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# Non-uniform B-spline curves with multiple shape parameters*

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**Abstract:** We introduce a kind of shape-adjustable spline curves defined over a non-uniform knot sequence. These curves not only have the many valued properties of the usual non-uniform B-spline curves, but also are shape adjustable under fixed control polygons. Our method is based on the degree elevation of B-spline curves, where maximum degrees of freedom are added to a curve parameterized in terms of a non-uniform B-spline. We also discuss the geometric effect of the adjustment of shape parameters and propose practical shape modification algorithms, which are indispensable from the user's perspective.

**Key words:** Non-uniform B-spline, Shape parameter, Degree elevation
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## 1 Introduction

In computer aided geometric design (CAGD), parametric curves are usually combinations of basis functions and control points. Once a set of bases and the corresponding control points are specified, curves are determined, and control polygons depict their rough shapes. A traditional way of adjusting the shapes of these parametric curves is moving the control points. However, in practical applications, when designers specify the position of the control points, they may already be satisfied with the rough shape of curves; yet, diverse candidate shapes are still needed for design purpose. Therefore, it is sometimes required to obtain different curves under fixed control points, while keeping the original rough shape. To achieve this goal, free parameters other than control points should be introduced into curve representation.

Rational parametric curves are a more flexible modeling tool compared with their non-rational counterparts. For rational curves, the so-called weights are specified to each control point, which can be used to change the curves by varying their values (Piegl, 1989; Au and Yuen, 1995; Juhász, 1999). However, due to their fractional expressions, the derivative and integral calculations of rational curves or surfaces become very complicated. What is more important is that the computation of rational splines proves unstable. For B-spline curves, the effect of knot modification on the shape of B-spline curves has been studied (Juhász and Hoffmann, 2001; 2003), and knots have also been considered as free parameters for shape modification (Hu *et al.*, 2001; Juhász and Hoffmann, 2004; Ye *et al.*, 2006). The C-curve is another shape-adjustable curve constructed in the algebraic space spanned by polynomial and trigonometrical functions. It was first proposed by Pottman (1993) and Zhang (1996; 1997; 1999) in the cu-

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bic case, and then extended to higher orders by an integral method in Lü *et al.* (2002b) and Chen and Wang (2003). The upper bound of a C-curve's domain is a free parameter that can be used to adjust the shape. Under fixed control points, a C-curve can have different shapes by changing the upper bound of the domain (Hoffmann *et al.*, 2006). Another analogous curve is the H-curve (Lü *et al.*, 2002a), whose basis consists of both polynomial and hyperbolic functions. The geometric effects of the shape parameters of the C-curve and H-curve have been discussed, and shape modeling algorithms have been proposed (Hoffmann *et al.*, 2006; Hoffmann and Juhász, 2008b; Liu *et al.*, 2009; 2010). Using a complex expression, the C-curve and H-curve are unified as an F-curve (Zhang and Krause, 2005; Zhang *et al.*, 2005), which is also shape-adjustable. The shape modification of the functional B-spline (FB-spline) curves has been discussed in Hoffmann and Juhász (2008a).

There are various shape-adjustable curves constructed by incorporating shape parameters into the usual spline basis. Barsky (1981) proposed the so-called Beta-spline, which is a generalization of the uniform cubic B-spline curve and has two degrees of free parameters (i.e., bias and tension) for shape adjustment. Han and Liu (2003) and Wang and Wang (2004a; 2004b; 2005a; 2005b; 2005c) presented several curves with a shape parameter, such as the polynomial uniform B-spline with a shape parameter, trigonometric polynomial uniform B-spline with a shape parameter, hyperbolic polynomial uniform B-spline with a shape parameter, and Bézier curve with a shape parameter. These curves not only have properties similar to their classical counterparts (i.e., the ones without shape parameters), but also are shape adjustable. Based on the investigation of the structure of a cubic B-spline extension in Han and Liu (2003) and a uniform B-spline curve with a parameter in Wang and Wang (2004b), Cao and Wang (2008) proposed a more general uniform B-spline curve with multi-shape parameters. Curves with shape parameters can easily be generalized to tensor product surfaces by the tensor product method. However, triangular patches with shape parameters cannot be obtained directly this way. Cao and Wang (2007) developed triangular Bézier surfaces with a shape parameter. Some alternative spline curves with shape parameters and their shape adjustment can be found in the literature (Papp and Hoffmann, 2007; Han *et al.*, 2009; Li *et al.*, 2009; Yan *et al.*, 2009; Yang and Zeng, 2009).

