Frontiers of Information Technology & Electronic Engineering
www.zju.edu.cn/jzus; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

H_∞ reference tracking control design for a class of nonlinear systems with time-varying delays*

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Received Feb. 10, 2015; Revision accepted May 23, 2015; Crosschecked Aug. 14, 2015

Abstract: This paper investigates the H_∞ trajectory tracking control for a class of nonlinear systems with time-varying delays by virtue of Lyapunov-Krasovskii stability theory and the linear matrix inequality (LMI) technique. A unified model consisting of a linear delayed dynamic system and a bounded static nonlinear operator is introduced, which covers most of the nonlinear systems with bounded nonlinear terms, such as the one-link robotic manipulator, chaotic systems, complex networks, the continuous stirred tank reactor (CSTR), and the standard genetic regulatory network (SGRN). First, the definition of the tracking control is given. Second, the H_∞ performance analysis of the closed-loop system including this unified model, reference model, and state feedback controller is presented. Then criteria on the tracking controller design are derived in terms of LMIs such that the output of the closed-loop system tracks the given reference signal in the H_∞ sense. The reference model adopted here is modified to be more flexible. A scaling factor is introduced to deal with the disturbance such that the control precision is improved. Finally, a CSTR system is provided to demonstrate the effectiveness of the established control laws.

Key words: H_∞ reference tracking, Nonlinear system, State feedback control, Time-varying delays, Unified model
doi:10.1631/FITEE.1500053 **Document code:** A **CLC number:** TP273

1 Introduction

The past two decades have witnessed great interest in the control of nonlinear systems, including H_∞ fuzzy control (Peng *et al.*, 2011; Li and Wang, 2012; Hsiao, 2013; Zhu *et al.*, 2014), neural network control (Lee *et al.*, 2013; Nodland *et al.*, 2013; Yang *et al.*, 2013), and robust adaptive control (Jin *et al.*, 2012; Yang *et al.*, 2012). Since its large-scale application in practical engineering processes, tracking controller design has been widely applied to industrial fields such as robotic trajectory tracking control, missile control, aircraft tracking control, sailing ship supply tracking, radar signal tracking for moving bodies, and high precision machining. However, most of the existing literature concentrates on the stabilization problems, except for some works (Zhang and Yu, 2010; Jin *et al.*, 2012; Liu and Chiang, 2012; Zhang H *et al.*, 2013a; 2013b). Jin *et al.* (2012) adopted the adaptive control method to design the tracking controller. In Zhang and Yu (2010), with the reference model introduced and reference input incorporated into the augmented disturbance, tracking control design conditions were derived for a class of linear neutral systems. Liu and Chiang (2012) investigated the output tracking problem in the case of an immeasurable system state using the Takagi-Sugeno (T-S) fuzzy model and virtual desired reference model. In Zhang H *et al.* (2013a), the authors

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* Project supported by the National Natural Science Foundation of China (Nos. 61222310, 61174142, and 61374021), the Zhejiang Provincial Natural Science Foundation of China (No. LZ14F030002), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Nos. 20120101110115 and 20130101110109), and the Fundamental Research Funds for the Central Universities, China (No. 2014XZZX003-12)

adopted the technique in predictive control and integral control to design the step tracking controller for networked systems, while an observer-based method was introduced later (Zhang H *et al.*, 2013b) to deal with the case where not all states are available in practice.

However, it is not easy to achieve the desired performance of the real physical controlled system. For one thing, the nonlinearity and uncertainty make it difficult or even infeasible to design controllers to guarantee the stabilization of the closed-loop system especially for those time-delayed or even time-varying delayed systems. For another, time delays always exist in real engineering such as physical and chemical systems and telecommunication neural networks due to the finite speed of information transition (Chen *et al.*, 2007; Chen and Zheng, 2015). Together with external noises or disturbances, time delays are considered a source of poor control performances or even instabilities, due to which control of such systems has received much attention. For the past few years, some researchers investigated the tracking control problem of the above-mentioned systems using the technique of parallel distributed compensation (PDC) based on the T-S fuzzy model (Zhang and Yu, 2010; Liu and Chiang, 2012; Zhang H *et al.*, 2013a; 2013b). However, since different nonlinear systems usually have different modeling methods, the PDC technique cannot provide a unified way to design the controller. As far as we know, there are no unified methods to deal with the control problems for different nonlinear systems.

Inspired by the discrete-time unified model (Liu *et al.*, 2014), we adopt a continuous-time unified model (CTUM), which is the interconnection of a linear delayed dynamic system and a bounded static nonlinear operator to investigate the H_∞ tracking control problem. The CTUM covers a lot of nonlinear systems such as the continuous stirred tank reactor (CSTR) (Cao and Frank, 2000; Liu and Chiang, 2012), chaotic systems (Guan and Chen, 2003; Yao *et al.*, 2011; Ahn, 2013; Liu *et al.*, 2013; Yang, 2013; Zhang G *et al.*, 2013), neural network control systems (Suykens *et al.*, 1996; Lee *et al.*, 2013), and Lur's systems (Feng *et al.*, 2013), on condition that the sector conditions are satisfied. We transform these different nonlinear systems into a unified model, and then design H_∞ tracking controllers in a unified way such that the output of the closed-loop system tracks the desired reference signal asymptotically and the influence of external disturbance on tracking precision is reduced to a lower level. The contributions of this paper include: (1) dealing with the stability and tracking control problems in a unified framework; (2) relaxing the need for the real reference input in conventional tracking control by introducing the model reference input; (3) improving tracking control precision by introducing a scaling factor to augmented disturbance.

Notations: $l_2[0,\infty)$ is the space of square integrable vectors. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices. $\boldsymbol{I}$ denotes the identity matrix of appropriate order. $*$ denotes the symmetric parts. diag() stands for a block-diagonal matrix. The notation $\boldsymbol{X}>\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are matrices of the same dimensions, means that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite. $C(\mathbb{R}^p;\mathbb{R}^q)$ denotes the space of all continuous functions mapping $\mathbb{R}^p\rightarrow\mathbb{R}^q$.

2 Preliminaries and problem formulation

Referring to the discrete-time model (Liu *et al.*, 2014), we adopt the following CTUM with inputs, outputs, and disturbances:

