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Controllability analysis of second-order multi-agent systems with directed and weighted interconnection*

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Abstract: This article investigates the controllability problem of multi-agent systems. Each agent is assumed to be governed by a second-order consensus control law corresponding to a directed and weighted graph. Two types of topology are considered. The first is concerned with directed trees, which represent the class of topology with minimum information exchange among all controllable topologies. A very simple necessary and sufficient condition regarding the weighting scheme is obtained for the controllability of double integrator multi-agent systems in this scenario. The second is concerned with a more general graph that can be reduced to a directed tree by contracting a cluster of nodes to a component. A similar necessary and sufficient condition is derived. Finally, several illustrative examples are provided to demonstrate the theoretical analysis results.

Key words: Multi-agent systems, Controllability, Directed tree
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1 Introduction

Recent years have witnessed a remarkable progress in the study of distributed control and coordination of networked multi-agent systems (Han *et al.*, 2015). Controllability analysis for leader-follower networks is a fundamental one in multi-agent systems. It is closely involved in complex network analysis (Liu *et al.*, 2011) and also related to formation control problems (Lin *et al.*, 2013; 2014; Zheng *et al.*, 2015). Basically, the controllability study of multi-agent systems aims at steering followers to any desired positions and/or velocities, through proper regulation of the leaders in the network. An interesting example from natural swarms is that we can make an artificial blackbeetle to guide the crowd of blackbeetles to a trap such that the use of insecticide is reduced.

Tanner (2004) introduced the controllability problem of multi-agent systems and presented a necessary and sufficient algebraic condition for it. Since then, the controllability study has received considerable attention in physics, mathematics, and control communities. Topological conditions were investigated, focusing mainly on undirected topologies with symmetric Laplacian matrices (Borsche and Attia, 2010; Martini *et al.*, 2010; Twu *et al.*, 2010; de la Croix and Egerstedt, 2012; Ji *et al.*, 2012). In Borsche and Attia (2010), the leader election problem was considered for the controllability purpose in multi-agent systems. In Ji *et al.* (2012), the leader election problem for multi-agent systems with a first-order consensus control law over an undirected tree was further investigated from the algebraic and graphical perspectives. Based on

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algebraic conditions of controllability, decentralized Laplacian spectrum estimation schemes were developed to verify the controllability of multi-agent systems (Franceschelli *et al.*, 2009; 2010). Moreover, controllability was shown to have a point to set property considering the interchangeable characteristic of agents (Twu *et al.*, 2010). Researchers also became interested in the controllability problem with more complex setups with switching topology and time delays (Liu *et al.*, 2008; 2014; Ji *et al.*, 2010).

Regarding weighting schemes, Jiang *et al.* (2009), Lou and Hong (2012), Liu *et al.* (2013), and Cai *et al.* (2014) brought structure controllability to multi-agent systems, which means that if there exist some weights such that a multi-agent system is controllable, then it is said to be structurally controllable (Lin, 1974). It was proved by Jiang *et al.* (2009) that a first-order multi-agent system is structurally controllable if and only if there exists at least one path from a leader to any follower. For the switching topology case, a first-order multi-agent system is structurally controllable if and only if the union graph is leader-follower connected (Liu *et al.*, 2013). By employing a weight-balanced partition method for a first-order multi-agent system with directed and weighting interconnection, structural controllability was analyzed by Lou and Hong (2012). Sundaram and Hadjicostis (2013) investigated structural controllability over finite fields and developed graph-theoretic conditions. Moreover, several decentralized methods were developed either to verify observability/controllability or to select an optimal leader, but they focused on the case of undirected control topologies and the results cannot be extended to directed control topologies (Franceschelli *et al.*, 2009; 2010; Borsche and Attia, 2010).

For multi-agent systems, there are two factors contributing to the controllability, i.e., the interaction topologies and the local interaction laws (Liu *et al.*, 2011). Most attention has been given to the characterization of controllable topologies. Despite the extensive work considering rather complicated issues, e.g., switching topologies and time delays, the derived algebraic and graphical conditions are not straightforward, which makes it not easy to extend the results to multi-agent systems with more complex local interaction laws. In contrast to the existing literature, we consider the controllability problem of a second-order multi-agent system with directed and weighting interconnection and aim to obtain easy-to-verify criteria for controllability checking in terms of the interaction weights. Towards this goal, two types of topology are considered. The first is concerned with directed trees, which represent the class of topology with minimum information exchange among all controllable topologies. A very simple necessary and sufficient condition regarding the weighting scheme is obtained in this scenario. The second is concerned with a more general graph that can be reduced to a directed tree by contracting a cluster of nodes to a component. A similar necessary and sufficient condition is derived. Our results differ from those in de la Croix and Egerstedt (2012), Ji *et al.* (2012), and Liu *et al.* (2014) in the aspects of both topologies and multi-agent system models. Moreover, compared with the results in Jiang *et al.* (2009), Lou and Hong (2012), Liu *et al.* (2013), and Cai *et al.* (2014) where the main concern is the existence of feasible weights such that a multi-agent system is controllable, our work, however, provides explicit conditions regarding the weighting scheme while the existence of feasible weights is ensured as a pre-condition.

Notations: In this paper, we denote $\boldsymbol{O}$ the matrix with all the elements zero, and $\boldsymbol{I}$ the identity matrix. If the dimension needs to be explicitly described, then $\boldsymbol{O}_n$ and $\boldsymbol{I}_n$ are used to indicate the zero matrix and the identity matrix of dimension $n \times n$, respectively. Moreover, $\mathbf{0}$ indicates a zero vector. We denote $\lambda(\boldsymbol{A})$ (or λ for simplicity) the eigenvalues of matrix $\boldsymbol{A}$ and '$\otimes$' the Kronecker product.

