}}(t)\right)\right). \quad (4)$$

Substituting Eq. (4) in Eq. (3), and using a routing matrix, we have

$$\dot{x}_i(t)=\kappa_i\left(x_i(t)U_i'\left(x_i(t)\right)-x_i\left(t-\tau_i(t)\right)\cdot\sum_{l\in L}R_{li}p_l\left(\sum_{j\in S}R_{lj}x_j\left(t-\tau_{i,j,l}(t)\right)\right)\right), \quad (5)$$

where $\tau_{i,j,l}(t)=\tau_{j,l}^{\mathrm{f}}(t)+\tau_{i,l}^{\mathrm{r}}(t).$

We are interested in the class of sources' utility functions of the form $U_i(x_i)=-1/(a_ix_i^{a_i}),$ where a_i>0 is the parameter of the utility function. In addition, let the price function be $p_l(y_l)=(y_l/c_l)^{b_l}$, where b_l>0 is the parameter of the price function. Therefore,

$$\dot{x}_i(t)=\kappa_i\left(\frac{1}{x_i(t)^{a_i}}-x_i\left(t-\tau_i(t)\right)\cdot\sum_{l\in L}R_{li}\left(\sum_{j\in S}\frac{R_{lj}x_j\left(t-\tau_{i,j,l}(t)\right)}{c_l}\right)^{b_l}\right). \quad (6)$$

In this paper, we consider a model of a network with general topology described by Eq. (6), and investigate the global stability analysis of this system. In addition, we assume that the time-varying delay functions $\tau_i(t)$ and $\tau_{i,j,l}(t)$ are bounded and continuously differentiable, $\tau_i(t)\geq0$, $\tau_{i,j,l}(t)\geq0$, $1-\dot{\tau}_i(t)>0,$ and $1-\dot{\tau}_{i,j,l}(t)>0,$ for i, j=1, 2, …, n, and l=1, 2, …, m.

Also, we define the upper bound on the time-varying delay parameter as follows:

$$r=\max_t\left(\max_{i,j,l}\left(\tau_i(t),\tau_{i,j,l}(t)\right)\right).$$

The model Eq. (6) is supplemented by an initial condition of the form $x_i(\theta)=\varphi_i(\theta),$ $\theta\in[-r,0],$ where $\varphi_i(\cdot)$ is a vector of continuous real-valued functions on $[-r, 0]$. When φ_i is strictly positive, we say the model has a nonzero initial condition.

3 Results

3.1 Preliminaries

To establish the condition of global stability in the model Eq. (6), we prepare a functional framework for our analysis. Then an analytical proof of the global stability will be explored.

Consider a functional delayed differential equation of the form

$$\begin{aligned}\dot{x}_i(t) = g_i\left(x_i(t)\right) - f_i\Big(&x_i\left(t-\tau_i(t)\right),\\ &x_1\left(t-\tau_{i,1,1}(t)\right), \ldots, x_n\left(t-\tau_{i,n,1}(t)\right),\\ &x_1\left(t-\tau_{i,1,2}(t)\right), \ldots, x_n\left(t-\tau_{i,n,2}(t)\right),\\ &\cdots,\\ &x_1\left(t-\tau_{i,1,m}(t)\right), \ldots, x_n\left(t-\tau_{i,n,m}(t)\right)\Big)\end{aligned} \quad (7)$$

with the following assumptions:

Assumption 1 Delay functions $\tau_i(t)$ and $\tau_{i,j,l}(t)$ are bounded and continuously differentiable with $\tau_i(t)\geq 0$, $\tau_{i,j,l}(t)\geq 0$, $1-\dot{\tau}_i(t)>0$, and $1-\dot{\tau}_{i,j,l}(t)>0$, for $t\in\Re_+$, i, j=1, 2, …, n, and l=1, 2, …, m.

Assumption 2 The nonlinear functions $g_i:\Re\to\Re$ and $f_i:\Re\times\Re^{nm}\to\Re$ (for i=1, 2, …, n) are real-valued continuous and bounded functions.

Assumption 3 There exists a non-negative, bounded and continuous function $\eta_i(t)$ defined on $\Re_+$ such that

$$\frac{g_i\left(x_i(t)\right)-g_i\left(\overline{x}_i(t)\right)}{x_i(t)-\overline{x}_i(t)} \leq -\eta_i(t)$$

for each $t\in\Re_+$, $x_i, \overline{x}_i\in\Re$, and i=1, 2, …, n.

Assumption 4 The function f_i satisfies a certain type of Lipschitz condition (Hale and Lunel, 1993); i.e., there exist non-negative, bounded and continuous functions $\delta_i(t)$ and $\gamma_{ijl}(t)$ defined on $\Re_+$, such that for each $t\in\Re_+$, $x_i, \overline{x}_i\in\Re$, and i=1, 2, …, n, we have

$$\begin{aligned}&\Big| f_i\left(x_i\left(t-\tau_i(t)\right), x_1\left(t-\tau_{i,1,1}(t)\right), \ldots, x_n\left(t-\tau_{i,n,m}(t)\right)\right)\\ &\quad - f_i\left(\overline{x}_i\left(t-\tau_i(t)\right), \overline{x}_1\left(t-\tau_{i,1,1}(t)\right), \ldots, \overline{x}_n\left(t-\tau_{i,n,m}(t)\right)\right)\Big|\\ &\leq \delta_i(t)\left|x_i\left(t-\tau_i(t)\right)-\overline{x}_i\left(t-\tau_i(t)\right)\right|\\ &\quad + \sum_{l=1}^{m}\sum_{j=1}^{n}\gamma_{ijl}(t)\left|x_j\left(t-\tau_{i,j,l}(t)\right)-\overline{x}_j\left(t-\tau_{i,j,l}(t)\right)\right|.\end{aligned}$$

We consider that the delayed differential Eq. (7) is supplemented with the initial condition of the form

$$x_i(t)=\varphi_i(t),\ t\in(-\infty, 0],\ i=1, 2, \ldots, n,$$

where $\varphi_i(\cdot)$ are real-valued functions continuous on $(-\infty, 0]$.