$$\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{A}_d\boldsymbol{x}(t-\tau(t))+\boldsymbol{B}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t))\\ \qquad\quad+\boldsymbol{B}_u\boldsymbol{u}(t)+\boldsymbol{B}_w\boldsymbol{w}(t),\\ \boldsymbol{\xi}(t)=\boldsymbol{C}_q\boldsymbol{x}(t)+\boldsymbol{C}_{qd}\boldsymbol{x}(t-\tau(t))+\boldsymbol{D}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t))\\ \qquad\quad+\boldsymbol{D}_u\boldsymbol{u}(t)+\boldsymbol{D}_w\boldsymbol{w}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}_y\boldsymbol{x}(t)+\boldsymbol{D}_{yu}\boldsymbol{u}(t)+\boldsymbol{D}_{yw}\boldsymbol{w}(t),\end{cases}\tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the system state, $\boldsymbol{u}(t)\in\mathbb{R}^m$ is the control input, $\boldsymbol{y}(t)\in\mathbb{R}^l$ is the measured output, $\boldsymbol{w}(t)\in\mathbb{R}^s$ is the process noise which belongs to $l_2[0,\infty)$, $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, $\boldsymbol{A}_d\in\mathbb{R}^{n\times n}$, $\boldsymbol{B}_p\in\mathbb{R}^{n\times L}$, $\boldsymbol{B}_w\in\mathbb{R}^{n\times s}$, $\boldsymbol{B}_u\in\mathbb{R}^{n\times m}$, $\boldsymbol{C}_q\in\mathbb{R}^{L\times n}$, $\boldsymbol{C}_{qd}\in\mathbb{R}^{L\times n}$, $\boldsymbol{D}_p\in\mathbb{R}^{L\times L}$, $\boldsymbol{D}_w\in\mathbb{R}^{L\times s}$, $\boldsymbol{D}_u\in\mathbb{R}^{L\times m}$, $\boldsymbol{C}_y\in\mathbb{R}^{l\times n}$, $\boldsymbol{D}_{yu}\in\mathbb{R}^{l\times m}$, and $\boldsymbol{D}_{yw}\in\mathbb{R}^{l\times s}$ are known matrices, $\boldsymbol{\xi}(t)\in\mathbb{R}^L$ is the input of the nonlinear continuous function $\boldsymbol{\phi}(\boldsymbol{\xi}(t))\in C(\mathbb{R}^L;\mathbb{R}^L)$ satisfying $\boldsymbol{\phi}(0)=0$, L is the number of nonlinear functions, $\tau(t)\in\mathbb{R}$ is the time-varying delay which is subject to $0\le\tau(t)\le d$, $\dot{\tau}(t)\le\mu\le\infty$, $\forall t\ge 0$, $\boldsymbol{\varpi}(t)$ is a given continuous function over $[-d,0]$, and the initial condition is described by $\boldsymbol{x}(t)=\boldsymbol{\varpi}(t)$, $\forall t\in[-d,0]$.

We assume that the nonlinear function in Eq. (1)

satisfies the following conditions:

$$0 \le \phi_i(\alpha)/\alpha \le h_i,\ \forall \alpha \in \mathbb{R}, \alpha \neq 0, \tag{2}$$

where $h_i > 0,\ i = 1, 2, \ldots, L$.

The reference model is given as follows:

$$\begin{cases} \dot{\boldsymbol{x}}_{\rm r}(t) = \boldsymbol{A}_{\rm r}\boldsymbol{x}_{\rm r}(t) + \boldsymbol{B}_{\rm r}\boldsymbol{r}(t), \\ \boldsymbol{y}_{\rm r}(t) = \boldsymbol{C}_{\rm r}\boldsymbol{x}_{\rm r}(t), \end{cases} \tag{3}$$

where $\boldsymbol{x}_{\rm r} \in \mathbb{R}^r$ is the reference state, $\boldsymbol{y}_{\rm r} \in \mathbb{R}^l$ is the reference output, $\boldsymbol{r}(t) \in \mathbb{R}^p$ is the energy bounded reference input, $\boldsymbol{A}_{\rm r} \in \mathbb{R}^{r\times r}$ (Hurwitz), $\boldsymbol{B}_{\rm r} \in \mathbb{R}^{r\times p}$, and $\boldsymbol{C}_{\rm r} \in \mathbb{R}^{l\times r}$ are constant matrices.

Remark 1 Compared with the reference model in Zhang and Yu (2010) and Liu and Chiang (2012), the model adopted here is more flexible due to the introduction of $\boldsymbol{B}_{\rm r}$ since the reference models used in Zhang and Yu (2010) and Liu and Chiang (2012) are special cases where $B_{\rm r} = 1$ in Zhang and Yu (2010) and $\boldsymbol{B}_{\rm r} = [0\ \ 1]^{\rm T}$ in Liu and Chiang (2012).

Based on the system state and reference state, we design the following state feedback controller:

$$\boldsymbol{u}(t) = \boldsymbol{K}_1\boldsymbol{x}(t) + \boldsymbol{K}_2\boldsymbol{x}_{\rm r}(t), \tag{4}$$

where $\boldsymbol{K}_1 \in \mathbb{R}^{m\times n}$ and $\boldsymbol{K}_2 \in \mathbb{R}^{m\times r}$ are the feedback gains. The closed-loop system including Eqs. (1), (3), and (4) is shown in Fig. 1.

Define the augmented state and augmented disturbance as follows:

$$\widetilde{\boldsymbol{x}}(t) = [\boldsymbol{x}^{\rm T}(t)\ \ \boldsymbol{x}_{\rm r}^{\rm T}(t)]^{\rm T},$$
$$\boldsymbol{v}(t) = [\boldsymbol{w}^{\rm T}(t)\ \ \boldsymbol{r}^{\rm T}(t)/k]^{\rm T}.$$

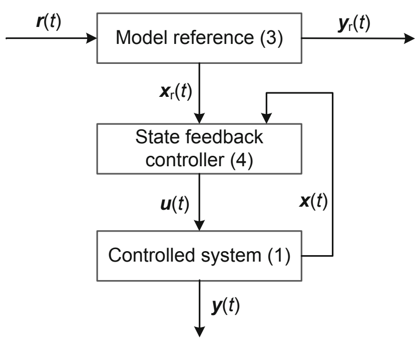

Fig. 1 Diagram of the closed-loop system

Then we can obtain the following augmented system:

$$\begin{cases} \dot{\widetilde{\boldsymbol{x}}}(t) = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{x}}(t) + \widetilde{\boldsymbol{A}}_d\widetilde{\boldsymbol{x}}(t-\tau(t)) + \widetilde{\boldsymbol{B}}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t)) \\ \qquad + \widetilde{\boldsymbol{B}}_w\boldsymbol{v}(t), \\ \boldsymbol{\xi}(t) = \widetilde{\boldsymbol{C}}_q\widetilde{\boldsymbol{x}}(t) + \widetilde{\boldsymbol{C}}_{qd}\widetilde{\boldsymbol{x}}(t-\tau(t)) + \boldsymbol{D}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t)) \\ \qquad + \widetilde{\boldsymbol{D}}_w\boldsymbol{v}(t), \\ \boldsymbol{e}(t) = \boldsymbol{y}_{\rm r}(t) - \boldsymbol{y}(t) = \widetilde{\boldsymbol{C}}_y\widetilde{\boldsymbol{x}}(t) + \widetilde{\boldsymbol{D}}_{yw}\boldsymbol{v}(t), \end{cases} \tag{5}$$

where

$$\widetilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} + \boldsymbol{B}_u\boldsymbol{K}_1 & \boldsymbol{B}_u\boldsymbol{K}_2 \\ \boldsymbol{0} & \boldsymbol{A}_{\rm r} \end{bmatrix}, \quad \widetilde{\boldsymbol{A}}_d = \begin{bmatrix} \boldsymbol{A}_d & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$
$$\widetilde{\boldsymbol{B}}_p = \begin{bmatrix} \boldsymbol{B}_p \\ \boldsymbol{0} \end{bmatrix}, \quad \widetilde{\boldsymbol{B}}_w = \begin{bmatrix} \boldsymbol{B}_w & \boldsymbol{0} \\ \boldsymbol{0} & k\boldsymbol{B}_{\rm r} \end{bmatrix},$$
$$\widetilde{\boldsymbol{C}}_q = \begin{bmatrix} \boldsymbol{C}_q + \boldsymbol{D}_u\boldsymbol{K}_1 & \boldsymbol{D}_u\boldsymbol{K}_2 \end{bmatrix},$$
$$\widetilde{\boldsymbol{C}}_{qd} = \begin{bmatrix} \boldsymbol{C}_{qd} & \boldsymbol{0} \end{bmatrix}, \ \widetilde{\boldsymbol{D}}_w = \begin{bmatrix} \boldsymbol{D}_w & \boldsymbol{0} \end{bmatrix},$$
$$\widetilde{\boldsymbol{C}}_y = \begin{bmatrix} -\boldsymbol{C}_y - \boldsymbol{D}_{yu}\boldsymbol{K}_1 & \boldsymbol{C}_r - \boldsymbol{D}_{yu}\boldsymbol{K}_2 \end{bmatrix},$$
$$\widetilde{\boldsymbol{D}}_{yw} = \begin{bmatrix} -\boldsymbol{D}_{yw} & \boldsymbol{0} \end{bmatrix},$$

in which k is called the scaling factor taking values from the set of all positive integers.