2 Preliminaries and problem formulation

In this section we present some basic notions from graph theory and then formulate the controllability problem studied in this paper.

2.1 Graph notions

A weighted digraph $\mathcal{G}$ consists of a node set $\mathcal{V}$ and an arc set $\mathcal{E}$. Each node represents an agent. Each arc in $\mathcal{E}$ is an ordered pair $(i, j), i, j \in \mathcal{V}$, which represents a directional sensing medial from i to j and is weighted by a positive number denoted by w_{ji}. If $(i, j) \in \mathcal{E}$, i is an in-neighbor of j, and j is an out-neighbor of i. An alternating sequence of distinct nodes and arcs in $\mathcal{G}$ is called a path. Node i

is said to be reachable from j if there exists at least one path from j to i.

An entry of the Laplacian matrix of $\mathcal{G}$, which is denoted as $\boldsymbol{\mathcal{L}}(\mathcal{G})$ or $\boldsymbol{\mathcal{L}}$ for simplicity, is defined as

$$\mathcal{L}_{ij} = \begin{cases} -w_{ij}, & (j,i) \in \mathcal{E}, \\ 0, & (j,i) \notin \mathcal{E}, \\ w_i, & i = j, \end{cases}$$

where $w_i = \sum_{j \neq i} w_{ij}$ is the degree of node i.

A digraph is called a directed tree if only one node called root has zero in-neighbor, and every other node has one in-neighbor and is reachable from the root. The root represents the leader.

In a directed tree, if two nodes are not reachable from each other, then they are said to be in different branches. In Fig. 1, nodes $j+1$ and $m+2$ are in different branches while nodes l and $l+1$ are in the same branch. In a directed tree, we call a node a fork node if it has multiple out-neighbors. Taking Fig. 1 as an example, nodes j, k, and the leader are all fork nodes. We call a node a leaf node if it has no out-neighbor. In Fig. 1, nodes $l+1$, $n+1$, $m+1$, and N are all leaf nodes. It is clear that the number of leaf nodes equals the number of branches. A stem is a simple directed tree with only one branch.

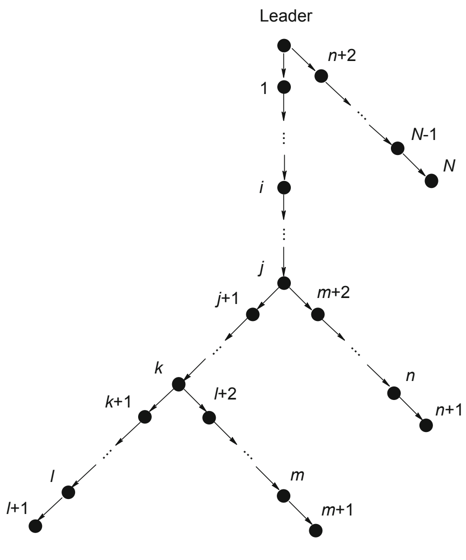

Fig. 1 A directed tree

We label the nodes of a directed tree according to the following rule. First, label the nodes in one branch in order starting from an out-neighbor of the leader to the leaf in the branch. For example, in Fig. 1 agents $1, 2, \ldots, i, \ldots, j, j+1, \ldots, k, k+1, \ldots, l, l+1$ are labeled where $i \leq j \leq k \leq l$. Second, label the nodes in another branch in order starting from the un-labeled out-neighbor of the labeled fork node with the largest number. For example, in Fig. 1 agents $l+2, \ldots, m, m+1$ are then labeled sequentially.

Furthermore, we introduce a contracted tree as depicted in Fig. 2. The ellipse element ($\mathcal{G}_i$, $i = 1, 2, \ldots, 9$) in the figure is called a component, which corresponds to a cluster of nodes. Similar to directed trees, if there is no path between $\mathcal{G}_i$ and $\mathcal{G}_j$, then they are in different branches. Note that if the ellipse elements in Fig. 2 are stems, then Fig. 2 becomes a directed tree. The definition of a leaf component is the same as the leaf node in a directed tree, which means that a leaf component has no out-neighboring component. We define a subgraph that is composed by components from the leader to a leaf component as LL-subgraph. For instance, there are six LL-subgraphs in Fig. 2, i.e., {leader, $\mathcal{G}_1, \mathcal{G}_4$}, {leader, $\mathcal{G}_1, \mathcal{G}_5$}, {leader, $\mathcal{G}_2$}, {leader, $\mathcal{G}_3, \mathcal{G}_6, \mathcal{G}_8$}, {leader, $\mathcal{G}_3, \mathcal{G}_7$}, and {leader, $\mathcal{G}_3, \mathcal{G}_6, \mathcal{G}_9$}.

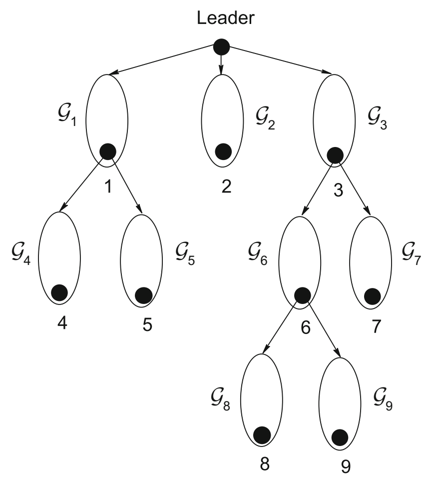

Fig. 2 A contracted tree

2.2 Controllability problem of multi-agent systems

A multi-agent system is considered in the leader-follower framework whose interaction graph is described by a digraph $\mathcal{G}$. Without loss of generality, assume there are $N+1$ agents, and agent $N+1$ serves as the leader, which is controlled by outside operators, while agents $1, 2, \ldots, N$ are followers.