The B-spline curve has a well-established and well-deserved place in the areas of data fitting, curve modeling, and ordinary differential equations solving, among others. Note that all kinds of B-spline curves with shape parameters mentioned above are generalizations of B-spline curves defined over uniform knot sequences. The non-uniform B-spline curve is the general type of B-spline curve that incorporates the uniform B-spline curve as a special case. Due to its non-uniform knot sequences, a non-uniform B-spline curve provides more flexibility for shape design. However, only a few papers have studied the spline construction of introducing a single local shape parameter into low-degree non-uniform B spline curves (Han, 2006; Juhász and Hoffmann, 2009). To complete the theory of curves with multiple shape parameters, we focus on introducing multiple shape parameters into non-uniform B-spline curves with arbitrary degree in this study. Our work is based on the fact that a $k$th order polynomial can be written in the form of a $(k+1)$th order polynomial and that one degree of freedom can be added to the representation. Inspired by the degree elevation of B-spline curves, we design a kind of spline and call it non-uniform B-spline with multiple shape parameters (NUBMP). Curves constructed by NUBMP (NUBMP curves) not only have properties similar to those of non-uniform B-spline curves, but also provide designers more handles (shape parameters) for shape adjustment under a fixed control polygon. We also analyze how the shape of the proposed curves will be affected if the shape parameters are adjusted. As 4th order B-spline curves are widely used in practical applications, we further study the properties of NUBMP curves of order four. Based on this, we provide a shape modification method essential for the practical use of the proposed curves in interactive design.

## 2 B-spline degree elevation

B-spline functions and curves have been exhaustedly studied. Thus, we only briefly review them to establish the notations of this paper. Please refer to Cox (1972), de Boor (1972), and Cohen *et al.* (1986) for more details on B-spline curves.

Given a positive integer $k$, knots $u_1 < u_2 < \ldots < u_m$ and integers $1 \leq z_i \leq k$, $i = 1, 2, \ldots, m$, suppose that

$$\begin{aligned} T &= \{t_1, t_2, \ldots, t_{n+k}\} \\ &= \{\underbrace{u_1, \ldots, u_1}_{z_1}, \underbrace{u_2, \ldots, u_2}_{z_2}, \ldots, \underbrace{u_m, \ldots, u_m}_{z_m}\}, \end{aligned} \quad (1)$$

where $n + k = \sum_{i=1}^{m} z_i$. Based on the knot sequence $T$, we define the usual $k$th order B-spline functions and denote them by $\widetilde{N}_{i,k}(t)$ (or $\widetilde{N}_{i,k}$ for simplicity) for $i = 1, 2, \ldots, n$, where $t$ is the parameter. Retaining the original knot order of sequence (1), a refined knot sequence can be defined by inserting every distinct knot once as

$$\begin{aligned} T^* &= \{t_1^*, t_2^*, ..., t_{n^*+k+1}^*\} \\ &= \{\underbrace{u_1, \ldots, u_1}_{z_1}, u_1, \underbrace{u_2, \ldots, u_2}_{z_2}, u_2, \ldots, \underbrace{u_m, \ldots, u_m}_{z_m}, u_m\}, \end{aligned} \quad (2)$$

where $n^* = n + m - 1$ and $t_j^*$ with $j = \sum_{i=1}^{s}(z_i + 1), s = 1, 2, ..., m$ are the newly added knots. For the later discussion, we introduce two notations as follows:

$\tau(i)$: for a knot $t_i = u_s \in T$, $t_{\tau(i)}^* \in T^*$ is the knot converted from knot $t_i$, i.e., $\tau(i) = i + s - 1$.

$\tau^*(j)$: if $t_j^*$ $(= u_s)$ is an old knot, then $\tau^*(j) = j - s + 1$; otherwise, $\tau^*(j) = \tau^*(j - 1)$.

We can now define the usual $(k + 1)$th order B-spline functions associated with knot sequence (2) and denote them by $N_{i,k+1}^*(t)$ (or $N_{i,k+1}^*$) for $i = 1, 2, ..., n^*$. The so-called degree raising of B-spline means that a B-spline expansion with respect to $k$th order B-spline functions associated with $T$ in Eq. (1) can be written as a B-spline series with respect to $(k + 1)$th order B-spline functions defined over an appropriately refined knot sequence $T^*$ in Eq. (2) (Cohen *et al.*, 1986). In particular, a $k$th order B-spline function defined over $T$ can be expressed as a linear combination of $(k + 1)$th B-spline functions associated with $T^*$ as

$$\widetilde{N}_{i,k} = \sum_{j=\tau(i)}^{\tau(i+k)-k} c_{j,k+1}^i N_{j,k+1}^*, \quad (3)$$

where $c_{j,k+1}^i$, $j = \tau(i), ..., \tau(i+k) - k$ are constant coefficients determined by knot vector $T^*$. This degree-raising method maintains the original continuity of the B-spline functions and of curves defined by them. For details, refer to Cohen *et al.* (1986).