Definition 1 (Liu and Chiang, 2012) Given a real number γ and a symmetric positive definite matrix $\boldsymbol{T}$, system (5) is said to be stable with γ-disturbance attenuation if the following conditions are satisfied:

1. With zero process disturbance and zero reference input (i.e., $\boldsymbol{v}(t) = 0$), system (5) is asymptotically stable;

2. With zero initial conditions, the following index holds:

$$J = \int_0^\infty [\boldsymbol{e}^{\rm T}(t)\boldsymbol{T}\boldsymbol{e}(t) - \gamma^2\boldsymbol{v}^{\rm T}(t)\boldsymbol{v}(t)]{\rm d}t \le 0 \tag{6}$$

for any $\boldsymbol{w}(t) \in l_2[0,\ \infty)$, $\boldsymbol{r}(t)$, and all admissible delays $\tau(t)$. The feedback controller (4) is said to be an H_∞ tracking controller with disturbance attenuation rate γ. This parameter γ is called the upper bound of the L_2 gain for system (5). If we find a minimal positive γ which satisfies the above two conditions, controller (4) is an optimal H_∞ tracking controller. Since $\boldsymbol{r}(t)$ is taken as a part of augmented disturbance, γ also represents the tracking error.

Remark 2 To enhance the tracking control precision, we introduce the positive scalar k into augmented disturbance. From Definition 1, we can see that applying k is equivalent to reducing the parameter γ by k times, which means the tracking error can be reduced by k times. Furthermore, k can be selected flexibly according to the specific control requirement.

Lemma 1 (Park *et al.*, 2011) For functions $f_i(t) \in C(\mathbb{R}^L;\mathbb{R}^L)$, $i=1,2,\ldots,N$, the corresponding reciprocal convex combination satisfies

$$(1)\ \sum_i \frac{1}{\alpha_i} f_i(t) \geq \sum_i f_i(t) + \sum_{i\neq j} g_{i,j}(t)$$
$$\forall g_{i,j}(t) \in C(\mathbb{R}^L;\mathbb{R}^L),\ g_{i,j}(t)=g_{j,i}(t),$$
$$(2)\ \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ * & f_j(t) \end{bmatrix} \geq 0,$$

provided that the following conditions are satisfied:

1. Functions $f_i(t), f_j(t) \in C(\mathbb{R}^L;\mathbb{R}^L)$, $i,j=1,\ 2,\ \ldots,\ N$ have positive values over an open subset of $\mathbb{R}^L$;

2. The reciprocal convex combination of these functions is a function of the form $1/\alpha_1+1/\alpha_2+\ldots+1/\alpha_N$ with $\alpha_i>0$, $\sum_{i=1}^N \alpha_i = 1$.

3 H_∞ tracking controller design for the unified model

First, we analyze the H_∞ performance of the closed-loop system (5) and derive the following theorem:

Theorem 1 Given $d\geq 0$, $\mu\geq 0$, and $\gamma>0$, if there exist symmetric positive definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\boldsymbol{R}$, a diagonal positive definite matrix $\boldsymbol{\Sigma}$, and a symmetric matrix $\boldsymbol{S}$ that satisfy the following linear matrix inequalities (LMIs):

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13} & \boldsymbol{M}_{14} & \boldsymbol{M}_{15} \\ * & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} & \boldsymbol{M}_{24} & \boldsymbol{M}_{25} \\ * & * & \boldsymbol{M}_{33} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & \boldsymbol{M}_{44} & \boldsymbol{M}_{45} \\ * & * & * & * & \boldsymbol{M}_{55} \end{bmatrix} < 0, \quad (7)$$

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{S} \\ * & \boldsymbol{R} \end{bmatrix} > 0, \quad (8)$$

where

$$\begin{aligned}
\boldsymbol{M}_{11} &= \boldsymbol{P}\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q}_1 + \boldsymbol{Q}_2 - \boldsymbol{R} \\
&\quad + \widetilde{\boldsymbol{C}}_y^{\mathrm{T}}\boldsymbol{T}\widetilde{\boldsymbol{C}}_y + d^2\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{A}}, \\
\boldsymbol{M}_{12} &= \boldsymbol{P}\widetilde{\boldsymbol{A}}_d + \boldsymbol{R} - \boldsymbol{S} + d^2\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{A}}_d, \\
\boldsymbol{M}_{13} &= \boldsymbol{S},\ \boldsymbol{M}_{14} = \boldsymbol{P}\widetilde{\boldsymbol{B}}_p + \widetilde{\boldsymbol{C}}_q^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{H} + d^2\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_p, \\
\boldsymbol{M}_{15} &= \widetilde{\boldsymbol{C}}_y^{\mathrm{T}}\boldsymbol{T}\widetilde{\boldsymbol{D}}_{yw} + \boldsymbol{P}\widetilde{\boldsymbol{B}}_w + d^2\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_w, \\
\boldsymbol{M}_{22} &= -(1-\mu)\boldsymbol{Q}_2 - 2\boldsymbol{R} + 2\boldsymbol{S} + d^2\widetilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{A}}_d, \\
\boldsymbol{M}_{23} &= \boldsymbol{R} - \boldsymbol{S},\ \boldsymbol{M}_{24} = \widetilde{\boldsymbol{C}}_{qd}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{H} + d^2\widetilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_p, \\
\boldsymbol{M}_{25} &= d^2\widetilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_w,\ \boldsymbol{M}_{33} = -\boldsymbol{Q}_1 - \boldsymbol{R}, \\
\boldsymbol{M}_{44} &= \boldsymbol{D}_p^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{H} + \boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{D}_p - 2\boldsymbol{\Sigma} + d^2\widetilde{\boldsymbol{B}}_p^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_p, \\
\boldsymbol{M}_{45} &= d^2\widetilde{\boldsymbol{B}}_p^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_w + \boldsymbol{H}\boldsymbol{\Sigma}\widetilde{\boldsymbol{D}}_w, \\
\boldsymbol{M}_{55} &= -\gamma^2\boldsymbol{I} + \widetilde{\boldsymbol{D}}_{yw}^{\mathrm{T}}\boldsymbol{T}\widetilde{\boldsymbol{D}}_{yw} + d^2\widetilde{\boldsymbol{B}}_w^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{B}}_w, \\
\boldsymbol{H} &= \mathrm{diag}(h_1,\ h_2,\ \ldots,\ h_L),
\end{aligned}$$

then system (5) with $\boldsymbol{v}(t)=\boldsymbol{0}$ is globally asymptotically stable, and the L_2 gain of system (5) is less than or equal to γ.