Each follower is governed by a double-integrator dynamics:

$$\ddot{z}_i = \sum_{(j,i)\in\mathcal{E}} w_{ij}(z_j - z_i) - \eta \dot{z}_i,\ i = 1, 2, \ldots, N, \quad (1)$$

where $w_{ij} \in \mathbb{R}_+$ is the weight of the arc from node j to i and $\eta > 0$ is the damping gain. The leader's state z_{N+1} is determined by outside operators. For simplicity, we consider only the case in which z_i is a one-dimensional real number. It is trivial to derive the same conclusion for the multi-dimensional case.

Denote $\boldsymbol{x}_i \triangleq [z_i \quad \dot{z}_i]^{\mathrm{T}}, i = 1, 2, \ldots, N$ and $\boldsymbol{x}_{N+1} = [z_{N+1} \ \dot{z}_{N+1}]^{\mathrm{T}}$. We can rewrite each follower's dynamics as

$$\dot{\boldsymbol{x}}_i = \tilde{\boldsymbol{A}}\boldsymbol{x}_i + \sum_{(j,i)\in\mathcal{E}} w_{ij}\tilde{\boldsymbol{B}}(\boldsymbol{x}_j - \boldsymbol{x}_i),$$

where

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} 0 & 1 \\ 0 & -\eta \end{pmatrix} \text{ and } \tilde{\boldsymbol{B}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The whole system can be written in the following compact form:

$$\dot{\boldsymbol{x}} = \underbrace{(\boldsymbol{I}_N \otimes \tilde{\boldsymbol{A}} + \boldsymbol{\mathcal{L}}_{11} \otimes \tilde{\boldsymbol{B}})}_{\boldsymbol{A}}\boldsymbol{x} + \underbrace{\boldsymbol{\mathcal{L}}_{12} \otimes \tilde{\boldsymbol{B}}}_{\boldsymbol{B}}\boldsymbol{u}, \quad (2)$$

where

$$\boldsymbol{x} = [\boldsymbol{x}_1^{\mathrm{T}} \quad \boldsymbol{x}_2^{\mathrm{T}} \quad \ldots \quad \boldsymbol{x}_N^{\mathrm{T}}]^{\mathrm{T}}, \boldsymbol{u} = \boldsymbol{x}_{N+1},$$

$$\boldsymbol{A} \in \mathbb{R}^{2N\times 2N}, \boldsymbol{B} \in \mathbb{R}^{2N\times 2},$$

$$\boldsymbol{\mathcal{L}}(\mathcal{G}) = -\begin{pmatrix} \boldsymbol{\mathcal{L}}_{11} & \boldsymbol{\mathcal{L}}_{12} \\ \boldsymbol{0}^{\mathrm{T}} & 0 \end{pmatrix}, \boldsymbol{\mathcal{L}}_{11} \in \mathbb{R}^{N\times N}, \boldsymbol{\mathcal{L}}_{12} \in \mathbb{R}^{N}.$$

Our problem is to determine whether the multi-agent system (2) is controllable for proper weights under certain topologies, i.e., whether it is possible to drive these follower agents to any configuration (states and velocities) by properly designed movement of the leader.

From Jiang *et al.* (2009) we know that a necessary condition for system (2) to be controllable is that all followers are reachable from the leader. It is clear that a directed tree with the leader to be the root satisfies the necessary condition. Moreover, in all the topologies that satisfy the necessary condition, a directed tree has the least number of arcs, which means that the information exchange among agents is minimal.

Thus, we first investigate the controllable condition for multi-agent system (2) over a directed tree depicted in Fig. 1. Second, we extend to investigate the controllable condition for multi-agent system (2) over a contracted tree depicted in Fig. 2.

We introduce a well-known controllability result of linear systems, which is the main tool for our theoretical development. For a linear system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times m}, \quad (3)$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector and $\boldsymbol{u}$ is the m-dimensional control input. The following lemma is classical (Antsaklis and Michel, 2006):

Lemma 1 The linear system (3) is controllable if and only if the matrix $[\boldsymbol{A} - \lambda\boldsymbol{I} \quad \boldsymbol{B}]$ has full row rank for any eigenvalue $\lambda(\boldsymbol{A})$.

3 Necessary and sufficient controllability conditions

3.1 Controllability of multi-agent systems over directed trees

For multi-agent systems over directed trees, we present our main result regarding the controllability below. In a directed tree, we know that there is only one weight for each node, i.e., $w_i = w_{ij}$. So, in this subsection, we will use w_i instead of w_{ij} for simplicity.

Theorem 1 The multi-agent system (2) over a directed tree $\mathcal{G}$ is controllable if and only if $w_i \neq w_j$ for agents i and j at different branches of $\mathcal{G}$.

The proof of Theorem 1 requires the following three technical lemmas.

The first lemma shows the explicit formula about the eigenvalues of the system matrix $\boldsymbol{A}$ of the multi-agent system (2) over a directed tree.