Throughout the paper we assume that Eq. (7) has solutions.

Definition 1 System Eq. (7) is said to be globally asymptotically stable if for any two solutions $x_i(t)$ and $\overline{x}_i(t)$ of Eq. (7), one has

$$\lim_{t\to+\infty}\sum_{i=1}^{n}\left(x_i(t)-\overline{x}_i(t)\right)^2 = 0.$$

Under Assumptions 1–4, we have the following lemma:

Lemma 1 Let x_i and $\overline{x}_i$ be any two solutions of Eq. (7). Then

$$\sum_{i=1}^{n}\left(x_i(t)-\overline{x}_i(t)\right)^2$$

is uniformly continuous on [0, +∞).

Proof By the boundedness of g_i and f_i in Eq. (7) as stated in Assumption 2, we know that $\dot{x}_i$ and $\dot{\overline{x}}_i$ are also bounded and therefore, the conclusion follows.

Lemma 2 (Slotine and Li, 1991) Let f be a non-negative, uniformly continuous, and integrable function on [0, +∞) then $\lim_{t\to\infty} f=0$.

3.2 Global stability analysis

We study the global stability of system Eq. (7) using the following theorem:

Theorem 1 Under Assumptions 1–4, the system Eq. (7) is globally asymptotically stable if there exists μ_i, for i=1, 2, …, n, such that

$$\inf_{t\in\Re_+}\left(\mu_i\left(\eta_i(t)-\frac{1}{2}\delta_i(t)-\frac{1}{2}\frac{\delta_i\left(\zeta_i^{-1}(t)\right)}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\right) -\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\mu_j\left(\gamma_{jil}(t)+\frac{\gamma_{jil}\left(\zeta_{j,i,l}^{-1}(t)\right)}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\right)\right)>0,$$

where $\zeta_i^{-1}(t)$ is the inverse function of $\zeta_i=t-\tau_i(t)$ and $\zeta_{i,j,l}^{-1}(t)$ is the inverse function of $\zeta_{i,j,l}=t-\tau_{i,j,l}(t)$.

Proof Let x_i and $\overline{x}_i$ be any two solutions of Eq. (7).

Consider the Lyapunov function $V(t)$ defined by

$$V(t)=\sum_{i=1}^{n}\mu_i\cdot\left(\frac{1}{2}\left(x_i(t)-\bar{x}_i(t)\right)^2+\frac{1}{2}\int_{\theta=t-\tau_i(t)}^{t}\frac{\delta_i(\zeta_i^{-1}(\theta))}{1-\dot{\tau}_i(\zeta_i^{-1}(\theta))}\left(x_i(\theta)-\bar{x}_i(\theta)\right)^2\mathrm{d}\theta+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\int_{\theta=t-\tau_{i,j,l}(t)}^{t}\frac{\gamma_{ijl}\left(\zeta_{i,j,l}^{-1}(\theta)\right)}{1-\dot{\tau}_{i,j,l}\left(\zeta_{i,j,l}^{-1}(\theta)\right)}\cdot\left(x_j(\theta)-\bar{x}_j(\theta)\right)^2\mathrm{d}\theta\right)$$

for $t\geq 0$. Considering Assumption 1, one can easily show that $V(0)<+\infty$.

Calculating the derivative of $V(t)$, and using Assumption 4, we can obtain

$$\begin{aligned}\dot{V}(t)\leq\sum_{i=1}^{n}\mu_i\Bigg(&\left(g_i\left(x_i(t)\right)-g_i\left(\bar{x}_i(t)\right)\right)\left(x_i(t)-\bar{x}_i(t)\right)\\&+\delta_i(t)\left|x_i\left(t-\tau_i(t)\right)-\bar{x}_i\left(t-\tau_i(t)\right)\right|\left(x_i(t)-\bar{x}_i(t)\right)\\&+\sum_{l=1}^{m}\sum_{j=1}^{n}\gamma_{ijl}(t)\left|x_j\left(t-\tau_{i,j,l}(t)\right)-\bar{x}_j\left(t-\tau_{i,j,l}(t)\right)\right|\\&\cdot\left(x_i(t)-\bar{x}_i(t)\right)+\frac{1}{2}\frac{\delta_i\left(\zeta_i^{-1}(t)\right)}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\left(x_i(t)-\bar{x}_i(t)\right)^2\\&-\frac{1}{2}\delta_i(t)\left(x_i\left(t-\tau_i(t)\right)-\bar{x}_i\left(t-\tau_i(t)\right)\right)^2\\&+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\frac{\gamma_{ijl}\left(\zeta_{i,j,l}^{-1}(t)\right)}{1-\dot{\tau}_{i,j,l}\left(\zeta_{i,j,l}^{-1}(t)\right)}\left(x_j(t)-\bar{x}_j(t)\right)^2\\&-\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\gamma_{ijl}(t)\left(x_j\left(t-\tau_{i,j,l}(t)\right)-\bar{x}_j\left(t-\tau_{i,j,l}(t)\right)\right)^2\Bigg).\end{aligned}$$