Proof Consider system (5) with $\boldsymbol{v}(t)=\boldsymbol{0}$, i.e.,

$$\begin{cases} \dot{\widetilde{\boldsymbol{x}}}(t) = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{x}}(t) + \widetilde{\boldsymbol{A}}_d\widetilde{\boldsymbol{x}}(t-\tau(t)) + \widetilde{\boldsymbol{B}}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t)), \\ \boldsymbol{\xi}(t) = \widetilde{\boldsymbol{C}}_q\widetilde{\boldsymbol{x}}(t) + \widetilde{\boldsymbol{C}}_{qd}\widetilde{\boldsymbol{x}}(t-\tau(t)) + \boldsymbol{D}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t)). \end{cases} \quad (9)$$

Since $\widetilde{\boldsymbol{x}}(t)=\boldsymbol{0}$ and $\boldsymbol{\xi}(t)=\boldsymbol{0}$ are solutions to Eq. (9), there exists at least one equilibrium point at the origin, i.e., $\widetilde{\boldsymbol{x}}_{\mathrm{eq}}(t)=\boldsymbol{0}$, $\boldsymbol{\xi}_{\mathrm{eq}}(t)=\boldsymbol{0}$. For system (9), we adopt the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \quad (10)$$

with

$$\begin{cases} V_1(t) = \widetilde{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\widetilde{\boldsymbol{x}}(t), \\ V_2(t) = \int_{t-d}^{t} \widetilde{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_1\widetilde{\boldsymbol{x}}(s)\mathrm{d}s, \\ V_3(t) = \int_{t-\tau(t)}^{t} \widetilde{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{Q}_2\widetilde{\boldsymbol{x}}(s)\mathrm{d}s, \\ V_4(t) = d\int_{-d}^{0}\int_{t+\theta}^{t} \dot{\widetilde{\boldsymbol{x}}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\widetilde{\boldsymbol{x}}}(s)\mathrm{d}s\mathrm{d}\theta, \end{cases}$$

where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}>0$, $\boldsymbol{Q}_1=\boldsymbol{Q}_1^{\mathrm{T}}>0$, $\boldsymbol{Q}_2=\boldsymbol{Q}_2^{\mathrm{T}}>0$, $\boldsymbol{R}=\boldsymbol{R}^{\mathrm{T}}>0$. Thus, for any $\widetilde{\boldsymbol{x}}(t)\neq\boldsymbol{0}$, $V(t)>0$ and $V(t)=0$ if and only if $\widetilde{\boldsymbol{x}}(t)=\boldsymbol{0}$.

The sector conditions in inequality (2) can be written as follows:

$$\phi_i^2(\xi_i(t)) - h_i\phi_i(\xi_i(t))\xi_i(t) \leq 0, \quad (11)$$

which is equivalent to

$$2\varepsilon_i\phi_i^2(\xi_i(t)) - 2\varepsilon_i h_i\phi_i(\xi_i(t))\xi_i(t) \leq 0, \quad (12)$$

where $\varepsilon_i>0$, $i=1,\ 2,\ \ldots,\ L$.

Applying Lemma 1, where

$$\begin{aligned}
f_1(t) =& [\widetilde{\boldsymbol{x}}(t-\tau(t)) - \widetilde{\boldsymbol{x}}(t-d)]^{\mathrm{T}} \\
&\cdot \boldsymbol{R}[\widetilde{\boldsymbol{x}}(t-\tau(t)) - \widetilde{\boldsymbol{x}}(t-d)], \\
f_2(t) =& [\widetilde{\boldsymbol{x}}(t) - \widetilde{\boldsymbol{x}}(t-\tau(t))]^{\mathrm{T}} \\
&\cdot \boldsymbol{R}[\widetilde{\boldsymbol{x}}(t) - \widetilde{\boldsymbol{x}}(t-\tau(t))], \\
g_{1,2}(t) =& g_{2,1}(t) = [\widetilde{\boldsymbol{x}}(t-\tau(t)) - \widetilde{\boldsymbol{x}}(t-d)]^{\mathrm{T}} \\
&\cdot \boldsymbol{S}[\widetilde{\boldsymbol{x}}(t) - \widetilde{\boldsymbol{x}}(t-\tau(t))],
\end{aligned}$$

we can obtain the following inequality:

$$\begin{aligned}
&-d\int_{t-d}^{t}\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}(s)\mathrm{d}s\\
=&-d\int_{t-d}^{t-\tau(t)}\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}(s)\mathrm{d}s-d\int_{t-\tau(t)}^{t}\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}(s)\mathrm{d}s\\
\le&-\frac{d}{d-\tau(t)}\left[\tilde{\boldsymbol{x}}(t-\tau(t))-\tilde{\boldsymbol{x}}(t-d)\right]^{\mathrm{T}}\boldsymbol{R}\left[\tilde{\boldsymbol{x}}(t-\tau(t))-\right.\\
&\left.\tilde{\boldsymbol{x}}(t-d)\right]-\frac{d}{\tau(t)}\left[\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(t-\tau(t))\right]^{\mathrm{T}}\boldsymbol{R}\left[\tilde{\boldsymbol{x}}(t)-\right.\\
&\left.\tilde{\boldsymbol{x}}(t-\tau(t))\right]\\
\le&-\begin{bmatrix}\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(t-\tau(t))\\ \tilde{\boldsymbol{x}}(t-\tau(t))-\tilde{\boldsymbol{x}}(t-d)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{R}&\boldsymbol{S}\\ *&\boldsymbol{R}\end{bmatrix}\\
&\cdot\begin{bmatrix}\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(t-\tau(t))\\ \tilde{\boldsymbol{x}}(t-\tau(t))-\tilde{\boldsymbol{x}}(t-d)\end{bmatrix}. \qquad (13)
\end{aligned}$$

The derivative of $V(t)$ along the solution to system (9) is as follows:

$$\begin{aligned}
&\dot{V}(t)\\
\le&2\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\tilde{\boldsymbol{x}}}(t)+\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)(\boldsymbol{Q}_1+\boldsymbol{Q}_2)\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}^{\mathrm{T}}(t-d)\\
&\cdot\boldsymbol{Q}_1\tilde{\boldsymbol{x}}(t-d)-(1-\mu)\tilde{\boldsymbol{x}}^{\mathrm{T}}(t-\tau(t))\boldsymbol{Q}_2\tilde{\boldsymbol{x}}(t-\tau(t))\\
&+d^2\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}}(t)\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}(t)-d\int_{t-d}^{t}\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}(s)\mathrm{d}s\\
\le&2\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\tilde{\boldsymbol{x}}}(t)+\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)(\boldsymbol{Q}_1+\boldsymbol{Q}_2)\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}^{\mathrm{T}}(t-d)\\
&\cdot\boldsymbol{Q}_1\tilde{\boldsymbol{x}}(t-d)-(1-\mu)\tilde{\boldsymbol{x}}^{\mathrm{T}}(t-\tau(t))\boldsymbol{Q}_2\tilde{\boldsymbol{x}}(t-\tau(t))\\
&+d^2\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}}(t)\boldsymbol{R}\dot{\tilde{\boldsymbol{x}}}(t)-\left[\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(t-\tau(t))\right]^{\mathrm{T}}\boldsymbol{R}\left[\tilde{\boldsymbol{x}}(t)\right.\\
&\left.-\tilde{\boldsymbol{x}}(t-\tau(t))\right]-\left[\tilde{\boldsymbol{x}}(t-\tau(t))-\tilde{\boldsymbol{x}}(t-d)\right]^{\mathrm{T}}\boldsymbol{R}\\
&\cdot\left[\tilde{\boldsymbol{x}}(t-\tau(t))-\tilde{\boldsymbol{x}}(t-d)\right]-2\left[\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(t-\tau(t))\right]^{\mathrm{T}}\\
&\cdot S\left[\tilde{\boldsymbol{x}}(t-\tau(t))-\tilde{\boldsymbol{x}}(t-d)\right]-2\boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\xi}(t))\boldsymbol{\Sigma}\boldsymbol{\phi}(\boldsymbol{\xi}(t))\\
&+2\boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\xi}(t))\boldsymbol{\Sigma}\boldsymbol{H}\boldsymbol{\xi}(t)\\
=&\boldsymbol{\Psi}^{\mathrm{T}}\widetilde{\boldsymbol{M}}\boldsymbol{\Psi}, \qquad (14)
\end{aligned}$$