## 3 Non-uniform B-spline with multi-shape parameters

In this section, we convert a usual B-spline function into a spline with free parameters called NUBMP. To raise the degree of a polynomial by one, one extra degree of freedom is added to the representation. Specifically, if we elevate the degree of B-spline functions by one, no less than one degree of freedom can be introduced into the expressions, as B-spline functions are piecewise polynomials. By exploiting all the available degrees of freedom in degree elevation, we can transform the usual B-spline functions to NUBMP. The main converting procedure consists of two steps. The first step is to express a $k$th order usual B-spline function associated with $T$ as a linear combination of $(k + 1)$th ones defined over the refined knot vector $T^*$ by the degree elevation (Eq. (3)) and then replace the constant coefficients $c_{j,k+1}^i$ in Eq. (3) with variable ones. At this stage, some degrees of freedom are brought into the B-spline functions representation. However, there is no guarantee that the transformed spline functions can form a basis. Therefore, in the second step, further constraints will be added to the variable coefficients to maintain the properties of the spline functions, such as non-negativity, local support, partition of unity, and linear independence. We denote the $k$th order NUBMPs defined over knot vector $T$ as $N_{i,k}(t)$, or simply as $N_{i,k}$. The details of the construction process of NUBMPs will be presented later in this section.

By replacing the constant coefficients $c_{j,k+1}^i$ with variable $a_{j,k+1}^i$ in Eq. (3), we roughly convert the usual B-spline functions $\tilde{N}_{i,k}$ into NUBMPs $N_{i,k}$ as

$$N_{i,k} = \sum_{j=\tau(i)}^{\tau(i+k)-k} a_{j,k+1}^i N_{j,k+1}^*. \quad (4)$$

Eq. (3) implies that spline functions $\tilde{N}_{i,k}$ defined by successive knots $t_i, t_{i+1}, ..., t_{i+k}$ can be expressed by $(k + 1)$th spline functions $N_{j,k+1}^*$ for $j = \tau(i), ..., \tau(i + k) - k$. In other words, a total of $\tau(i+k) - \tau(i) - k$ degrees of freedom are brought into the modified spline functions $N_{i,k}$ defined in Eq. (4). Variable coefficients $a_{j,k+1}^i$ are linearly independent, and further constraints on them are required to guarantee the basic properties of the spline functions.

First, to guarantee the non-negativity property of the transformed spline functions, all variable coefficients should naturally be non-negative, i.e.,

$$a^i_{j,k+1} \geq 0, j=\tau(i),...,\tau(i+k)-k, i=1,2,...,n-k. \quad (5)$$

Using constraints (5), it is easy to verify that the transformed spline functions maintain the properties of local support and order of continuity.

Second, note that a spline $N^*_{j,k+1}$ is defined over successive knots $t^*_j, t^*_{j+1}, ..., t^*_{j+k+1}$. Then Eq. (4) infers that a $(k+1)$th order spline $N^*_{j,k+1}$ simultaneously occurs in the expressions of spline functions $N_{i,k}$ associated with knots $t_i, i=\tau^*(j),...,\tau^*(j+k+1)$. In particular, spline $N_{j,k+1}$ participates in the expressions of $N_{\tau^*(j+k+1)-k,k},...,N_{\tau^*(j),k}$ with the corresponding parameters $a^{\tau^*(j+k+1)-k}_{j,k+1},...,a^{\tau^*(j)}_{j,k+1}$. Thus, to guarantee the spline functions' property of partition of unity, all parameters corresponding to the same spline $N^*_{j,k+1}$ must sum to 1, i.e.,

$$\sum_{i=\tau^*(j+k+1)-k}^{\tau^*(j)} a^i_{j,k+1} = 1. \quad (6)$$

Finally, by following the linearly independent property of usual B-spline functions, we can easily obtain the same property of NUBMPs. The following is the formal definition of NUBMPs.

**Definition 1** Order $k\,(\geq 2)$ NUBMPs defined over knot vector $T$ are defined as

$$N_{i,k}(t) = \sum_{j=\tau^*(i)}^{\tau(i+k)-k} a^i_{j,k+1} N^*_{j,k+1},$$

where $N^*_{j,k+1}$ are the $(k+1)$th order non-uniform B-spline functions defined over refined knot vector $T^*$, and coefficients $a^i_{j,k+1} \geq 0$ satisfy constraint conditions in Eq. (6).

**Definition 2** Order $k\,(\geq 2)$ NUBMP curves associated with knot vector $T$ are defined as

$$\boldsymbol{P}(t) = \sum_{i=1}^{n} N_{i,k}(t)\boldsymbol{P}_i, \quad (7)$$

where $\boldsymbol{P}_i \in \mathbb{R}^3$ are control points. When shape parameters $a^i_{j,k+1}$ contained in the spline functions are equal to $c^i_{j,k+1}$ in Eq. (3), curve (7) degenerates to a usual B-spline curve called the standard curve of curve (7).

Examples of NUBMP curves associated with knot sequence $T=\{0,0,0,0,0.30,0.50,0.89,1,1,1,1\}$ and fixed control points $\boldsymbol{P}_0, \boldsymbol{P}_1, ..., \boldsymbol{P}_6$ (black points) are shown in Fig. 1a (see p.806), where the yellow one is the standard curve. We classify shape parameters $a^i_{j,k+1}$ with the same subscript '$j,k+1$' into a group and denote it by $A_{j,k+1}$, i.e., $a^i_{j,k+1} \in A_{j,k+1}$. The size of $A_{j,k+1}$ is denoted by $\#A_{j,k+1}$. Then Eq. (6) implies that NUBMPs defined over knot sequence $T$ have a total of

$$\sum_{j=1}^{m+n^*} (\#A_{j,k+1} - 1) \quad (8)$$

free shape parameters for shape adjustment. This implies that for a spline to be defined in the order $k$ and a knot sequence $T$, the lower is the multiplicity of each knot, the more shape parameters will the constructed NUBMPs have.