Using the inequality $2ab\leq a^2+b^2$, we have

$$\begin{aligned}\dot{V}(t)\leq\sum_{i=1}^{n}\mu_i\Bigg(&\left(g_i\left(x_i(t)\right)-g_i\left(\bar{x}_i(t)\right)\right)\left(x_i(t)-\bar{x}_i(t)\right)\\&+\frac{1}{2}\delta_i(t)\left(x_i(t)-\bar{x}_i(t)\right)^2+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\gamma_{ijl}(t)\left(x_j(t)-\bar{x}_j(t)\right)^2\\&+\frac{1}{2}\frac{\delta_i\left(\zeta_i^{-1}(t)\right)}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\left(x_i(t)-\bar{x}_i(t)\right)^2\\&+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\frac{\gamma_{ijl}\left(\zeta_{i,j,l}^{-1}(t)\right)}{1-\dot{\tau}_{i,j,l}\left(\zeta_{i,j,l}^{-1}(t)\right)}\left(x_j(t)-\bar{x}_j(t)\right)^2\Bigg).\end{aligned}$$

Thus,

$$\begin{aligned}\dot{V}(t)\leq\sum_{i=1}^{n}\Bigg(&\Bigg(-\mu_i\eta_i(t)+\frac{1}{2}\mu_i\delta_i(t)\frac{1}{2}\frac{\mu_i\delta_i\left(\zeta_i^{-1}(t)\right)}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\Bigg)\\&\cdot\left(x_i(t)-\bar{x}_i(t)\right)^2+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\mu_j\gamma_{jil}(t)\left(x_i(t)-\bar{x}_i(t)\right)^2\\&+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\frac{\mu_j\gamma_{jil}\left(\zeta_{j,i,l}^{-1}(t)\right)}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\left(x_i(t)-\bar{x}_i(t)\right)^2\Bigg)\\\leq\sum_{i=1}^{n}\Bigg(&-\mu_i\eta_i(t)+\frac{1}{2}\mu_i\delta_i(t)+\frac{1}{2}\frac{\mu_i\delta_i\left(\zeta_i^{-1}(t)\right)}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\\&+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\mu_j\gamma_{jil}(t)+\frac{\mu_j\gamma_{jil}\left(\zeta_{j,i,l}^{-1}(t)\right)}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\Bigg)\cdot\left(x_i(t)-\bar{x}_i(t)\right)^2.\end{aligned}$$

Now, let

$$c_1=\min_{1\leq i\leq n}\inf_{t\in\Re_+}\left(\mu_i\left(\eta_i(t)-\frac{1}{2}\delta_i(t)—\frac{1}{2}\frac{\delta_i(\zeta_i^{-1}(t))}{1-\dot{\tau}_i(\zeta_i^{-1}(t))}\right)-\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\mu_j\left(\gamma_{jil}(t)+\frac{\gamma_{jil}\left(\zeta_{j,i,l}^{-1}(t)\right)}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\right)\right)>0.$$

Then, the above estimate leads to

$$\dot{V}(t)\leq -c_1\sum_{i=1}^{n}\left(x_i(t)-\bar{x}_i(t)\right)^2. \tag{8}$$

Integrating both sides of Eq. (8) with respect to t gives

$$V(t)+c_1\int_0^t\sum_{i=1}^{n}\left(x_i(\theta)-\bar{x}_i(\theta)\right)^2\mathrm{d}\theta\leq V(0)<+\infty,\ t\geq 0,$$

which implies $\sum_{i=1}^{n}\left(x_i(t)-\bar{x}_i(t)\right)^2\in L^1[0,+\infty)$.

However, by Lemmas 1 and 2 we can conclude that

$$\lim_{t\to+\infty}\sum_{i=1}^{n}\left(x_i(t)-\bar{x}_i(t)\right)^2=0,$$

which proves that system Eq. (7) is globally asymptotically stable.

4 Application to a congestion control system

In this section, we determine the global stability condition for the network congestion control system described by Eq. (6). We construct a Lyapunov function satisfying the stability condition of Theorem 1 for this system. First, we prove that each of flows $x_i(t)$ in Eq. (6) has strictly positive upper and lower bounds. We define the upper bound on b_l as $B_i = \max_{l\in r_i} b_l = \max_{l\in L} R_{li} b_l$.

Proposition 1 Suppose that the network model in Eq. (6) has a nonzero initial condition φ_i for all i and that the constant γ satisfies $\max_{\theta\in[-r,0]}|\varphi_i(\theta)| \le x_i^* \mathrm{e}^{\gamma}$ for each i. If $a_i>B_i+1$, then each of the flows x_i satisfies $x_i^* \mathrm{e}^{-\gamma} \le x_i(t) \le x_i^* \mathrm{e}^{\gamma}$ for all time $t\ge -r$.

Proof We will prove this by contradiction. Note from the initial condition that the bound on $x_i(t)$ holds in the interval $[-r, 0]$ for all i. Suppose that some flow x_j violates the inequality on the interval $(0, \infty)$. Then, by continuity of the functions $x_i(t)$ and their derivatives, there must exist a time $t\ge 0$ such that:

1. For each i, the inequality $x_i^* \mathrm{e}^{-\gamma} \le x_i(t) \le x_i^* \mathrm{e}^{\gamma}$ holds on $[-r, t]$.

2. At time t, one of the following conditions holds:

$$x_j(t) = x_j^* \mathrm{e}^{\gamma},\ \dot{x}_j(t) > 0, \tag{9}$$

$$x_j(t) = x_j^* \mathrm{e}^{-\gamma},\ \dot{x}_j(t) > 0. \tag{10}$$

We will show that this leads to a contradiction. Suppose that Eq. (9) holds. Then

$$x_j(t) = x_j^* \mathrm{e}^{\gamma},\ \dot{x}_j(t) > 0.$$

But from Eq. (6), we have

$$\dot{x}_j(t) \le \kappa_j \left(\frac{1}{x_j^{*a_j} \mathrm{e}^{a_j\gamma}} - x_j\left(t-\tau_j(t)\right) \cdot \sum_{l\in L} R_{lj} \left(\sum_{i\in S} R_{li} x_j\left(t-\tau_{j,i,l}(t)\right) \Big/ c_l \right)^{b_l} \right).$$