where $\boldsymbol{\Sigma}=\mathrm{diag}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_L)>0$, $\boldsymbol{\Psi}=\left[\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\ \tilde{\boldsymbol{x}}^{\mathrm{T}}(t-\tau(t))\ \ \tilde{\boldsymbol{x}}^{\mathrm{T}}(t-d)\ \ \boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\xi}(t))\right]^{\mathrm{T}}$,

$$\widetilde{\boldsymbol{M}}=\begin{bmatrix}\widetilde{\boldsymbol{M}}_{11}&\boldsymbol{M}_{12}&\boldsymbol{M}_{13}&\boldsymbol{M}_{14}\\ *&\boldsymbol{M}_{22}&\boldsymbol{M}_{23}&\boldsymbol{M}_{24}\\ *&*&\boldsymbol{M}_{33}&\boldsymbol{0}\\ *&*&*&\boldsymbol{M}_{44}\end{bmatrix}$$

with $\widetilde{\boldsymbol{M}}_{11}=\boldsymbol{P}\widetilde{\boldsymbol{A}}+\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{Q}_2-\boldsymbol{R}+d^2\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\widetilde{\boldsymbol{A}}$. By virtue of the Schur complement (Boyd *et al.*, 1994), $\boldsymbol{M}<0$ is equivalent to inequality (15) (at the top of the next page). $\widetilde{\boldsymbol{M}}$ is the principal minor of the left side of inequality (15). We have $\widetilde{\boldsymbol{M}}<0$, i.e., $\dot{V}(t)<0$. So, system (5) with $\boldsymbol{v}(t)=\boldsymbol{0}$ is globally asymptotically stable. With zero initial conditions, J in Eq. (6) for system (5) is equivalent to

$$\begin{aligned}
J=&\int_0^{\infty}\left[\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{T}\boldsymbol{e}(t)-\gamma^2\boldsymbol{v}^{\mathrm{T}}(t)\boldsymbol{v}(t)+\dot{V}(t)\right]\mathrm{d}t\\
&-V(\tilde{\boldsymbol{x}}(\infty))+V(\tilde{\boldsymbol{x}}(0))\\
\le&\int_0^{\infty}\left[\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{T}\boldsymbol{e}(t)-\gamma^2\boldsymbol{v}^{\mathrm{T}}(t)\boldsymbol{v}(t)+\dot{V}(t)\right]\mathrm{d}t\\
=&\int_0^{\infty}\left\{\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\Psi}+2\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\widetilde{\boldsymbol{B}}_w\boldsymbol{v}(t)+2\boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\xi}(t))\boldsymbol{\Sigma}\boldsymbol{H}\right.\\
&\cdot\widetilde{\boldsymbol{D}}_w\boldsymbol{v}(t)+\left[\widetilde{\boldsymbol{C}}_y\tilde{\boldsymbol{x}}(t)+\widetilde{\boldsymbol{D}}_{yw}\boldsymbol{v}(t)\right]^{\mathrm{T}}\boldsymbol{T}\\
&\left.\cdot\left[\widetilde{\boldsymbol{C}}_y\tilde{\boldsymbol{x}}(t)+\widetilde{\boldsymbol{D}}_{yw}\boldsymbol{v}(t)\right]-\gamma^2\boldsymbol{v}^{\mathrm{T}}(t)\boldsymbol{v}(t)\right\}\mathrm{d}t\\
=&\int_0^{\infty}\boldsymbol{\Upsilon}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\Upsilon}\mathrm{d}t, \qquad (16)
\end{aligned}$$

where

$$\boldsymbol{\Upsilon}=[\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\ \tilde{\boldsymbol{x}}^{\mathrm{T}}(t-\tau(t))\ \tilde{\boldsymbol{x}}^{\mathrm{T}}(t-d)\ \boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\xi}(t))\ \boldsymbol{v}^{\mathrm{T}}(t)]^{\mathrm{T}}.$$

Since $\boldsymbol{M}<0$ by Theorem 1, we have $J<0$ for any $\boldsymbol{\Upsilon}\neq\boldsymbol{0}$. Due to Definition 1, the L_2 gain of system (5) is less than or equal to γ. Thus, the proof is completed.

We can rewrite $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{C}}_q$, and $\widetilde{\boldsymbol{C}}_y$ in Eq. (7) as $\widetilde{\boldsymbol{A}}=\bar{\boldsymbol{A}}+\bar{\boldsymbol{B}}_u\boldsymbol{K}$, $\widetilde{\boldsymbol{C}}_q=\bar{\boldsymbol{C}}_q+\boldsymbol{D}_u\boldsymbol{K}$, $\widetilde{\boldsymbol{C}}_y=\bar{\boldsymbol{C}}_y-\boldsymbol{D}_{yu}\boldsymbol{K}$, where

$$\boldsymbol{K}=[\boldsymbol{K}_1\quad\boldsymbol{K}_2],\ \bar{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{A}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{A}_{\mathrm{r}}\end{bmatrix},\ \bar{\boldsymbol{B}}_u=\begin{bmatrix}\boldsymbol{B}_u\\ \boldsymbol{0}\end{bmatrix},$$
$$\bar{\boldsymbol{C}}_q=[\boldsymbol{C}_q\quad\boldsymbol{0}],\quad\bar{\boldsymbol{C}}_y=[-\boldsymbol{C}_y\quad\boldsymbol{C}_{\mathrm{r}}].$$

Theorem 2 Given $d\ge 0$, $\mu\ge 0$, and $\gamma>0$, if there exist symmetric positive definite matrices $\boldsymbol{X}$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Y}_3$, $\boldsymbol{Y}_4$, a diagonal positive definite matrix $\boldsymbol{V}$, and a matrix $\boldsymbol{W}$ that satisfy Eq. (17) (see the next page) and the following LMIs:

$$\begin{bmatrix}\boldsymbol{Y}_3&\boldsymbol{Y}_4\\ *&\boldsymbol{Y}_3\end{bmatrix}>0, \qquad (18)$$

then system (5) with $\boldsymbol{v}(t)=\boldsymbol{0}$ is globally asymptotically stable, and the L_2 gain of system (5) is less than or equal to γ. Furthermore, the feedback gain of controller (4) is

$$\boldsymbol{K}=\boldsymbol{W}\boldsymbol{X}^{-1}. \qquad (19)$$

Proof Pre- and post-multiplying the left and right sides of inequality (15) (see the next page)

$$\begin{bmatrix} \boldsymbol{\Xi}_1 & \boldsymbol{P}\widetilde{\boldsymbol{A}}_d+\boldsymbol{R}-\boldsymbol{S} & \boldsymbol{S} & \boldsymbol{P}\widetilde{\boldsymbol{B}}_p+\widetilde{\boldsymbol{C}}_q^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{H} & \boldsymbol{P}\widetilde{\boldsymbol{B}}_w & d\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R} & \widetilde{\boldsymbol{C}}_y^{\mathrm{T}}\boldsymbol{T} \\ * & -(1-\mu)\boldsymbol{Q}_2-2\boldsymbol{R}+2\boldsymbol{S} & \boldsymbol{R}-\boldsymbol{S} & \widetilde{\boldsymbol{C}}_{qd}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{H} & \boldsymbol{0} & d\widetilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R} & \boldsymbol{0} \\ * & * & -\boldsymbol{Q}_1-\boldsymbol{R} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & \boldsymbol{\Xi}_2 & \boldsymbol{H}\boldsymbol{\Sigma}\widetilde{\boldsymbol{D}}_w & d\widetilde{\boldsymbol{B}}_p^{\mathrm{T}}\boldsymbol{R} & \boldsymbol{0} \\ * & * & * & * & -\gamma^2\boldsymbol{I} & d\widetilde{\boldsymbol{B}}_w^{\mathrm{T}}\boldsymbol{R} & \widetilde{\boldsymbol{D}}_{yw}^{\mathrm{T}}\boldsymbol{T} \\ * & * & * & * & * & -\boldsymbol{R} & \boldsymbol{0} \\ * & * & * & * & * & * & -\boldsymbol{T} \end{bmatrix} < 0, \quad (15)$$