## 4 Shape modification by adjusting shape parameters

In this section, the geometric meaning of shape parameters is studied. At first, we deduce an equivalent form of NUBMP curves. Based on this, we analyze the effect of the modification of a single shape parameter and multiple shape parameters on the shape of curves, respectively. Hereinafter, we suppose that the standard curve is the initial curve on which the shape adjusting is based.

Following Eq. (4), an NUBMP curve of order $k$ (Eq. (7)) can be rewritten as a usual B-spline curve of order $k+1$ associated with the refined knot sequence $T^*$ (Eq. (2)) as

$$\boldsymbol{P}(t) = \sum_{j=1}^{m+n^*} N^*_{j,k+1}(t)\boldsymbol{P}^*_j, \quad (9)$$

where the control points of a higher order curve satisfy $\boldsymbol{P}^*_j = \sum_{a^i_{j,k+1}\in A_{j,k+1}} a^i_{j,k+1}\boldsymbol{P}_i$, i.e., $\boldsymbol{P}^*_j$ is located in the convex hull of $\boldsymbol{P}_i$. We call curve (9), control points $\boldsymbol{P}^*_j$, and B-spline functions $N^*_{j,k+1}$ the dual curve, dual control points, and dual spline functions of curve (7), respectively. As shown in Fig. 1a, the yellow points are the dual control points of the standard curve.

We denote the variation of shape parameter $a^i_{j,k+1}$ as $\Delta a^i_{j,k+1}$. Varying one or more of the shape parameters, we call the moving locus of a point in an NUBMP curve a 'path' or a 'region'. Due to the restrictive condition that all parameters of $A_{j,k+1}$ (with $\#A_{j,k+1} \geq 2$) should sum to 1, there are in total $\#A_{j,k+1} - 1$ free shape parameters in group $A_{j,k+1}$. In theory, we can choose any one shape parameter in $A_{j,k+1}$ for shape adjustment. For simplicity, we can fix any $\#A_{j,k+1} - 2$ of the shape parameters and leave one degree of freedom for shape adjustment. In particular, if we choose two indexes $i_0, i_1$ with $i_0, i_1 < \#A_{j,k+1}$ and specify

$$a_{j,k+1}^{i} = c_{j,k+1}^{i} \quad \text{for} \quad 0 \leq i < \#A_{j,k+1}, \quad i \neq i_0, i_1, \tag{10}$$

then the remaining two shape parameters $a_{j,k+1}^{i_0}$ and $a_{j,k+1}^{i_1}$ become linearly dependent; i.e., $a_{j,k+1}^{i_0}$ and $a_{j,k+1}^{i_1}$ are mutually determined by each other. For example, we can choose $a_{j,k+1}^{i_0}$ to be adjusted, and then $a_{j,k+1}^{i_1}$ is determined by $a_{j,k+1}^{i_0}$. From the view of a dual curve, dual control point $\boldsymbol{P}_j^*$ is translated by the translation vector

$$\boldsymbol{d} = \Delta a_{j,k+1}^{i_0} \boldsymbol{P}_{i_0} + \Delta a_{j,k+1}^{i_1} \boldsymbol{P}_{i_1} = \Delta a_{j,k+1}^{i_0} (\boldsymbol{P}_{i_0} - \boldsymbol{P}_{i_1}). \tag{11}$$

Therefore, a curve's shape is modified in the following form:

$$\widetilde{\boldsymbol{P}}(t) = \boldsymbol{P}(t) + N_{j,k+1}^{*}(t)\boldsymbol{d}, \tag{12}$$

and the paths of points of a curve are straight line segments parallel to the vector $\boldsymbol{P}_{i_0} - \boldsymbol{P}_{i_1}$. This is the property of all curves that are a combination of control points and basis functions (Hoffmann and Juhász, 2008a). As shown in the example in Fig. 1a, dual control point $\boldsymbol{P}_1^*$ (of the standard curve) is originally located on $\boldsymbol{P}_0\boldsymbol{P}_1$. In this example, $\#A_{1,5} = 2$; hence, no shape parameters are needed to be fixed in advance. When $a_{1,5}^0$ (resp. $a_{1,5}^1$) is adjusted in $[0, 1]$, $a_{1,5}^1$ (resp. $a_{1,5}^0$) is determined, and $\boldsymbol{P}_1^*$ moves along line segment $\boldsymbol{P}_0\boldsymbol{P}_1$. Hence, paths of curve points are line segments parallel to vector $\boldsymbol{P}_0\boldsymbol{P}_1$ (see the path of point at $t = 0.15$, red line). Blue curves at the top and bottom correspond to the NUBMP curves with $a_{1,5}^0$ (resp. $a_{1,5}^1$) setting to 0 (resp. 1) and 1 (resp. 0), respectively.