From condition 1, we have

$$\dot{x}_j(t) \le \kappa_j \left(\frac{1}{x_j^{*a_j} \mathrm{e}^{a_j\gamma}} - x_j^* \mathrm{e}^{-\gamma} \sum_{l\in L} R_{lj} \left(\sum_{i\in S} \frac{R_{li} x_i^* \mathrm{e}^{-\gamma}}{c_l} \right)^{b_l} \right).$$

We know that the equilibrium point satisfies

$$\sum_{l\in L} R_{lj} \left(\sum_{i\in S} R_{li} x_i^* \Big/ c_l \right)^{b_l} = \frac{1}{(x_j^*)^{a_j+1}}.$$

Thus, for $a_i>B_i+1$, we have

$$\dot{x}_j(t) \le \frac{\kappa_j}{x_j^{*a_j}} \left(\mathrm{e}^{-a_j\gamma} - \mathrm{e}^{-\gamma}\mathrm{e}^{-B_j\gamma}\right) < 0.$$

Thus, Eq. (9) cannot hold. Similarly, we can show that a contradiction arises when Eq. (10) holds. Hence, we conclude that no such time exists and thus the sought bounds must be valid for all flows for all time.

In practice, an upper bound on $x_i(t)$ results from a physical limitation such as finite receiver buffer size. This is to prevent packet losses at the receiver buffer, as packets are lost because of buffer overflow when the sender transmits packets at a rate faster than the receiver can process them. In addition, a lower bound results from the fact that the source needs to probe the congestion level of the network by continually transmitting packets.

Owing to the complexity of this model, we first study the case of homogeneous sources with a time-varying delay.

4.1 Homogeneous network

First, consider the following equation as a model of a single source/link network:

$$\dot{x}(t) = \kappa\left(x(t)U'\left(x(t)\right) - x\left(t-\tau(t)\right) p\left(x\left(t-\tau(t)\right)\right)\right). \tag{11}$$

Let the utility function be $U(x)=-1/(ax^a)$ and the price function be $p(x)=(x/c)^b$, where a and b are positive parameters of utility and price functions, respectively. We can rewrite Eq. (11) as

$$\dot{x}(t) = \kappa\left(1/x(t)^a - x\left(t-\tau(t)\right)^{b+1} \Big/ c^b \right). \tag{12}$$

Suppose that there exists a constant γ satisfying $\max_{\theta\in[-r,0]}|\varphi(\theta)|\le x^*\mathrm{e}^{\gamma}$. Applying Theorem 1, we obtain the following corollary:

Corollary 1 If $a>b+1$, $\tau(t)\ge 0$ for $t\ge 0$, and $\inf_{t\in\Re_+}\left((1-\dot\tau(t))/(1-\dot\tau(t)/2)\right)>\mathrm{e}^{2a\gamma}$, then the system Eq. (12) is globally asymptotically stable.

Proof Since $a>b+1$, according to Proposition 1, $x(t)$ is bounded for $t\ge -r$; i.e., $x^*\mathrm{e}^{-\gamma}\le x(t)\le x^*\mathrm{e}^{\gamma}$, for $t\ge -r$. Therefore, defining $y(t)=x(t-\tau(t))$, $g(x)=\kappa/x^a$, and $f(y)=\kappa y^{b+1}/c^b$, Assumption 2 holds. Assuming $\bar x<x$ and according to the mean value theorem, there exists an $\varepsilon\in(\bar x,x)$ such that

$$\frac{g(x)-g(\bar x)}{x-\bar x}=g'(\varepsilon)=-\kappa a\varepsilon^{-(a+1)}$$
$$<-\kappa a x^{-(a+1)}\le -\kappa a(x^*)^{-(a+1)}\mathrm{e}^{-(2a+1)\gamma}\mathrm{e}^{a\gamma}.$$

Also, assuming $\bar y<y$ and according to the mean value theorem, there exists a $\rho\in(\bar y,y)$ such that

$$\frac{f(y)-f(\bar y)}{y-\bar y}=f'(\rho)=\kappa(b+1)(\rho/c)^b$$
$$<\kappa(b+1)(y/c)^b\le\kappa(b+1)(x^*/c)^b\mathrm{e}^{b\gamma}.$$

Thus, Assumptions 3 and 4 hold. Defining $\eta(t)=\kappa a(x^*)^{-(a+1)}\mathrm{e}^{-(2a+1)\gamma}\mathrm{e}^{a\gamma}$ and $\delta(t)=\kappa(b+1)(x^*/c)^b\mathrm{e}^{b\gamma}$, if $\inf_{t\in\Re_+}\left((1-\dot\tau(t))/(1-\dot\tau(t)/2)\right)>\mathrm{e}^{2a\gamma}$, we have

$$\inf_{t\in\Re_+}\left(\eta(t)-\frac{1}{2}\delta(t)-\frac{1}{2}\frac{\delta\left(\xi^{-1}(t)\right)}{1-\dot\tau\left(\xi^{-1}(t)\right)}\right)$$
$$=\inf_{t\in\Re_+}\left(\kappa\left(a(x^*)^{-(a+1)}\mathrm{e}^{-(2a\gamma+\gamma)}\mathrm{e}^{a\gamma}-\frac{1}{2}(b+1)\left(\frac{x^*}{c}\right)^b\mathrm{e}^{b\gamma}\right.\right.$$
$$\left.\left.-\frac{1}{2}\frac{(b+1)(x^*/c)^b\mathrm{e}^{b\gamma}}{1-\dot\tau\left(\xi^{-1}(t)\right)}\right)\right)$$
$$=\inf_{t\in\Re_+}\left(\kappa(x^*)^{-(a+1)}\mathrm{e}^{-\gamma}\right.$$
$$\left.\cdot\left(a\mathrm{e}^{-2a\gamma}\mathrm{e}^{a\gamma}-(b+1)\mathrm{e}^{(b+1)\gamma}\frac{1-\dot\tau\left(\xi^{-1}(t)\right)/2}{1-\dot\tau\left(\xi^{-1}(t)\right)}\right)\right)>0.$$

Thus, according to Theorem 1, the system Eq. (12) is globally asymptotically stable.