$$\boldsymbol{\Xi}_1=\boldsymbol{P}\widetilde{\boldsymbol{A}}+\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_1+\boldsymbol{Q}_2-\boldsymbol{R},\quad \boldsymbol{\Xi}_2=-2\boldsymbol{\Sigma}+\boldsymbol{D}_p^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{H}+\boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{D}_p.$$

$$\boldsymbol{N}_1=\begin{bmatrix} \boldsymbol{\Xi}_3 & \widetilde{\boldsymbol{A}}_d\boldsymbol{X}+\boldsymbol{Y}_3-\boldsymbol{Y}_4 & \boldsymbol{Y}_4 & \boldsymbol{\Xi}_4 & \widetilde{\boldsymbol{B}}_w & \boldsymbol{\Xi}_6 & \boldsymbol{\Xi}_7 \\ * & -(1-\mu)\boldsymbol{Y}_2-2\boldsymbol{Y}_3+2\boldsymbol{Y}_4 & \boldsymbol{Y}_3-\boldsymbol{Y}_4 & \boldsymbol{X}\widetilde{\boldsymbol{C}}_{qd}^{\mathrm{T}}\boldsymbol{H} & \boldsymbol{0} & d\boldsymbol{X}\widetilde{\boldsymbol{A}}_d^{\mathrm{T}} & \boldsymbol{0} \\ * & * & -\boldsymbol{Y}_1-\boldsymbol{Y}_3 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & \boldsymbol{\Xi}_5 & \boldsymbol{H}\widetilde{\boldsymbol{D}}_w & d\boldsymbol{V}\widetilde{\boldsymbol{B}}_p^{\mathrm{T}} & \boldsymbol{0} \\ * & * & * & * & -\gamma^2\boldsymbol{I} & d\widetilde{\boldsymbol{B}}_w^{\mathrm{T}} & \widetilde{\boldsymbol{D}}_{yw}^{\mathrm{T}}\boldsymbol{T} \\ * & * & * & * & * & -\boldsymbol{X}-\boldsymbol{X}^{\mathrm{T}}+\boldsymbol{Y}_3 & \boldsymbol{0} \\ * & * & * & * & * & * & -\boldsymbol{T} \end{bmatrix} < 0, \quad (17)$$

$$\boldsymbol{\Xi}_3=(\bar{\boldsymbol{A}}\boldsymbol{X}+\bar{\boldsymbol{B}}_u\boldsymbol{W})^{\mathrm{T}}+\bar{\boldsymbol{A}}\boldsymbol{X}+\bar{\boldsymbol{B}}_u\boldsymbol{W}+\boldsymbol{Y}_1+\boldsymbol{Y}_2-\boldsymbol{Y}_3,\quad \boldsymbol{\Xi}_4=\widetilde{\boldsymbol{B}}_p\boldsymbol{V}+\boldsymbol{X}\bar{\boldsymbol{C}}_q^{\mathrm{T}}\boldsymbol{H}+\boldsymbol{W}^{\mathrm{T}}\boldsymbol{D}_u^{\mathrm{T}}\boldsymbol{H},$$

$$\boldsymbol{\Xi}_5=-2\boldsymbol{V}+\boldsymbol{V}\boldsymbol{D}_p^{\mathrm{T}}\boldsymbol{H}+\boldsymbol{H}\boldsymbol{D}_p\boldsymbol{V},\quad \boldsymbol{\Xi}_6=d\boldsymbol{X}\bar{\boldsymbol{A}}^{\mathrm{T}}+d\boldsymbol{W}^{\mathrm{T}}\bar{\boldsymbol{B}}_u^{\mathrm{T}},\quad \boldsymbol{\Xi}_7=\boldsymbol{X}\bar{\boldsymbol{C}}_y^{\mathrm{T}}\boldsymbol{T}-\boldsymbol{W}^{\mathrm{T}}\boldsymbol{D}_{yu}^{\mathrm{T}}\boldsymbol{T}.$$

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by $\mathrm{diag}(\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{\Sigma}^{-1},\boldsymbol{R}^{-1},\boldsymbol{I},\boldsymbol{I})$ respectively and defining

$$\begin{cases} \boldsymbol{P}^{-1}=\boldsymbol{X},\ \boldsymbol{\Sigma}^{-1}=\boldsymbol{V},\ \boldsymbol{W}=\boldsymbol{K}\boldsymbol{X}, \\ \boldsymbol{Y}_1=\boldsymbol{P}^{-1}\boldsymbol{Q}_1\boldsymbol{P}^{-1},\ \boldsymbol{Y}_2=\boldsymbol{P}^{-1}\boldsymbol{Q}_2\boldsymbol{P}^{-1}, \\ \boldsymbol{Y}_3=\boldsymbol{P}^{-1}\boldsymbol{R}\boldsymbol{P}^{-1},\ \boldsymbol{Y}_4=\boldsymbol{P}^{-1}\boldsymbol{S}\boldsymbol{P}^{-1}, \end{cases} \quad (20)$$

we have Eq. (21) (at the top of the next page). Due to the fact that

$$(\boldsymbol{R}^{-1}-\boldsymbol{X})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{R}^{-1}-\boldsymbol{X})\geq 0, \quad (22)$$

we have

$$-\boldsymbol{R}^{-1}\leq \boldsymbol{X}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{X}-\boldsymbol{X}-\boldsymbol{X}^{\mathrm{T}}=\boldsymbol{Y}_3-\boldsymbol{X}-\boldsymbol{X}^{\mathrm{T}}. \quad (23)$$

Then based on inequality (21) and Eq. (23), we obtain

$$\overline{\boldsymbol{N}}_1\leq \boldsymbol{N}_1. \quad (24)$$

Since $\boldsymbol{N}_1<0$, we have $\overline{\boldsymbol{N}}_1<0$; i.e., $\boldsymbol{M}<0$ holds.

Similarly, pre- and post-multiplying the left and right sides of inequality (8) by $\mathrm{diag}(\boldsymbol{P}^{-1},\boldsymbol{P}^{-1})$, we have inequality (18). In turn, we derive inequality (8) once inequality (18) holds. Theorem 2 not only guarantees the stability of the controlled system but also determines the tracking controller gain. Thus, the proof is completed.