In general, we can modify more than one shape parameter simultaneously. Analogously, one or more dual control points may be adjusted mediately. Specifically, if the modified shape parameters $a_{j,k+1}^i$ belong to the same group $A_{j,k+1}$, only one dual control point $\boldsymbol{P}_{j,k+1}$ is adjusted, and the translation vector $\boldsymbol{d}$ as given in Eq. (11) becomes

$$\boldsymbol{d} = \sum_{a_{j,k+1}^i \in A_{j,k+1}} \Delta a_{j,k+1}^i \boldsymbol{P}_i, \tag{13}$$

where $\Delta a_{j,k+1}^i$ may be equal to zero or not, depending on whether $a_{j,k+1}^i$ is fixed. In other words, dual control point $\boldsymbol{P}_j^*$ is translated in the convex hull:

$$\sum_{a_{j,k+1}^i \in A_{j,k+1}} a_{j,k+1}^i \boldsymbol{P}_i, \qquad \sum_{a_{j,k+1}^i \in A_{j,k+1}} a_{j,k+1}^i = 1, \tag{14}$$

and regions of curves are convex hulls similar to the convex hull (14) with a scale factor $N_{j,k+1}^*(t)$. The geometric effect of the simultaneous alteration of shape parameters from different groups is the cumulative effect of the alteration of the individual group, as shape parameters from different groups are linearly independent.

There is a dual advantage of using shape parameter modification. First, similar to the usual spline curves, the adjusted curves are located inside the convex hull of control points. Second, for a $k$th order NUBMP curve, one shape parameter can affect only $\lceil (k+1)/2 \rceil$ curve segments at most. It is different from the case of control point repositioning, where $k$ curve segments may be affected. The reason is that the adjustment of a single shape parameter only mediately gives rise to the modification on one dual control point, whereas a dual spline function of a $k$th order NUBMP curve has at most $\lceil (k+1)/2 \rceil$ nonzero interval. For example, a shape parameter of a 4th order NUBMP curve affects at most three curve segments, as shown in Fig. 1, where black crosses represent the points connecting the curve segments. Curves in Fig. 1b are obtained by adjusting $a_{4,5}^2$, where the ones on the top and bottom correspond to the dual control point $\boldsymbol{P}_4^*$ coinciding with points $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

## 5 NUBMP curves of the fourth order

In many applications of geometric modeling, piecewise polynomials of low order are preferred due to their simplicity and robustness. However, for certain applications, higher order polynomials are necessary or advantageous because smoothness is required or because higher order polynomials enable better approximation. Order four is a natural starting point because it gives $C^2$ smoothness, which is sufficient for many applications, while keeping the increased complexity to a minimum. In this section, we further study NUBMPs of the 4th order in two aspects to make this set of bases possible to be applied in applications. First, the variation diminishing (VD) property is one of the important properties of B-spline curves; therefore, it is also desired for NUBMP curves. In the first subsection, we give the conditions of shape parameters that maintain the VD property of NUBMP curves. Second, the constrained modification of curves is important in practical shape design systems. Therefore, based on the conditions imposed on the parameters, a constrained shape modification algorithm is designed in the second subsection. NUBMPs of an order lower than four is not specifically discussed, but they are analogous to and simpler than the case of order four.

### 5.1 Conditions for the VD property

Completely analogous to the usual B-spline curves, the VD property of NUBMP curves means that an NUBMP curve twists no more than its control polygon. An intuitive illustration of this is that a straight line will intersect the NUBMP curve no more times than it does the corresponding control polygon. The same relation holds true for a plane with a 3D space NUBMP curve. For convenience of discussion, we first define the inflection edge of planar polygon.

**Definition 3** For the four consecutive vertices $\boldsymbol{P}_i ... \boldsymbol{P}_{i+3}$ of a planar polygon, if points $\boldsymbol{P}_i$ and $\boldsymbol{P}_{i+3}$ are on opposite sides of the edge $\boldsymbol{P}_{i+1}\boldsymbol{P}_{i+2}$, then the edge $\boldsymbol{P}_{i+1}\boldsymbol{P}_{i+2}$ is an inflection edge of the planar polygon.

To endow NUBMP curves with the VD property, we use the theorem below:

**Theorem 1** For 4th order NUBMP curves defined over knot sequence 1, if their shape parameters satisfy

$$\frac{a_{j,5}^i}{a_{j-1,5}^i}+\frac{a_{j,5}^{i+2}}{a_{j+1,5}^{i+2}}\leq 1 \quad \text{for} \quad A_{j,5}=\{a_{j,5}^i, a_{j,5}^{i+1}, a_{j,5}^{i+2}\}, \tag{15}$$

where $1\leq j\leq m+n^*$ and $m$ and $n^*$ are defined in Section 2, then the VD property holds.