Lemma 2 For the multi-agent system (2) over a directed tree, there is

$$\lambda(\boldsymbol{A}) = \frac{-\eta \pm \sqrt{\eta^2 - 4w_1}}{2}, \ldots, \frac{-\eta \pm \sqrt{\eta^2 - 4w_N}}{2}.$$

Proof With the labeling method described in the previous section, the system matrix $\boldsymbol{A}$ of the multi-agent system (2) over a directed tree is a lower triangular matrix. Thus, it can be simply obtained that $|\boldsymbol{A} - \lambda\boldsymbol{I}| = |\tilde{\boldsymbol{A}} - w_1\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2| \cdot |\tilde{\boldsymbol{A}} - w_2\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2| \ldots |\tilde{\boldsymbol{A}} - w_N\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2|$, from which the conclusion follows.

Lemma 2 gives a controllable condition for the multi-agent system (2) over a stem, which is the simplest directed tree. An example of a stem is depicted in Fig. 3.

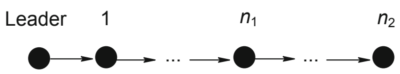

Fig. 3 A stem

Lemma 3 The multi-agent system (2) over a stem is controllable for any weight.

Proof Note that for the multi-agent system (2) over a stem, the system matrix $\boldsymbol{A}$ has the form given in Eq. (4) and

$$\boldsymbol{B} = [w_1\tilde{\boldsymbol{B}} \quad \boldsymbol{O} \quad \ldots \quad \boldsymbol{O}]^{\mathrm{T}}. \tag{6}$$

Thus, $[\boldsymbol{A} - \lambda\boldsymbol{I} \quad \boldsymbol{B}]$ has the form given in Eq. (5). Note that columns $1, 2, \ldots, 2(n_2 - 1), 2n_2 + 1, 2n_2 + 2$ of Eq. (5) form a $2n_2 \times 2n_2$ matrix. By using the Laplace cofactor expansion along the last two columns, we find that the determinant of this square matrix is nonzero. Hence, Eq. (5) is of full row rank and thus the multi-agent system (2) over a stem is controllable for any weight by Lemma 1.

The third lemma is concerned with the controllability of the multi-agent system (2) over a directed tree, for which the leader has only one out-neighbor. An example is shown in Fig. 4.

Lemma 4 The multi-agent system (2) over a directed tree $\mathcal{G}$ with the leader having one out-neighbor is controllable if and only if $w_i \neq w_j$ for agents i and j at different branches of $\mathcal{G}$.

Proof First, we consider the case in which $\mathcal{G}$ has only two branches. In this case, the system matrix

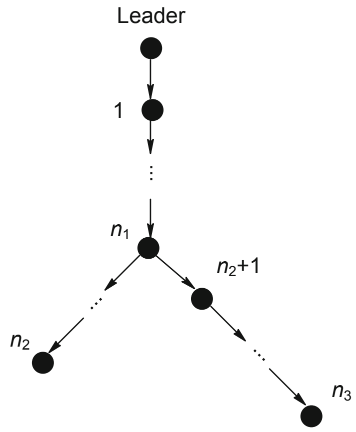

Fig. 4 A directed tree: the leader has one out-neighbor

$\boldsymbol{A}$ and input matrix $\boldsymbol{B}$ of the multi-agent system (2) over $\mathcal{G}$ are of the following form:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & & \\ \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{L} & \boldsymbol{A}_2 & \\ \boldsymbol{B}_3^{\mathrm{T}}\boldsymbol{L} & & \boldsymbol{A}_3 \end{pmatrix}, \tag{7}$$

$$\boldsymbol{B} = [\boldsymbol{B}_1^{\mathrm{T}} \quad \boldsymbol{O}\ldots\boldsymbol{O} \quad \boldsymbol{O}\ldots\boldsymbol{O}]^{\mathrm{T}} \in \mathbb{R}^{2n_3\times 2}, \tag{8}$$

where

$$\boldsymbol{A}_1 \in \mathbb{R}^{2n_1\times 2n_1},$$
$$\boldsymbol{A}_2 \in \mathbb{R}^{2(n_2-n_1)\times 2(n_2-n_1)},$$
$$\boldsymbol{A}_3 \in \mathbb{R}^{2(n_3-n_2)\times 2(n_3-n_2)}$$

are similar to Eq. (4), $\boldsymbol{B}_1$ is given as Eq. (6),

$$\boldsymbol{B}_2 = [w_{n_1+1}\tilde{\boldsymbol{B}}^{\mathrm{T}}\ \boldsymbol{O}\ \ldots\ \boldsymbol{O}]^{\mathrm{T}} \in \mathbb{R}^{2(n_2-n_1)\times 2},$$
$$\boldsymbol{B}_3 = [w_{n_2+1}\tilde{\boldsymbol{B}}^{\mathrm{T}}\ \boldsymbol{O}\ \ldots\ \boldsymbol{O}]^{\mathrm{T}} \in \mathbb{R}^{2(n_3-n_2)\times 2},$$
$$\boldsymbol{L} = [\boldsymbol{O}\ \boldsymbol{O}\ \ldots\ \boldsymbol{I}_2] \in \mathbb{R}^{2\times 2n_1}.$$

$$\begin{pmatrix} \tilde{\boldsymbol{A}} - w_1\tilde{\boldsymbol{B}} & & & & \\ w_2\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}} - w_2\tilde{\boldsymbol{B}} & & & \\ & & \ddots & & \\ & & & \tilde{\boldsymbol{A}} - w_{n_2-1}\tilde{\boldsymbol{B}} & \\ & & & w_{n_2}\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}} - w_{n_2}\tilde{\boldsymbol{B}} \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} \tilde{\boldsymbol{A}} - w_1\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2 & & & & & w_1\tilde{\boldsymbol{B}} \\ w_2\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}} - w_2\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2 & & & & \\ & & \ddots & & & \\ & & & \tilde{\boldsymbol{A}} - w_{n_2-1}\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2 & & \\ & & & w_{n_2}\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}} - w_{n_2}\tilde{\boldsymbol{B}} - \lambda\boldsymbol{I}_2 & \end{pmatrix} \tag{5}$$