Remark 3 If $\boldsymbol{A}_d=\boldsymbol{C}_{qd}=\boldsymbol{0}$ or $\tau(t)=0$, system (1) is a non-delayed system as follows:

$$\begin{cases} \dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t))+\boldsymbol{B}_u\boldsymbol{u}(t)+\boldsymbol{B}_w\boldsymbol{w}(t), \\ \boldsymbol{\xi}(t)=\boldsymbol{C}_q\boldsymbol{x}(t)+\boldsymbol{D}_p\boldsymbol{\phi}(\boldsymbol{\xi}(t))+\boldsymbol{D}_u\boldsymbol{u}(t)+\boldsymbol{D}_w\boldsymbol{w}(t), \\ \boldsymbol{y}(t)=\boldsymbol{C}_y\boldsymbol{x}(t)+\boldsymbol{D}_{yu}\boldsymbol{u}(t)+\boldsymbol{D}_{yw}\boldsymbol{w}(t). \end{cases} \quad (25)$$

Theorem 2 is still valid for the design of feedback controller (4) and the following corollary gives the algorithm in detail:

Corollary 1 Given $d\geq 0$, $\mu\geq 0$, and $\gamma>0$, if there exist a symmetric positive definite matrix $\boldsymbol{X}$, a diagonal positive definite matrix $\boldsymbol{V}$, and a matrix $\boldsymbol{W}$ that satisfy Eq. (26) (see the next page), then the closed-loop system including Eqs. (25), (3), and (4) with $\boldsymbol{v}(t)=\boldsymbol{0}$ is globally asymptotically stable, and the L_2 gain of this closed-loop system is less than or equal to γ. Furthermore, the feedback gain can still be obtained from $\boldsymbol{K}=\boldsymbol{W}\boldsymbol{X}^{-1}$.

Remark 4 Apart from varying-time delayed systems, Theorem 2 is also applicable to constant-time

$$\overline{\boldsymbol{N}}_1 = \begin{bmatrix} \boldsymbol{\varXi}_3 & \widetilde{\boldsymbol{A}}_d\boldsymbol{X}+\boldsymbol{Y}_3-\boldsymbol{Y}_4 & \boldsymbol{Y}_4 & \boldsymbol{\varXi}_4 & \widetilde{\boldsymbol{B}}_w & \boldsymbol{\varXi}_6 & \boldsymbol{\varXi}_7 \\ * & -(1-\mu)\boldsymbol{Y}_2-2\boldsymbol{Y}_3+2\boldsymbol{Y}_4 & \boldsymbol{Y}_3-\boldsymbol{Y}_4 & \boldsymbol{X}\widetilde{\boldsymbol{C}}_{qd}^{\mathrm{T}}\boldsymbol{H} & \boldsymbol{0} & d\boldsymbol{X}\widetilde{\boldsymbol{A}}_d^{\mathrm{T}} & \boldsymbol{0} \\ * & * & -\boldsymbol{Y}_1-\boldsymbol{Y}_3 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & \boldsymbol{\varXi}_5 & \boldsymbol{H}\widetilde{\boldsymbol{D}}_w & d\boldsymbol{V}\widetilde{\boldsymbol{B}}_p^{\mathrm{T}} & \boldsymbol{0} \\ * & * & * & * & -\gamma^2\boldsymbol{I} & d\widetilde{\boldsymbol{B}}_w^{\mathrm{T}} & \widetilde{\boldsymbol{D}}_{yw}^{\mathrm{T}}\boldsymbol{T} \\ * & * & * & * & * & -\boldsymbol{R}^{-1} & \boldsymbol{0} \\ * & * & * & * & * & * & -\boldsymbol{T} \end{bmatrix} < 0. \tag{21}$$

$$\boldsymbol{N}_2 = \begin{bmatrix} (\overline{\boldsymbol{A}}\boldsymbol{X}+\overline{\boldsymbol{B}}_u\boldsymbol{W})^{\mathrm{T}}+\overline{\boldsymbol{A}}\boldsymbol{X}+\overline{\boldsymbol{B}}_u\boldsymbol{W} & \boldsymbol{\varXi}_4 & \widetilde{\boldsymbol{B}}_w & \boldsymbol{\varXi}_7 \\ * & \boldsymbol{\varXi}_5 & \boldsymbol{H}\widetilde{\boldsymbol{D}}_w & \boldsymbol{0} \\ * & * & -\gamma^2\boldsymbol{I} & \widetilde{\boldsymbol{D}}_{yw}^{\mathrm{T}}\boldsymbol{T} \\ * & * & * & -\boldsymbol{T} \end{bmatrix} < 0. \tag{26}$$

delayed systems just by setting $\mu=0$ with $\tau(t)=d$.

Remark 5 The optimal H_∞ tracking controller of the form Eq. (4) can be obtained by solving the eigenvalue problem (EVP) as follows:

Delayed:

$$\text{minimize } \gamma \tag{27}$$

$$\text{s.t. inequalities (17) and (18);} \tag{28}$$

Non-delayed:

$$\text{minimize } \gamma \tag{29}$$

$$\text{s.t. inequality (26).} \tag{30}$$

4 Numerical example

In this section, the continuous stirred tank reactor system is provided to demonstrate the effectiveness of the design methods proposed in Section 3. Since our method is based on the unified model, it is first necessary to transform the controlled system into the unified model.

Consider the following varying-time delayed continuous stirred tank reactor (Cao and Frank, 2000; Liu and Chiang, 2012):

$$\begin{cases} \dot{x}_1(t) = -\dfrac{1}{\lambda}x_1(t) + (\dfrac{1}{\lambda}-1)x_1(t-\tau(t)) \\ \qquad\qquad + D_a f(x_1,x_2) + w(t), \\ \dot{x}_2(t) = -(\dfrac{1}{\lambda}+\beta)x_2(t) + (\dfrac{1}{\lambda}-1)x_2(t-\tau(t)) \\ \qquad\qquad + mD_a f(x_1,x_2) + \beta u(t) + w(t), \\ y = x_2(t), \end{cases} \tag{31}$$

where $f(x_1,x_2) = (1-x_1(t))\exp\left(\dfrac{x_2(t)}{1+x_2(t)/\gamma_0}\right)$, the process noise $w(t)$ is chosen as a uniformly random noise, the state $0 \le x_1(t) \le 1$ corresponds to the conversion rate of the reaction, and $x_2(t) \ge 0$ is the dimensionless temperature. The parameters are $\gamma_0 = 20,\ m=8,\ D_a=0.072,\ \beta=0.3,\ \lambda=0.8$. We take $\tau(t)=0.1+0.1\sin t$, and thus $d=0.2$, $\mu=0.1$.

Referring to Liu and Chiang (2012), we set $0.1 \le x_2(t) \le 6$. Then we have $0 \le f(x_1(t),x_2(t))/x_2(t) \le 16$, i.e., $H=16$. Before applying Theorem 2 to develop the controller, we convert system (31) into the unified model (1), where

$$\boldsymbol{A} = \begin{bmatrix} -\frac{1}{\lambda} & 0 \\ 0 & -\frac{1}{\lambda}-\beta \end{bmatrix},\ \boldsymbol{A}_d = \begin{bmatrix} \frac{1}{\lambda}-1 & 0 \\ 0 & \frac{1}{\lambda}-1 \end{bmatrix},$$

$$\boldsymbol{B}_p = \begin{bmatrix} D_a & mD_a \end{bmatrix}^{\mathrm{T}},\ \boldsymbol{B}_u = \begin{bmatrix} 0 \\ \beta \end{bmatrix},\ \boldsymbol{B}_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\boldsymbol{C}_q = \begin{bmatrix} 0 & 1 \end{bmatrix},\ \boldsymbol{C}_{qd} = \boldsymbol{0}_{1\times 2},\ \boldsymbol{D}_p = \boldsymbol{0}_{1\times 2},$$

$$\boldsymbol{D}_u = \boldsymbol{0}_{1\times 1},\ \boldsymbol{D}_w = \boldsymbol{0}_{1\times 1},\ \boldsymbol{C}_y = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$\boldsymbol{D}_{yu} = \boldsymbol{D}_{yw} = \boldsymbol{0}_{1\times 1},$$

$$\phi(\xi(t)) = (1-x_1(t))\exp\left(\frac{\xi(t)}{1+\xi(t)/\gamma_0}\right),\ \xi(t)=x_2(t).$$