Now, let us consider a multi-source/link rate-based congestion control system with a time-varying delay given by

$$\dot x_i(t)=\kappa_i\left(x_i(t)U_i'\left(x_i(t)\right)-x_i\left(t-\tau(t)\right)\right.$$
$$\left.\cdot\sum_{l\in L}R_{li}p_l\left(\sum_{j\in S}R_{lj}x_j\left(t-\tau(t)\right)\right)\right).\quad(13)$$

Let the utility function be of the form $U_i(x_i)=-1/(ax_i^{a_i})$ and the price function be $p_l(y_l)=(y_l/c_l)^{b_l}$, where a_i and b_i are the parameters of utility and price functions, respectively. Therefore,

$$\dot x_i(t)=\kappa_i\left(\frac{1}{x_i(t)^{a_i}}-x_i\left(t-\tau(t)\right)\right.$$
$$\left.\cdot\sum_{l\in L}R_{li}\left(\sum_{j\in S}R_{lj}x_j\left(t-\tau(t)\right)\Big/c_l\right)^{b_l}\right).\quad(14)$$

Similarly, suppose that there exists a constant γ satisfying $\max_{\theta\in[-r,0]}|\varphi_i(\theta)|\le x_i^*\mathrm{e}^{\gamma}$, for each i.

In addition, we define

$$R_i=\max_{l\in r_i}\sum_{j\in S}R_{lj}=\max_{l\in L}R_{li}\left(\sum_{j\in S}R_{lj}\right),$$
$$F_{li}=\left(R_{li}/c_l^{b_l}\right)\left(\sum_{j\in S}R_{lj}\rho_j\right)^{b_l-1}.$$

By applying Theorem 1, we obtain the following corollary:

Corollary 2 Let $a_i>B_i+1$, $\tau(t)\ge 0$, $1-\dot\tau(t)>0$, for $i=1, 2, \ldots, n$, and $t\ge 0$. The system Eq. (14) is globally asymptotically stable, if

$$\inf_{t\in\Re_+}\left(\sum_{i=1}^{n}\mu_i\left(\kappa_i(x_i^*)^{-(a_i+1)}\left(a_i\mathrm{e}^{-(a_i+1)\gamma}-\mathrm{e}^{B_i\gamma}\frac{1-\dot\tau\left(\zeta^{-1}(t)\right)/2}{1-\dot\tau\left(\zeta^{-1}(t)\right)}\right)\right.\right.$$
$$\left.\left.-\sum_{l=1}^{m}\sum_{j=1}^{n}\left(\mu_j\kappa_jB_jR_jx_j^*F_{lj}\mathrm{e}^{B_j\gamma}\frac{1-\dot\tau\left(\zeta^{-1}(t)\right)/2}{1-\dot\tau\left(\zeta^{-1}(t)\right)}\right)\right)\right)>0.$$

Proof Suppose that $g_i(x_i)=\kappa_i / x_i^{a_i}$ and

$$f_i(y_1, y_2, \ldots, y_n)=\kappa_i y_i \sum_{l\in L} R_{li}\left(\left(\sum_{j\in S} R_{lj} y_j\right)\Big/ c_l\right)^{b_l},$$

where $y_i= x_i(t-\tau(t))$, for i=1, 2, …, n, and t≥0. As $a_i>B_i+1$, according to Proposition 1, $x_i(t)$ is bounded; i.e., $x_i^* \mathrm{e}^{-\gamma} \le x_i(t) \le x_i^* \mathrm{e}^{\gamma}$ for i=1, 2, …, n and $t\ge -r$. Therefore, Assumption 2 holds. Assuming $\bar{x}_i < x_i$, for i=1, 2, …, n, and according to the mean value theorem, there exists an $\varepsilon_i \in (\bar{x}_i, x_i)$ such that

$$\begin{aligned}\frac{g_i(x_i)-g_i(\bar{x}_i)}{x_i-\bar{x}_i} &= \frac{\mathrm{d}g_i}{\mathrm{d}x_i}\cdot \varepsilon_i = -\kappa_i a_i \varepsilon_i^{-(a_i+1)} \\ &< -\kappa_i a_i x_i^{-(a_i+1)} < -\kappa_i a_i (x_i^*)^{-(a_i+1)} \mathrm{e}^{-(a_i+1)\gamma}.\end{aligned}$$

Also, there exist $\rho_i \in (\bar{y}_i, y_i)$, for i=1, 2, …, n, such that

$$\begin{aligned}&\left| f_i(y_1, y_2, \ldots, y_n) - f_i(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)\right| \\ &= \sum_{j\in S} \frac{\mathrm{d}f_i}{\mathrm{d}x_j}(\rho_1, \rho_2, \ldots, \rho_n)\left|y_j-\bar{y}_j\right| \\ &\le \kappa_i \sum_{l\in L}\left(\frac{R_{li}}{c_l^{b_l}}\left(\sum_{k\in S} R_{lk}\rho_k\right)^{b_l}\right)\left|y_i-\bar{y}_i\right| \\ &\quad + \sum_{j\in S}\left(\kappa_i B_i \rho_i \sum_{l\in L} R_{lj}\frac{R_{li}}{c_l^{b_l}}\left(\sum_{k\in S} R_{lk}\rho_k\right)^{b_l-1}\left|y_j-\bar{y}_j\right|\right) \\ &\le \kappa_i \mathrm{e}^{B_i\gamma}\left((x_i^*)^{-(a_i+1)}\left|y_i-\bar{y}_i\right| + B_i R_i x_i^* \sum_{l\in L}\sum_{j\in S} F_{li}\left|y_j-\bar{y}_j\right|\right).\end{aligned}$$