**Proof** The dual curve of a 4th order NUBMP curve with control points $\boldsymbol{P}_i$ is a usual 5th order B-spline curve with dual control points $\boldsymbol{P}_j^*$ in the convex hull of the original control points $\boldsymbol{P}_i$. As discussed in Section 3, each dual spline function $N_{j,k+1}^*$ participates in the expressions of spline functions $N_{\tau^*(j+k+1)-k,k}, ..., N_{\tau^*(j),k}$. Specifically, for NUBMP curves of order four, dual spline function $N_{j,5}^*$ is involved in the expression of at most three NUBMPs, i.e., $\#A_{j,5}\leq 3$ always holds. In particular, for the position of a dual control point $\boldsymbol{P}_j^*$ we have only the following possibilities:

1. If $A_{j,5}=\{a_{j,5}^i\}$, $\boldsymbol{P}_j^*$ will coincide with control point $\boldsymbol{P}_i$;

2. If $A_{j,5}=\{a_{j,5}^i, a_{j,5}^{i+1}\}$, $\boldsymbol{P}_j^*$ will be on the line segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$; and

3. If $A_{j,5}=\{a_{j,5}^i, a_{j,5}^{i+1}, a_{j,5}^{i+2}\}$, $\boldsymbol{P}_j^*$ will be in the convex hull of $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{P}_{i+2}$.

From the point of view of a dual curve, if its dual control polygon formed by $\boldsymbol{P}_j^*$ has no more inflection edges than the original control polygon formed by $\boldsymbol{P}_i$, then the VD property of an NUMBP curve is sufficiently guaranteed. For three successive groups $A_{j,5}, A_{j+1,5}$, and $A_{j+2,5}$, if all have a size smaller than three, then their corresponding dual control points $\boldsymbol{P}_j^*$, $\boldsymbol{P}_{j+1}^*$, and $\boldsymbol{P}_{j+2}^*$ will be on the successive legs of the original control polygon. Hence, the dual control polygon will not have more inflection edges than the original control polygon has. The only case where the inflection edges of a dual control polygon may be increased is when some group $A_{j,k+1}$ has a size of three. Similarly, the dual control point $\boldsymbol{P}_j^*$ is in the convex hull of three successive original control points. Suppose $A_{j,5}=\{a_{j,5}^i, a_{j,5}^{i+1}, a_{j,5}^{i+2}\}$. Then we have

$$\boldsymbol{P}_j^*=a_{j,5}^i\boldsymbol{P}_i+a_{j,5}^{i+1}\boldsymbol{P}_{i+1}+a_{j,5}^{i+2}\boldsymbol{P}_{i+2}, \tag{16}$$

where $a_{j,5}^i+a_{j,5}^{i+1}+a_{j,5}^{i+2}=1$. Moreover, it is simple to verify that if $\#A_{j,5}=3$, both $\#A_{j-1,5}$ and $\#A_{j+1,5}$ are equal to 2. Thus, we have

$$\begin{cases} \boldsymbol{P}_{j-1}^*=a_{j-1,5}^i\boldsymbol{P}_i+a_{j-1,5}^{i+1}\boldsymbol{P}_{i+1}, \\ \boldsymbol{P}_{j+1}^*=a_{j+1,5}^{i+1}\boldsymbol{P}_{i+1}+a_{j,5}^{i+2}\boldsymbol{P}_{i+2}. \end{cases} \tag{17}$$

Therefore, to prevent the increase of inflection edges of the dual control polygon, $\boldsymbol{P}_j^*$ should be located in the convex hull of triangle $\boldsymbol{P}_{j-1}^*\boldsymbol{P}_{i+1}\boldsymbol{P}_{j+1}^*$. An example of $j=3, i=1$ is shown in Fig. 2, where the permissible position of $\boldsymbol{P}_3^*$ is the region in gray including its boundary. If we represent $\boldsymbol{P}_j^*$ as

$$\boldsymbol{P}_j^*=u\boldsymbol{P}_{j-1}^*+(1-u-v)\boldsymbol{P}_{i+1}+v\boldsymbol{P}_{j+1}^*, \tag{18}$$

where $(u, v, 1-u-v)$ are the barycentric coordinates of $\boldsymbol{P}_j^*$ with respect to triangle $\boldsymbol{P}_{j-1}^*\boldsymbol{P}_{i+1}\boldsymbol{P}_{j+1}^*$, then