The reference model is given as follows:

$$\begin{cases} \dot{\boldsymbol{x}}_{\mathrm{r}}(t) = \boldsymbol{A}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{B}_{\mathrm{r}}\boldsymbol{r}(t), \\ \boldsymbol{y}_{\mathrm{r}}(t) = \boldsymbol{C}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}}(t), \end{cases} \tag{32}$$

where

$$\boldsymbol{A}_{\mathrm{r}} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{r}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_{\mathrm{r}} = \begin{bmatrix} 4 & 0 \end{bmatrix}.$$

In this example, we choose the following two kinds of reference inputs to verify the effectiveness of the established control laws:

$$r(t) = \sin(0.5t) + 3, \tag{33}$$

$$r(t) = \begin{cases} 1, & 0 \le t < 8, \\ 2, & 8 \le t < 16, \\ 3, & 16 \le t < 24, \\ 4, & 24 \le t < 32, \\ 2, & \text{otherwise}. \end{cases} \tag{34}$$

Note that γ and $\boldsymbol{T}$ here are equal to $1/\rho$ and $\boldsymbol{R}$ in Liu and Chiang (2012), respectively. To show the advantages of our method over that adopted in Liu and Chiang (2012), we take $\gamma = 1/\rho = 0.826$, $\boldsymbol{T} = \mathrm{diag}(0.01,\ 0.01)$, and $w(t)$ as a uniformly random noise with amplitude 0.01. Then according to Theorem 2, we obtain the feedback gains with the two reference inputs (Eqs. (33) and (34)) in the cases of $k = 8$ and $k = 15$, respectively, as follows:

k=8:

$$\boldsymbol{K}_1 = [-142.4924 \quad -157.075],$$
$$\boldsymbol{K}_2 = [581.1610 \quad 17.8439];$$

k=15:

$$\boldsymbol{K}_1 = [-157.3569 \quad -171.7296],$$
$$\boldsymbol{K}_2 = [641.8698 \quad 18.0441].$$

The simulation results with the reference input taken as Eq. (33) are depicted in Figs. 2 and 3. Fig. 2 demonstrates the system output and reference input described by Eq. (33), while the tracking errors in the cases of $k = 8$ and $k = 15$ are shown in Fig. 3. From Figs. 2 and 3, we can see that the tracking control performance is well achieved in the H_∞ sense.

Similarly, when the reference input is selected as the multi-step signal, the system output and reference input described by Eq. (34) are depicted in Fig. 4, while Fig. 5 displays the tracking errors when $k = 8$ and $k = 15$. The results in Figs. 4 and 5 validate the developed tracking controller in the form of Eq. (4). With the increase of k, the tracking error can be even smaller such that the tracking controller still exists. The value of k can be selected flexibly according to the desired performance. In Liu and Chiang (2012), however, the improvement for the tracking precision was not considered.

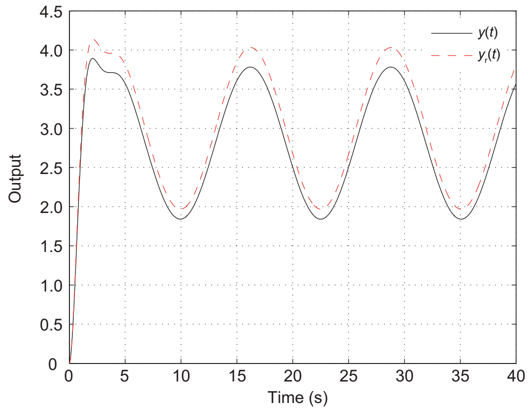

Fig. 2 The output of system (31) and the reference input (33) when $k = 8$

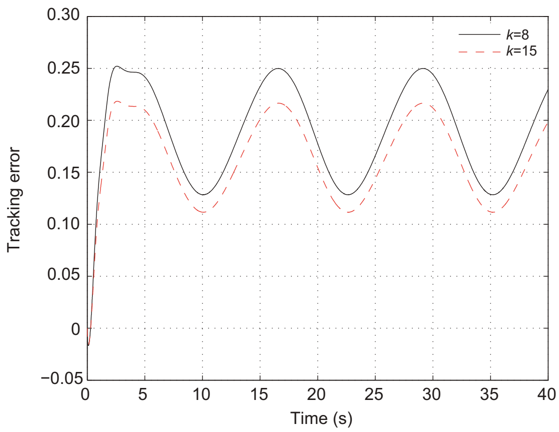

Fig. 3 Tracking errors in the cases of $k = 8$ and $k = 15$, respectively, with reference input (33)

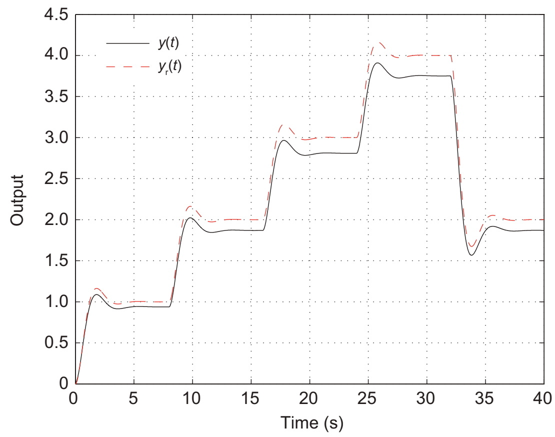

Fig. 4 The output of system (31) and the reference input (34) when $k = 8$

Furthermore, Liu and Chiang (2012) claimed to replace the time-varying delay with a suitable value approximately determined by trial-and-error. Though it is acceptable in practice, it is better in

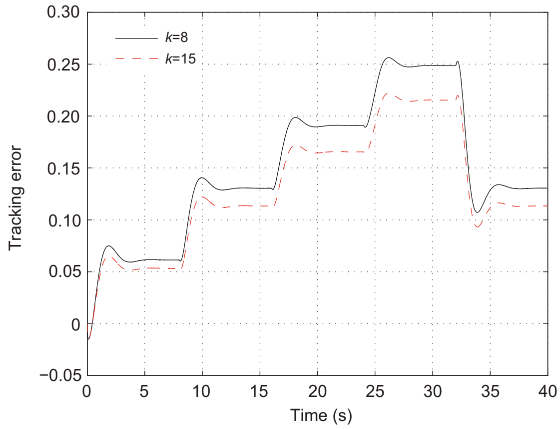

Fig. 5 Tracking errors in the cases of $k = 8$ and $k = 15$, respectively, with reference input (34)

our opinion to use the time-varying delay directly as is presented in our work. Compared with Liu and Chiang (2012), no extra uncertainties are introduced when we model the controlled system. Also, the method of trial-and-error in Liu and Chiang (2012) is time-consuming and will increase the workload of controller design. In this sense, the proposed method in our work is more convenient.

5 Conclusions

In this paper, the tracking control problem for a class of continuous varying-time delayed nonlinear systems is investigated via a unified model which consists of a linear delayed dynamic system and a static nonlinear operator. Based on this model, a state feedback controller is designed, which guarantees that the closed-loop system is stable and that the output of the system tracks the model reference trajectory in the H_∞ sense even in the case of unknown time-varying delays. The reference model used here is more flexible than previous ones. The scaling factor, a positive integer, is introduced to reduce the tracking error. Finally, a CSTR system is provided to demonstrate the effectiveness of the developed tracking control method.

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