Thus, Assumptions 3 and 4 hold. Defining $\eta_i(t) =\kappa_i a_i (x_i^*)^{-(a_i+1)}\mathrm{e}^{-(a_i+1)\gamma}$, $\delta_i(t)=\kappa_i (x_i^*)^{-(a_i+1)}\mathrm{e}^{B_i\gamma}$, and $\gamma_{ijl}(t)=\kappa_i B_i R_i x_i^* F_{li}\mathrm{e}^{B_i\gamma}$, we have

$$\begin{aligned}&\inf_{t\in\Re_+}\left(\sum_{i=1}^{n}\left(\mu_i\left(\eta_i(t)-\frac{1}{2}\delta_i(t)-\frac{1}{2}\frac{\delta_i\left(\zeta^{-1}(t)\right)}{1-\dot{\tau}\left(\zeta^{-1}(t)\right)}\right)\right.\right. \\ &\qquad \left.\left.-\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\mu_j\left(\gamma_{jil}(t)+\frac{\gamma_{jil}(\zeta^{-1}(t))}{1-\dot{\tau}(\zeta^{-1}(t))}\right)\right)\right) \\ &= \inf_{t\in\Re_+}\left(\sum_{i=1}^{n}\mu_i\left(\kappa_i (x_i^*)^{-(a_i+1)}\left(a_i\mathrm{e}^{-(a_i+1)\gamma}-\mathrm{e}^{B_i\gamma}\frac{1-\dot{\tau}\left(\zeta^{-1}(t)\right)/2}{1-\dot{\tau}\left(\zeta^{-1}(t)\right)}\right)\right.\right. \\ &\qquad \left.\left.-\sum_{l=1}^{m}\sum_{j=1}^{n}\left(\mu_j\kappa_j B_j R_j x_j^* F_{lj}\mathrm{e}^{B_j\gamma}\frac{1-\dot{\tau}\left(\zeta^{-1}(t)\right)/2}{1-\dot{\tau}\left(\zeta^{-1}(t)\right)}\right)\right)\right)>0.\end{aligned}$$

Thus, according to Theorem 1, the system Eq. (14) is globally asymptotically stable.

4.2 Heterogeneous network

Consider the general form of a rate-based congestion control system for a network with multi-source multi-link topology and time-varying delays, as described by Eq. (6). As in the case of a homogeneous network, we assume that there exists a constant γ satisfying $\max_{\theta\in[-r,0]}\left|\varphi_i(\theta)\right| \le x_i^*\mathrm{e}^{\gamma}$ for each i.

Applying Theorem 1, we have the following corollary:

Corollary 3 Let $a_i>B_i+1$, $\tau_i(t)\ge 0$, $\tau_{i,j,l}(t)\ge 0$, $1-\dot{\tau}_i(t)>0$, $1-\dot{\tau}_{i,j,l}(t)>0$, for i, j=1, 2, …, n, l=1, 2, …, m, and t≥0. The system Eq. (6) is globally asymptotically stable, if

$$\begin{aligned}&\inf_{t\in\Re_+}\left(\sum_{i=1}^{n}\left(\mu_i\kappa_i (x_i^*)^{-(a_i+1)}\left(a_i\mathrm{e}^{-(a_i+1)\gamma}-\mathrm{e}^{B_i\gamma}\frac{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)/2}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\right)\right.\right. \\ &\qquad \left.\left.-\sum_{l=1}^{m}\sum_{j=1}^{n}\left(\mu_j\kappa_j B_j R_j F_{lj} x_j^*\mathrm{e}^{B_j\gamma}\frac{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)/2}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\right)\right)\right)>0.\end{aligned}$$

Proof The proof is similar to the proof of Corollary 2. Suppose that $g_i(x_i)=\dfrac{\kappa_i}{x_i^{a_i}}$ and

$$f_i(z_{i11}, z_{i12}, \ldots, z_{inm})=\kappa_i y_i \sum_{l\in L} R_{li}\left(\frac{1}{c_l}\sum_{j\in S} R_{lj} z_{ijl}\right)^{b_l},$$

where $y_i=x_i(t-\tau_i(t))$, $z_{ijl}=x_j(t-\tau_{i,j,l}(t))$, for i, j=1, 2, …, n, l=1, 2, …, m and t≥0. As $a_i>B_i+1$, according to Proposition 1, $x_i(t)$ is bounded; i.e., $x_i^*\mathrm{e}^{-\gamma}\le x_i(t) \le x_i^*\mathrm{e}^{\gamma}$ for i=1, 2, …, n and $t\ge -r$. Therefore, Assumption 2 holds. Assuming $\bar{x}_i < x_i$, for i=1, 2, …, n, and according to the mean value theorem, there exists an $\varepsilon_i \in (\bar{x}_i, x_i)$ such that

$$\frac{g_i(x_i)-g_i(\overline{x}_i)}{x_i-\overline{x}_i}=\frac{\mathrm{d}g_i}{\mathrm{d}x_i}\cdot\varepsilon_i=-\kappa_i a_i \varepsilon_i^{-(a_i+1)}<-\kappa_i a_i x_i^{-(a_i+1)}$$
$$<-\kappa_i a_i (x_i^*)^{-(a_i+1)}\mathrm{e}^{-(a_i+1)\gamma}.$$