Similar to Eq. (5), $(\boldsymbol{A}_1, \boldsymbol{B}_1)$, $(\boldsymbol{A}_2, \boldsymbol{B}_2)$, and $(\boldsymbol{A}_3, \boldsymbol{B}_3)$ are all controllable pairs. We now check the rank of

$$[\boldsymbol{A} - \lambda \boldsymbol{I} \quad \boldsymbol{B}] =$$

$$\left(\begin{array}{c|c|c|c} \boldsymbol{A}_1 - \lambda \boldsymbol{I} & & & \boldsymbol{B}_1 \\ \hline \boldsymbol{B}_2^{\mathrm{T}} \boldsymbol{L} & \boldsymbol{A}_2 - \lambda \boldsymbol{I} & & \\ \hline \boldsymbol{B}_3^{\mathrm{T}} \boldsymbol{L} & & \boldsymbol{A}_3 - \lambda \boldsymbol{I} & \end{array}\right). \tag{9}$$

Sufficiency: If the agents in different branches of $\mathcal{G}$ have different weights, then according to Lemma 2, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ have different eigenvalues. Note that for any λ, at most one of $\boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)}$ and $\boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)}$ is not of full row rank. If $\boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)}$ is not of full row rank, then $\boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)}$ is of full row rank and

$$\left(\begin{array}{c|c|c} \boldsymbol{A}_1 - \lambda \boldsymbol{I}_{2n_1} & & \boldsymbol{B}_1 \\ \hline \boldsymbol{B}_2^{\mathrm{T}} \boldsymbol{L} & \boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)} & \end{array}\right)$$

is the same as Eq. (5). Again, using the Laplace cofactor expansion along the last two columns of a square matrix similar to Eq. (5), we find that it has full row rank. Hence, Eq. (9) is of full row rank. The conclusion is also true when $\boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)}$ is not of full row rank for some λ. If both $\boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)}$ and $\boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)}$ are of full row rank, then $[\boldsymbol{A}_1 - \lambda \boldsymbol{I}_{2n_1} \quad \boldsymbol{B}_1]$, $\boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)}$, and $\boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)}$ are all of full row rank. Thus, Eq. (9) is of full row rank and therefore, the conclusion follows from Lemma 1.

Necessity: If two agents in different branches have a common weight, then according to Lemma 2, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ have one identical eigenvalue. So, $\operatorname{rank}(\boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)}) \le 2(n_2 - n_1) - 1$ and $\operatorname{rank}(\boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)}) \le 2(n_3 - n_2) - 1$. Note

$$\begin{pmatrix} \boldsymbol{B}_2^{\mathrm{T}} \boldsymbol{L} \\ \boldsymbol{B}_3^{\mathrm{T}} \boldsymbol{L} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & w_{n_1+1} & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & w_{n_2+1} & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}. \tag{10}$$

Thus, the rank of

$$\left(\begin{array}{c|c|c} \boldsymbol{B}_2^{\mathrm{T}} \boldsymbol{L} & \boldsymbol{A}_2 - \lambda \boldsymbol{I}_{2(n_2-n_1)} & \\ \hline \boldsymbol{B}_3^{\mathrm{T}} \boldsymbol{L} & & \boldsymbol{A}_3 - \lambda \boldsymbol{I}_{2(n_3-n_2)} \end{array}\right)$$

is less than or equal to $2(n_3 - n_2) - 1$. Hence, Eq. (9) is not possible to be of full row rank. According to Lemma 1, it follows that the multi-agent system (2) is not controllable.

Second, if there are more than two branches in $\mathcal{G}$, then the same argument can be applied repeatedly and the same conclusion follows.

Next, we come to prove Theorem 1 using Lemmas 2–4.

Proof of Theorem 1 For a general directed tree with an example given in Fig. 5, the system matrix $\boldsymbol{A}$ and input matrix $\boldsymbol{B}$ of multi-agent system (2) are of the following forms, respectively:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_5 & \\ & \boldsymbol{A}_6 \end{pmatrix}, \ \boldsymbol{B} = [\boldsymbol{B}_5^{\mathrm{T}} \quad \boldsymbol{B}_6^{\mathrm{T}}]^{\mathrm{T}},$$

where $\boldsymbol{A}_5$ is similar to Eq. (7), $\boldsymbol{A}_6 \in \mathbb{R}^{2(n_4-n_3)\times 2(n_4-n_3)}$ has the similar form to Eq. (4), $\boldsymbol{B}_5$ is given in Eq. (8), and $\boldsymbol{B}_6 \in \mathbb{R}^{2(n_4-n_3)\times 2}$ has the similar form to Eq. (6).