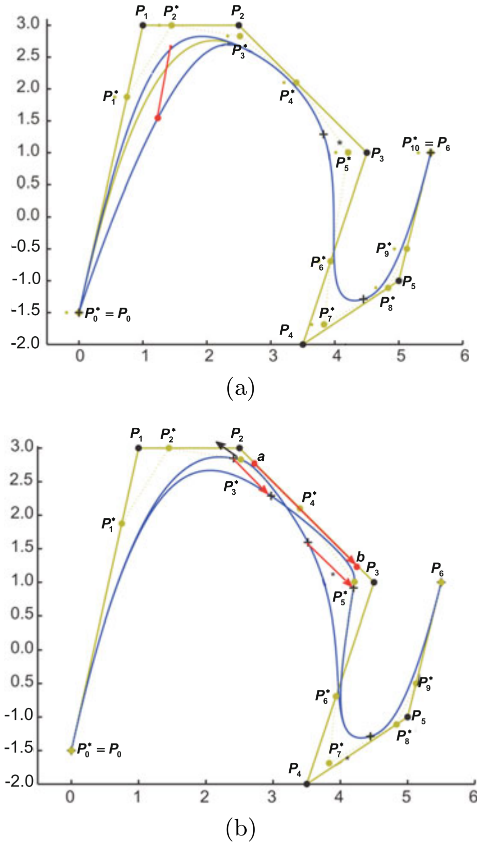


**Fig. 1 Fourth order NUBMP curves defined over $T = \{0, 0, 0, 0, 0.30, 0.50, 0.89, 1, 1, 1, 1\}$ with fixed control points (black) $P_0, P_1, ..., P_6$. (a) Modifying shape parameter $a^1_{1,5} \in [0, 1]$. The yellow curve and points are the standard curve and corresponding dual control points, respectively. (b) Modifying shape parameter $a^3_{4,5} \in [0.1245, 0.9017]$**

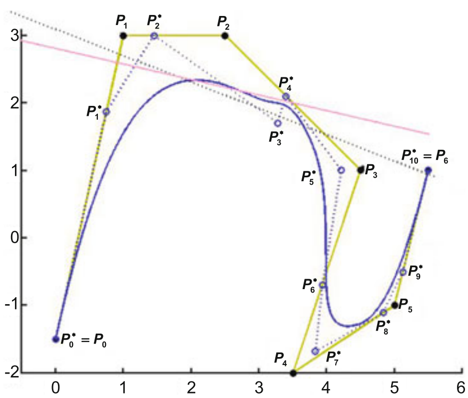

**Fig. 3 An example of an NUBMP curve without the variation diminishing (VD) property, where a red line intersects the control polygon and curve twice and four times, respectively. The shape parameters are inappropriately chosen such that $P^*_3$ is located on the line segment of $P_1P_3$, which is outside the permissible region (i.e., triangle $P^*_2P_2P^*_4$), as shown in Fig. 2. Hence, there is no guarantee for the VD property of curves**

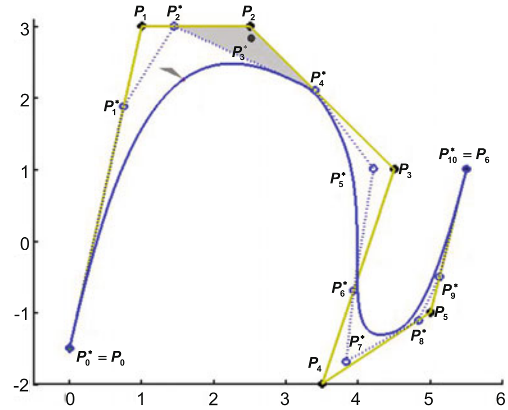

**Fig. 2 Region of a curve point at $t = 0.15$ on a 4th order NUBMP curve by modifying group $A_{3,5}$**

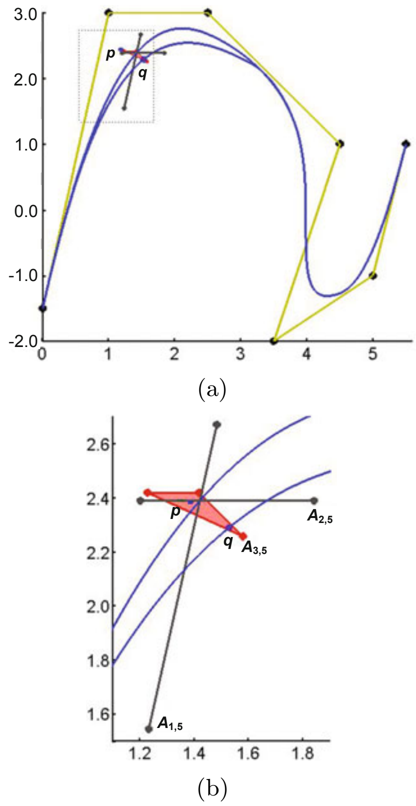


**Fig. 4 Constraint shape modification by alteration of a shape parameter group. (a) Valid region corresponding to different shape parameter groups; (b) Constraint shape modification by alteration of a shape parameter group**

to avoid the increase of inflection edges, barycentric coordinates in Eq. (18) should satisfy $0 \le u, v \le 1, 0 \le u + v \le 1$. By substituting Eq. (17) into Eq. (18), and comparing the coefficients of $\boldsymbol{P}_i$, $\boldsymbol{P}_{i+1}$, and $\boldsymbol{P}_{i+2}$ with the ones in Eq. (16), we have

$$a^i_{j,5} = ua^i_{j-1,5} \quad \text{and} \quad a^{i+2}_{j,5} = va^{i+2}_{j+1,5},$$

i.e.,

$$u = \frac{a^i_{j,5}}{a^i_{j-1,5}} \quad \text{and} \quad v = \frac{a^{i+2}_{j,5}}{a^{i+2}_{j+1,5}}.$$

Following $u + v \leq 1$, we complete the proof.