In addition, there exist $\rho_i\in(\overline{y}_i, y_i)$ and $\rho_{ijl}\in(\overline{z}_{ijl}, z_{ijl})$ such that

$$\begin{aligned}
&\left|f_i(y_i, z_{i11}, \ldots, z_{inm})-f_i(\overline{y}_i, \overline{z}_{i11}, \ldots, \overline{z}_{inm})\right| \\
&=\frac{\mathrm{d}f_i}{\mathrm{d}y_i}(\rho_i, \rho_{i11}, \ldots, \rho_{inm})\left|y_i-\overline{y}_i\right| \\
&\quad+\sum_{j\in S}\sum_{l\in L}\frac{\mathrm{d}f_i}{\mathrm{d}z_{ijl}}(\rho_i, \rho_{i11}, \ldots, \rho_{inm})\left|z_{ijl}-\overline{z}_{ijl}\right| \\
&\le \kappa_i\sum_{l\in L}\left(\frac{R_{li}}{c_l^{b_l}}\right)\left(\sum_{j\in S}R_{lj}\rho_{ijl}\right)^{b_l}\left|y_i-\overline{y}_i\right| \\
&\quad+\kappa_i B_i\rho_i\sum_{l\in L}\sum_{j\in S}\left(\frac{R_{li}}{c_l^{b_l}}\right)R_{lj}\left(\sum_{k\in S}R_{lk}\rho_{ikl}\right)^{b_l-1}\left|z_{ijl}-\overline{z}_{ijl}\right| \\
&\le \kappa_i \mathrm{e}^{B_i\gamma}x_i^{*-(a_i+1)}\left|y_i-\overline{y}_i\right| \\
&\quad+\kappa_i \mathrm{e}^{B_i\gamma}B_i x_i^*\sum_{l\in L}\left(\frac{R_{li}}{c_l^{b_l}}\right)\left(\sum_{k\in S}R_{lk}\rho_{ikl}\right)^{b_l-1}\sum_{j\in S}\left(R_{lj}\left|z_{ijl}-\overline{z}_{ijl}\right|\right) \\
&\le \kappa_i \mathrm{e}^{B_i\gamma}\left(x_i^{*-(a_i+1)}\left|y_i-\overline{y}_i\right|+B_i R_i x_i^*\sum_{l\in L}\sum_{j\in S}F_{li}\left|z_{ijl}-\overline{z}_{ijl}\right|\right).
\end{aligned}$$

Thus, Assumptions 3 and 4 hold. Defining $\eta_i(t)=\kappa_i a_i x_i^{*-(a_i+1)}\mathrm{e}^{-(a_i+1)\gamma}$, $\delta_i(t)=\kappa_i x_i^{*-(a_i+1)}\mathrm{e}^{B_i\gamma}$, and $\gamma_{ijl}(t)=\kappa_i B_i R_i x_i^* F_{li}\mathrm{e}^{B_i\gamma}$, we have

$$\begin{aligned}
&\inf_{t\in\Re_+}\left(\sum_{i=1}^{n}\left(\mu_i\left(\eta_i(t)-\frac{1}{2}\delta_i(t)-\frac{1}{2}\frac{\delta_i\left(\zeta_i^{-1}(t)\right)}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\right)\right.\right. \\
&\quad\left.\left.-\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{n}\mu_j\left(\gamma_{jil}(t)+\frac{\gamma_{jil}\left(\zeta_{j,i,l}^{-1}(t)\right)}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\right)\right)\right) \\
&=\inf_{t\in\Re_+}\left(\sum_{i=1}^{n}\left(\mu_i\kappa_i(x_i^*)^{-(a_i+1)}\left(a_i\mathrm{e}^{-(a_i+1)\gamma}-\mathrm{e}^{B_i\gamma}\frac{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)/2}{1-\dot{\tau}_i\left(\zeta_i^{-1}(t)\right)}\right)\right.\right. \\
&\quad\left.\left.-\sum_{l=1}^{m}\sum_{j=1}^{n}\left(\mu_j\kappa_j B_j R_j F_{lj}x_j^*\mathrm{e}^{B_j\gamma}\frac{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)/2}{1-\dot{\tau}_{j,i,l}\left(\zeta_{j,i,l}^{-1}(t)\right)}\right)\right)\right)>0.
\end{aligned}$$

Thus, according to Theorem 1, the system Eq. (6) is globally asymptotically stable.

5 Discussion

We have presented analytic methodology and results on the stability of distributed algorithms of a network with elastic traffic with time-varying delays. The construction of Lyapunov functions in our analysis is more efficient than that of other methods, since nonlinearity and delay effects have been taken into account. The stability conditions derived from the analysis of the linearized or undelayed versions can provide only limited information about the dynamic behavior of a network. Therefore, we considered delay-dependent stability conditions in the present study. According to the results of Corollaries 1–3, the stability conditions include the users' data such as κ_i, a_i, the information of delays $\tau_i(t)$ and $\dot{\tau}_i(t)$, for i=1, 2, …, n, the links' data such as b_l, c_l, for l=1, 2, …, m, and the routing matrix $\boldsymbol{R}$. Therefore, the global stability of the whole network system is achieved under some conditions on users' or links' parameters. We can design the implementations of these algorithms using network information that can be accessed by users and links.

To the best of our knowledge, no global stability result on the multi-dimensional case of a Kelly primal algorithm in the presence of heterogeneous and time-varying delays has previously been published. Compared with the results of previous studies, the main advantages of our analysis are as follows:

1. The network model is assumed to be multi-dimensional and heterogeneous.

2. Delays are considered as time-varying parameters. Delays in the network are usually not known exactly and they are time-varying in nature because a part of most delays is caused by queue latencies and routers which change frequently according to their congestion levels.