Sufficiency: If the agents in different branches have different weights, then from Lemmas 3 and 4 and their proofs, we know that $(\boldsymbol{A}_5, \boldsymbol{B}_5)$ and $(\boldsymbol{A}_6, \boldsymbol{B}_6)$ are both controllable pairs, and that $\boldsymbol{A}_5$ and $\boldsymbol{A}_6$ have no identical eigenvalue. If $\lambda(\boldsymbol{A}) = \lambda(\boldsymbol{A}_5)$, then $\boldsymbol{A}_6 - \lambda \boldsymbol{I}_{2(n_4-n_3)}$ and $[\boldsymbol{A}_5 - \lambda \boldsymbol{I}_{2n_3} \quad \boldsymbol{B}_5]$ are both of full row rank. Thus, $[\boldsymbol{A} - \lambda \boldsymbol{I} \quad \boldsymbol{B}]$ is of full row rank. If $\lambda(\boldsymbol{A}) = \lambda(\boldsymbol{A}_6)$, then $\boldsymbol{A}_5 - \lambda \boldsymbol{I}_{2n_3}$ and $[\boldsymbol{A}_6 - \lambda \boldsymbol{I}_{2(n_4-n_3)} \quad \boldsymbol{B}_6]$ are both of full row rank. Thus, $[\boldsymbol{A} - \lambda \boldsymbol{I} \quad \boldsymbol{B}]$ is of full row rank. Hence, multi-agent system (2) is controllable.

Necessity: Suppose that two agents in different branches have a common weight. First, we consider the case in which these two agents are in the sets $\{n_1 + 1, \dots, n_2\}$ and $\{n_2 + 1, \dots, n_3\}$, respectively. Then it can be known that $(\boldsymbol{A}_5, \boldsymbol{B}_5)$ is not controllable. Thus, $[\boldsymbol{A} - \lambda \boldsymbol{I} \quad \boldsymbol{B}]$ cannot be of full row rank. Second, we consider the case in which these two agents are in the sets $\{1, 2, \dots, n_3\}$ and $\{n_3 + 1, n_3 + 2, \dots, n_4\}$, respectively. Then it can be known that $\boldsymbol{A}_5$ and $\boldsymbol{A}_6$ have an identical eigenvalue. So, it follows that $\operatorname{rank}(\boldsymbol{A}_5 - \lambda \boldsymbol{I}_{n_3}) \le 2n_3 - 1$ and $\operatorname{rank}(\boldsymbol{A}_6 - \lambda \boldsymbol{I}_{2(n_4-n_3)}) \le 2(n_4 - n_3) - 1$. Note that $\boldsymbol{B}$ has the similar form to Eq. (10). Thus, the rank of $[\boldsymbol{A} - \lambda \boldsymbol{I} \quad \boldsymbol{B}]$ is less than or equal to $2n_4 - 1$, which is not of full row rank. In conclusion, for both cases, multi-agent system (2) is not controllable.

If there are more branches, the proof is similar.

Remark 1 In a directed tree, if the agents with the same in-neighbor have the same weight, e.g., in Fig. 5 agents 1 and n_3+1 choose the same weight, i.e.,

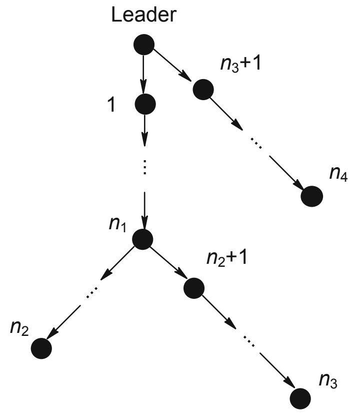

Fig. 5 A directed tree

$w_1 = w_{n_3+1}$, it is clear that agents 1 and n_3+1 have exactly the same dynamics. Thus, the multi-agent system is not controllable as the states of agents 1 and n_3+1 cannot be driven to different ones. Such a case is excluded by our condition in Theorem 1 as agents 1 and n_3+1 are in different branches, which leads to the requirement of different weights for them.

In Theorem 1, the same damping parameter η is used for all the follower agents in control law (1). Indeed, the following corollary shows the similar result when different damping gains η_i are used in control law (1).

Corollary 1 Suppose that the follower agents have different damping gains, i.e., $\eta_i \neq \eta_j$ for $i \neq j$. Then the multi-agent system (2) over a directed tree $\mathcal{G}$ is controllable if and only if the agents in different branches of $\mathcal{G}$ have different $\lambda(\boldsymbol{M}_i)$, where

$$\boldsymbol{M}_i = \begin{pmatrix} 0 & 1 \\ -w_i & -\eta_i \end{pmatrix}.$$

Proof Based on exactly the same argument as for Theorem 1, we can obtain the above conclusion.

Remark 2 If every agent chooses its weight randomly and independently in a real line according to some probability distribution, then the probability for agents in different branches to have a common weight is 0. Hence, by doing so, the multi-agent system (2) is controllable with probability 1.

3.2 Controllability of multi-agent systems over contracted trees

Next, we come to investigate the controllability of multi-agent systems over a class of more general class of directed graphs, called contracted trees.

Theorem 2 The multi-agent system (2) over a contracted tree $\mathcal{G}$ is controllable if and only if the following two conditions hold:

1. Every LL-subgraph of $\mathcal{G}$ is controllable.
2. The components $\mathcal{G}_i$ in different branches of $\mathcal{G}$ have different eigenvalues.

The proof of Theorem 2 needs the following lemma:

Lemma 5 The system eigenvalues of the multi-agent system (2) over a contracted tree $\mathcal{G}$ are composed of the eigenvalues of all the sub-systems corresponding to all the components in $\mathcal{G}$.