Fig. 1b shows an example of adjusting the shape parameter $a^4_{4,5}$. To maintain the VD property, the permissible range of $a^4_{4,5}$ becomes $[0.1245, 0.9017]$ instead of $[0, 1]$. Similarly, the allowable region of the dual control point $\boldsymbol{P}^*_4$ is on line segment $\boldsymbol{ab}$ instead of $\boldsymbol{P}_2\boldsymbol{P}_3$. Another example of adjusting the elements of group $A_{3,5}$ simultaneously is shown in Fig. 2. To maintain the VD property, the dual control point $\boldsymbol{P}^*_3$ should be restricted in the triangle $\boldsymbol{P}^*_2\boldsymbol{P}_2\boldsymbol{P}^*_4$ (Fig. 2, gray region). The small triangle similar to $\boldsymbol{P}^*_2\boldsymbol{P}_2\boldsymbol{P}^*_4$ corresponds to a region of curve point at $t = 0.15$ when the elements of the parameter group $A_{3,5}$ are simultaneously adjusted. The curve in Fig. 2 corresponds to the dual control point $\boldsymbol{P}^*_3$ coinciding with $\boldsymbol{P}^*_4$. A counterexample is shown in Fig. 3, where the shape parameters are inappropriately chosen, such that $\boldsymbol{P}^*_3$ is located outside the admissible area (triangle $\boldsymbol{P}^*_2\boldsymbol{P}_2\boldsymbol{P}^*_4$). Hence, the VD property is not guaranteed. The red line intersects the control polygon twice but intersects the curve four times. Note that condition (15) is sufficient only for maintaining the VD property of the curves. Hence, it does not completely or uniquely determine the permissible ranges of the shape parameters or permissible regions of the dual control points.

### 5.2 Constrained shape modification

Constrained shape modification of a curve is important for practical shape design. For example, allowing a curve to pass through a specified point is usually required in CAGD systems. According to previous analysis, an NUBMP curve can be modified as follows: a specified point on the modified curve is moved to a target position by adjusting a group of shape parameters, where the target position is restricted to a well-defined region, i.e., a line segment or a triangle region. An example of shape modification under position constraints by adjusting a single group of shape parameters is illustrated in Fig. 4. The implementing procedure is as follows.

Step 1: Specify a point $\boldsymbol{p}$ to be moved on the curve. Namely, specify parameter $\tau$ such that $\boldsymbol{p} = \boldsymbol{P}(\tau)$ (in Fig. 4, $\tau = 1.5$). The system shows the list of group $A_{j,5}$, which can affect the position of point at $\tau$. There will be at most three groups for the 4th order NUBMP curves.

Step 2: Choose the group to be modified. The system shows the corresponding path or region of the selected points. As shown in Fig. 4, $A_{4,5}$ is chosen. The selected group has a size of three, so the permissible position of the selected points shown by the system is a triangle. For clarity, the paths corresponding to the remaining two choices of the group are also shown.

Step 3: Specify a point $\boldsymbol{q}$ on the path or region as the target position of the selected curve point $\boldsymbol{p}$. In Fig. 4, $\boldsymbol{p}$ and $\boldsymbol{q}$ are denoted by the blue points. The system displays the curves interpolating $\boldsymbol{q}$.

An alternative practical control scenario is that we first select a shape parameter group and a point $\boldsymbol{q}$ in the space. Without specifying a point on the curve, we want the modified curve to pass through point $\boldsymbol{q}$. If the chosen group has only two parameters, point $\boldsymbol{q}$ can be in only the cylinder generated by the paths. Specifically, this cylinder degenerates to a planar region if the modified curve is planar. Then the corresponding parameter $\tau$ of the curve can be determined by finding the path that interpolates the specified point. If the chosen group has three elements, the position of $\boldsymbol{q}$ can be specified on the volume swept by the triangle as parameter $\tau$ varies in the curve definition interval. For planar curves, the volume degenerates to a planar region. In this case, there usually exists more than one solution because point $\boldsymbol{q}$ can be located in an infinite number of triangles. Hence, a range $[\tau_0, \tau_1]$ of $\tau$ can be specified, such that the adjustment from point $\boldsymbol{P}(\tau)$ to $\boldsymbol{q}$ can be achieved. In other words, there is an extra free parameter to meet the additional restricted condition; e.g., we can impose a restraint on the curve's tangent direction at $\boldsymbol{q}$. For spatial curves, a unique parameter $\tau$ can similarly be determined by finding the generator that passes through the specified points. After obtaining the parameter $\tau$, we can proceed by following the aforementioned algorithm.

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