3. A common approach to analyzing the stability of NDDE models is to use L-K or L-R stability theorems. However, it is extremely difficult to obtain L-K or L-R functions that are suitable for analysis of the considered model. Moreover, it appears that the analytical results based on L-K or L-R methods are more or less conservative. We exploit the inherent properties of the model in constructing functions, and thus the analysis is very simple in practice.

Although the present method has some advantages, the stability conditions depend on the time derivative of delays. This is undesirable because the delays in networks can be jittery. Furthermore, the implementation of the algorithm has extra communication costs, because each user or link needs to collect relevant information from the other users or links. Also, as expected, the proposed technique leads to more conservation and more restriction, as a result of the worst case conditions assumed in the network dynamics.

6 Simulations

In this section, we carry out two scenarios of simulation to verify the validity of the global stability conditions presented in Corollaries 1–3. First, consider a single-source single-link network of the form Eq. (11) with the gain parameter κ=1 and link capacity c=5. The utility function is $U(x(t))=-1/(ax(t)^a)$ with a=1.5 and the price function is $p(x(t))=(x(t)/c)^b$ with b=0.1 (Ranjan *et al.*, 2006). The delay parameter is $\tau(t)$=10+0.5sin$t$ and $\dot{\tau}(t)=0.5\cos t$. Therefore, for $a>b+1$, the system is bounded. The solution of the optimization problem is given by x^*=1. Fig. 2a shows the convergence of the user rate to the optimal rate. Now, we select the utility parameter a=1 such that $a<b+1$, the system is not bounded, and the stability conditions in Corollary 1 do not hold. Fig. 2b shows the unstable behavior of the system.

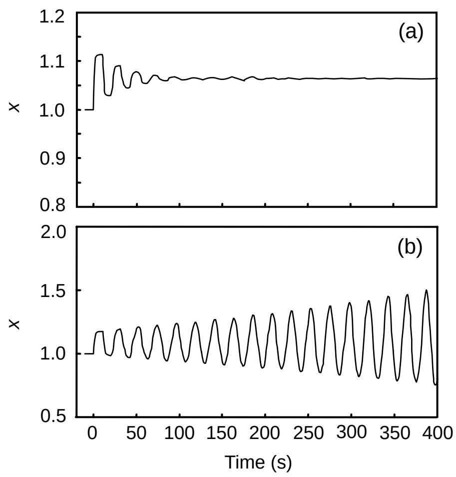

Fig. 2 (a) Stable and (b) unstable behaviors of the single-source single-link network

Now, consider a simple network topology with three users which share two links. The first link is shared by users 1 and 3, while the second link is shared by users 2 and 3 (Fig. 3).

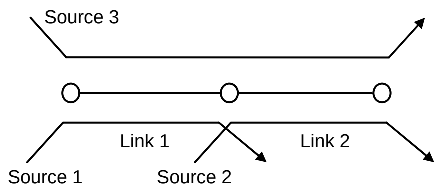

Fig. 3 A network topology with three sources and two links

The routing matrix will be as follows:

$$\boldsymbol{R}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}.$$

The utility functions are of the form $U_i(x)=-1/(a_i x^{a_i})$ for i=1, 2, 3 with $a_1=a_2$=3 and a_3=4. The price functions are of the form $p_j(x)=(x/c_j)^{b_j}$ for j=1, 2 with $b_1=b_2$=1.9. The link capacities are set to be $[c_1\ c_2]^{\mathrm{T}}=[5\ 4]^{\mathrm{T}}$. We assume that all delays lie in the reverse path from the receivers to the senders. The feedback delays are

$$[\tau_1\ \tau_2\ \tau_3]=[28+0.5\sin t\ \ 43+0.5\sin t\ \ 77+0.5\sin t].$$

Therefore, for all users, the system is globally asymptotically stable. The solution of the optimization problem is given by

$$\boldsymbol{x}^*=[1.1921\quad 1.1809\quad 1.0944]^{\mathrm{T}}.$$

In addition, we obtain $B_1=B_2$=1.9, $R_1=R_2$ =2, and

$$\boldsymbol{F}=\begin{bmatrix}0.67 & 0 & 2.68\\ 0 & 0.29 & 1.45\end{bmatrix}.$$

Fig. 4a shows the convergence of the user rates to the optimal rates. Now, we select the resource price parameters $b_1=b_2$=3.5 such that the stability conditions in Corollary 3 do not hold. Fig. 4b shows the unstable behavior of the system.

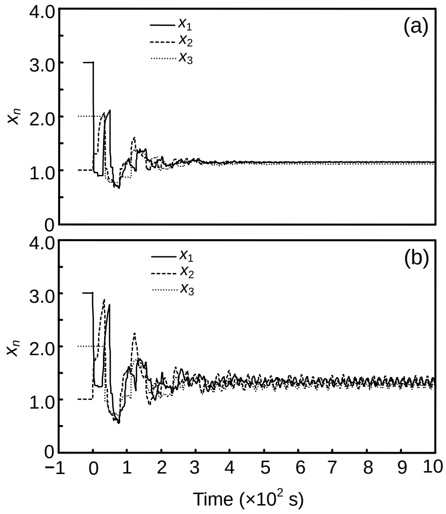

Fig. 4 (a) Stable and (b) unstable behaviors of the multi-source multi-link network

It is obvious that changing the price function parameters can lead to instability and oscillations in the users' rates. Under this condition the system boundedness condition does not hold, and therefore the stability condition cannot be satisfied.

7 Conclusion

In this paper, we derived some conditions for global stability in a general model of a congestion control system for a heterogeneous network with arbitrary topology and time-varying communication delays. The conditions obtained in this paper include certain restrictions on the systems parameters, network topology, upper bounds on rates and communication delays. We have applied the results to the various scenarios that may exist in computer communication networks. The results offer some guidelines for designing source or AQM controllers for the network with time-varying delays. Control algorithms can be designed such that the restrictions on the system parameters hold.

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