Proof Suppose that there are n components in $\mathcal{G}$. Then similar to the proof of Lemma 2, we can write the system matrix $\boldsymbol{A}$ of multi-agent system (2) in the following form:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & & & \boldsymbol{O} \\ & \boldsymbol{A}_2 & & \\ & \boldsymbol{C} & \ddots & \\ & & & \boldsymbol{A}_n \end{pmatrix},$$

where each $\boldsymbol{A}_i$ $(i = 1, 2, \ldots, n)$ is the block corresponding to each component $\mathcal{G}_i$ in $\mathcal{G}$. Thus, $\lambda(\boldsymbol{A}) = \{\lambda(\boldsymbol{A}_1), \lambda(\boldsymbol{A}_2), \ldots, \lambda(\boldsymbol{A}_n)\}$.

Proof of Theorem 2 Similar to the proof of Theorem 1, we can check the rank of $[\boldsymbol{A} - \lambda\boldsymbol{I} \quad \boldsymbol{B}]$ and show that the system is controllable if and only if the two conditions in the theorem are fulfilled.

In the following, we consider two particular contracted trees, for which the controllability conditions will become simpler.

The first is concerned with a class of contracted trees, for which each component is a stem with multiple feedback arcs. An example of one component is given in Fig. 6. Note that in Fig. 6, there exists a path $1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow N$, and there also exist multiple arcs from i to j, where $i > j$.

We call it a contracted tree with feedback arcs. Now we are ready to present our main result for this case.

Theorem 3 The multi-agent system (2) over a contracted tree $\mathcal{G}$ with feedback arcs is controllable if and only if the components $\mathcal{G}_i$ in different branches of $\mathcal{G}$ have different eigenvalues.

The proof of Theorem 3 requires the following result concerning the controllability of multi-agent systems over a stem with feedback arcs.

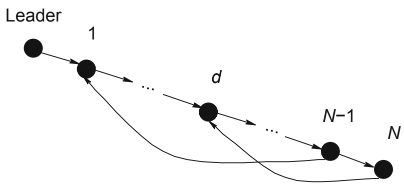

Fig. 6 A stem with feedback arcs

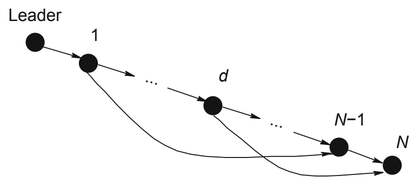

Fig. 7 A stem with feedforward arcs

Lemma 6 The multi-agent system (2) over a stem with feedback arcs is controllable for any weight.

Proof We can see that the system matrix $\boldsymbol{A}$ of the multi-agent system (2) over a stem with feedback arcs has the form given in Eq. (11), where '$*$' may or may not be a zero matrix depending on whether there exist arcs, and

$$\boldsymbol{B} = [w_l\tilde{\boldsymbol{B}} \quad \boldsymbol{O} \quad \ldots \quad \boldsymbol{O}]^{\mathrm{T}} \in \mathbb{R}^{2N\times 2}.$$

We check the rank of matrix $[\boldsymbol{A}-\lambda\boldsymbol{I} \quad \boldsymbol{B}]$. Similar to the stem without feedback arcs, columns 1, 2, ..., $2(N-1)$, $2N+1$, $2N+2$ of $[\boldsymbol{A}-\lambda\boldsymbol{I} \quad \boldsymbol{B}]$ form a $2N\times 2N$ matrix with a nonzero determinant. Hence, $[\boldsymbol{A}-\lambda\boldsymbol{I} \quad \boldsymbol{B}]$ is always of full row rank for any λ.

Proof of Theorem 3 Consider any LL-subgraph of $\mathcal{G}$. It can be checked that condition 1 of Theorem 2 is satisfied according to Lemma 6. Thus, condition 2 of Theorem 2 becomes the necessary and sufficient condition for this class of contracted trees.

The second is concerned with another class of contracted trees, for which each component is a stem with multiple feedforward arcs. An example is given in Fig. 7. As we can see in Fig. 7, there exists a path $1\to 2\to 3\to\ldots\to N$, but in this case there also exist multiple arcs from i to j where $i<j$. We call it a contracted tree with feedforward arcs. Next, we present the controllability result for this case.

Theorem 4 The multi-agent system (2) over a contracted tree $\mathcal{G}$ with feedforward arcs is controllable if and only if the following two conditions are satisfied:

1. Every stem with feedforward arcs from the leader to a leaf agent is controllable.
2. Agents in different branches of $\mathcal{G}$ have different weighted degrees $w_i=\sum_{(j,i)\in\mathcal{E}} w_{ij}$.

Proof A stem with feedforward arcs from the leader to a leaf agent is an LL-subgraph. If it is controllable, then condition 1 of Theorem 2 is fulfilled. Similar to Lemma 2, we can obtain that $\lambda(\boldsymbol{A})=\lambda(\tilde{\boldsymbol{A}}-w_i\tilde{\boldsymbol{B}})$. Hence, condition 2 of Theorem 2 holds if and only if agents in different branches of $\mathcal{G}$ have different weighted degrees.

4 Numerical examples

In this section, we give several examples to illustrate the theoretical results developed in previous sections.

4.1 Illustration of Theorem 1

The first example is depicted in Fig. 8. The leader has three out-neighbors. Let $w_1=w_2=w_3=w_4=1$, $w_5=w_6=2$, $w_7=w_8=w_9=3$, $w_{10}=w_{11}=w_{12}=4$, and $w_{13}=w_{14}=5$. Thus, the agents in different branches have different weights. Using Lemma 1 to verify the controllability, we find that $[\boldsymbol{A}-\lambda\boldsymbol{I} \quad \boldsymbol{B}]$ has full row rank for all $\lambda(\boldsymbol{A})$. Hence, the multi-agent system in this example is controllable.

Now let $w_1=1$, $w_2=2$, $w_3=3$, $w_4=4$, $w_5=5$, $w_6=6$, $w_7=7$, $w_8=8$, $w_9=9$, $w_{10}=10$, $w_{11}=11$, $w_{12}=w_4=4$, $w_{13}=13$, and $w_{14}=14$. We can see that agents 4 and 12 are in different branches but they have the same weight. Again, using Lemma 1, we find that $[\boldsymbol{A}-\lambda\boldsymbol{I} \quad \boldsymbol{B}]$ is rank

$$\begin{pmatrix} \tilde{\boldsymbol{A}}-w_1\tilde{\boldsymbol{B}} & * & * & \cdots & * \\ w_{21}\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}}-w_2\tilde{\boldsymbol{B}} & * & \cdots & * \\ & w_{32}\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}}-w_3\tilde{\boldsymbol{B}} & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & w_{N,N-1}\tilde{\boldsymbol{B}} & \tilde{\boldsymbol{A}}-w_N\tilde{\boldsymbol{B}} \end{pmatrix} \in \mathbb{R}^{2N\times 2N} \tag{11}$$

deficient. Hence, the multi-agent system in this example is not controllable.

The above two situations for Example 1 (Fig. 8) show that our result (Theorem 1) is correct.

4.2 Illustration of Theorem 3

Fig. 9 is an example of a contracted tree with feedback arcs. In this example, we let all the weights in $\mathcal{G}_1$ be 1, let all the weights in $\mathcal{G}_2$ be 2, and let all the weights in $\mathcal{G}_3$ be 3. According to Theorem 3, the multi-agent system is controllable since $\mathcal{G}_3$ has different eigenvalues with both $\mathcal{G}_1$ and $\mathcal{G}_2$

Now we change the weights in $\mathcal{G}_3$ to 2. We can see that $\mathcal{G}_2$ and $\mathcal{G}_3$ are of the same topology with the same weights. Therefore, they have common eigenvalues. The multi-agent system is not controllable in this case.

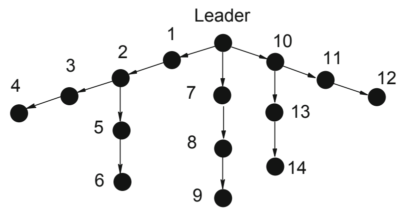

Fig. 8 Example 1: a directed tree

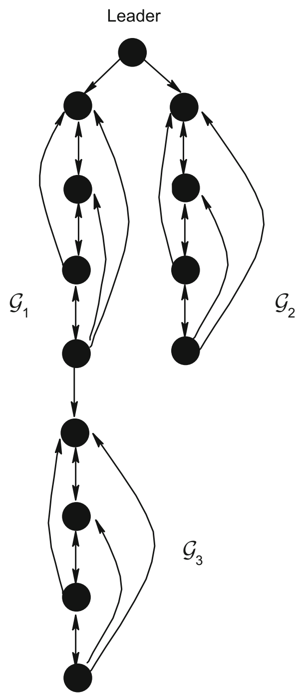

Fig. 9 Example 2: a contracted tree with feedback arcs

4.3 Illustration of Theorem 4

Fig. 10 is an example of a contracted tree with feedforward arcs. In this example, we first let $w_1 = w_{21} = w_{43} = w_{54} = w_{65} = 2$, $w_{31} = w_{32} = 3$, $w_{64} = 1$, $w_{73} = 4$, $w_{87} = 5$, $w_{97} = 2$, and $w_{98} = 4$. It can be checked that the LL-subgraph composed of agents $1, 2, 3, 7, 8, 9$ is not controllable, making the whole system not controllable.

Second, we change w_{73} from 4 to 1. The two conditions in Theorem 4 are fulfilled. Hence, the multi-agent system over the graph given in Fig. 10 is controllable with the weights above.

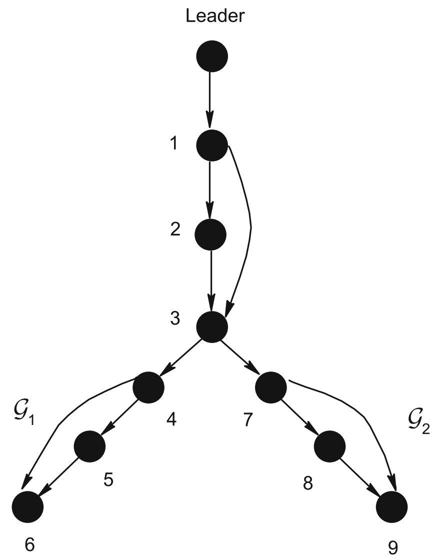

Fig. 10 Example 3: a contracted tree with feedforward arcs

5 Conclusions

This article investigates the controllability problem of second-order multi-agent systems over directed trees and contracted trees. Necessary and sufficient controllable conditions are obtained, which are easy to verify. That is, by choosing different weights for agents at different branches, the controllability can be ensured, which also implies that the multi-agent system is controllable with probability one if the agents randomly and independently select their weights in their local interaction laws. In this vein, many interesting problems can be further studied, including extension to generic linear time-invariant models and more complex graphs.

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